Some Results On Super Edge Magic n-Stars

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Abstract

For integer $n \geq 2$, let $a_1, a_2, a_3, \ldots, a_n$ be an increasing sequence of nonnegative integers, and define the n-star $St(a_1, a_2, \ldots, a_n)$ as the disjoint union of the n star graphs $K(1, a_1)$, $K(1, a_2), \ldots, K(1, a_n)$. In this paper we have partially settled the conjecture by Lee and Kong [4] that says for any odd $n \geq 3$, the n-star $St(a_1, a_2, \ldots, a_n)$ is super edge magic. We solve the two cases

1. The n-star $St(a_1, a_2, \ldots, a_n)$ is super edge magic where $a_i = 1 + (i-1)d$ for all integers $1 \leq i \leq n$ and d is any positive integer.
2. An n-star $St(a_1, a_2, \ldots, a_n)$ is not super edge magic when $a_1 = 0$.

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1 Introduction

In 1967, Rosa [5] introduced graph labelings, the effects of which can be found in [2]. In 1970, the concept of edge magic total labeling was introduced by Kotzig and Rosa [3]. Motivated by the concept of edge magic total labelings, in 1998 Enomoto et al. [1] defined the concept of super edge magic labelings.

In this paper we consider finite simple (p,q)-graphs $G(V,E)$ with p vertices and q edges. A graph G is called edge magic total if there is a bijection $f : V \cup E \to \{1,2,\ldots,p+q\}$ such that $f(u)+f(v)+f(e) = c_f$, is constant for each $e=(u,v) \in E$ [3]. A edge magic total graph is called super edge magic if $f(V) = \{1,2,\ldots,p\}$ and $f(E) = \{p+1, p+2, \ldots, p+q\}$ [1]. For any $n \geq 2$, let $a_1, a_2, a_3, \ldots, a_n$ be an increasing sequence of nonnegative integers and define the n-star $St(a_1,a_2,a_3,\ldots,a_n)$ as the disjoint union of the star graph $K(1,a_1), K(1,a_2), \ldots, K(1,a_n)$. The graph $St(a_1,a_2,a_3,\ldots,a_n)$ is shown in Figure 1.

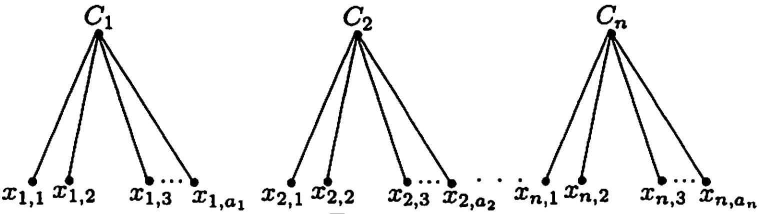


Figure 1

S. M. Lee and M. C. Kong [4] conjectured that for any odd $n \geq 3$ the n-star $St(a_1,a_2,a_3,\ldots,a_n)$ is super edge magic. In this paper the conjecture is partially solved. We solve the two cases

1. The n-star $St(a_1,a_2,\ldots,a_n)$ is super edge magic where $a_i = 1+(i-1)d$ for all integers $1 \leq i \leq n$ and d is any positive integer.

2. An n-star $St(a_1,a_2,\ldots,a_n)$ is not super edge magic when $a_1 = 0$.

2 Main Result

Result 2.1. *For any odd integer $n \geq 3$, if $a_1,a_2,a_3\ldots,a_n$ is an increasing sequence of positive integers where $a_i = 1+(i-1)d$ for integers $1 \leq i \leq n$ and any positive integer d, then the n-star $St(a_1,a_2,a_3,\ldots,a_n)$ is super edge magic.*

Proof. Let $n \geq 3$ be an odd integer and let $a_1,a_2,\ldots,a_n$ be an increasing sequences of positive integers, where $a_i = 1+(i-1)d$ for integers $1 \leq i \leq n$ and any positive integer d. Denote the central vertices of the n-star $G = St(a_1,a_2,a_3,\ldots,a_n)$ by $c_1,c_2,c_3,\ldots,c_n$, the pendant vertices by $x_{i,j}$ and incident edges by $e_{i,j} = (c_i, x_{i,j})$ for integers $1 \leq i \leq n$ and $1 \leq j \leq 1+(n-1)d$. Define a bijection $f : V(G) \to \{1,2,3,\ldots,(2n+\frac{n(n-1)}{2}d)\}$ and $f : E(G) \to \{2n+\frac{n(n-1)}{2}d+1, 2n+\frac{n(n-1)}{2}d+2, 2n+\frac{n(n-1)}{2}d+3, \ldots, (3n+n(n-1)d)\}$ as follows.

For the central vertices, we have

$$f(c_i) = \begin{cases} \frac{3n+1-2i}{2} + (d-1)(n-2) & \text{if } i \text{ is odd,} \\ & 1 \le i \le n-2, n \ge 3 \\ \frac{3n-3+2i}{2} + (d-1)(n-2) & \text{if } i \text{ is even,} \\ & 1 < i < n-2, n \ge 5 \\ \frac{5i+2}{2} + (d-1)(n-2) & \text{if } i = n-1 \\ \frac{3i-3}{2} + (d-1)(n-2) & \text{if } i = n. \end{cases}$$

For the pendant vertices, we have

$$f(x_{1,1}) = \begin{cases} d+4 & \text{if } n = 3 \\ -d+2+(n-1)(d+1) & \text{if } n \ge 5, n = 4k+1, k \ge 1 \\ -d+1+(n-1)(d+1) & \text{if } n \ge 5, n = 4k+3, k \ge 1 \end{cases}$$

$$f(x_{i,1}) = \begin{cases} -d+1+(n-1)(d+\frac{3}{2}) & \text{if } n \ge 5, i = 2 \\ -d+2+(n-1)(d+\frac{3}{2}) + \frac{(i-1)}{2} & \text{if } n \ge 5, i = 2k+1, k \ge 1, \\ & 3 \le i \le n-2 \\ -d+\frac{3}{2}+(n-1)(d+\frac{3}{2}) - \frac{(i-1)}{2} & \text{if } n \ge 7, i = 4k, k \ge 1, \\ & 4 \le i \le n-2 \\ -d+\frac{3}{2}+(n-1)(d+\frac{1}{2}) - \frac{(i-1)}{2} & \text{if } n \ge 9, i = 4k+2, k \ge 1, \\ & 6 \le i \le n-2 \end{cases}$$

$$f(x_{i,j}) = \begin{cases} -d+(t-1)i+(n-1)(d+2) & \text{if } n \ge 5, \\ \quad +j+\frac{d(i-2)(i-1)}{2} & 1 < i \le n-2, i \text{ is even,} \\ & 1+(i-1)(t-1) < j \\ & \le 1+(i-1)t, 1 \le t \le d \\ -(t+(i-1)(d-t)) & \text{if } n \ge 5, \\ \quad +(n-1)(d+2)+j+\frac{d(i-2)(i-1)}{2} & 1 < i \le n-2, i \text{ is odd,} \\ & 1+(i-1)(t-1) < j \\ & \le 1+(i-1)t, 1 \le t \le d \end{cases}$$

$$f(x_{n-1,j}) = \begin{cases} j & \text{if } 1 \le j \le 1+(n-2)d, n=3 \\ j & \text{if } 1 \le j \le \frac{n+3}{2}+(n-2)(d-1), \\ & n \ge 5 \\ \frac{4j-n-3}{2}-(n-2)(d-1) & \text{if } \frac{n+3}{2}+(n-2)(d-1) < j \\ & \le \frac{3n+3}{4}+(n-2)(d-1), n \ge 5, \\ & n = 4k+3, k \ge 1 \\ 2j-2-(n-2)(d-1) & \text{if } \frac{3n+3}{4}+(n-2)(d-1) < j \\ & \le 1+(n-2)d, n \ge 5, \\ & n=4k+3, k \ge 1 \\ \frac{4j-n-3}{2}-(n-2)(d-1) & \text{if } \frac{n+3}{2}+(n-2)(d-1) < j \\ & \le \frac{3n+1}{4}+(n-2)(d-1), n \ge 5, \\ & n=4k+5, k \ge 1 \\ 2j-1-(n-2)(d-1) & \text{if } \frac{3n+1}{4}+(n-2)(d-1) < j \\ & \le 1+(n-2)d, n \ge 5, \\ & n=4k+5, k \ge 1 \end{cases}$$

$$f(x_{n,j}) = \frac{n^2+n+2j}{2} + \frac{(n-2)(n-1)}{2}(d-1) \quad \text{if } 1 \le j \le 1+(n-1)d.$$

For the edges, we have

$$f(e_{1,1}) = \begin{cases} n^2+3n-5+(n^2-n-1)(d-1) & \text{if } n=3 \\ n^2+n+1+(n^2-2n+2)(d-1) & \text{if } n \ge 5, n=4k+1, k \ge 1 \\ n^2+n+2+(n^2-2n+2)(d-1) & \text{if } n \ge 5, n=4k+3, k \ge 1 \end{cases}$$

$$f(e_{i,1}) = \begin{cases} \frac{2n^2+n+3}{2}+(d-1)(n^2-2n+2) & \text{if } n \ge 5, i=2 \\ \frac{2n^2+n+2+i}{2}+(d-1)(n^2-2n+2) & \text{if } n \ge 5, i=2k+1, k \ge 1, \\ & 3 \le i \le n-2 \\ \frac{2n^2+n+5-i}{2}+(d-1)(n^2-2n+2) & \text{if } n \ge 7, i=4k, k \ge 1, \\ & 4 \le i \le n-2 \\ \frac{2n^2+3n+3-i}{2}+(d-1)(n^2-2n+2) & \text{if } n \ge 9, i=4k+2, k \ge 1, \\ & 6 \le i \le n-2 \end{cases}$$

$$f(e_{i,j}) = \begin{cases} n^2+2+i+t+(i-1)(d-t) & \text{if } n \geq 5, 1 < i \leq n-2, \\ +(d-1)(n^2-2n+1) & i \text{ is odd}, \\ -j-\frac{d(i-2)(i-1)}{2} & (1+(i-1)(t-1)) < j \\ & \leq (1+(i-1)t), 1 \leq t \leq d \\ n^2+5-it & \text{if } n \geq 5, 1 < i \leq n-2, \\ +(d-1)(n^2-2n+2)-j & i \text{ is even}, \\ -\frac{d(i-2)(i-1)}{2} & 1+(i-1)(t-1) < j \\ & \leq 1+(i-1)t, 1 \leq t \leq d \end{cases}$$

$$f(e_{n-1,j}) = \begin{cases} (n+1)^2-j+(d-1)(n^2-n) & \text{if } 1 \leq j \leq 1+(n-2)d, \\ & n=3 \\ (n+1)^2-j+(d-1)(n^2-n) & \text{if } 1 \leq j \leq \frac{n+3}{2} \\ & +(n-2)(d-1), n \geq 5 \\ \frac{2n^2+5n+5-4j}{2}+(d-1)(n^2-2) & \text{if } \frac{n+3}{2}+(n-2)(d-1) \\ & < j \leq \frac{3n+3}{4} \\ & +(n-2)(d-1), n \geq 5, \\ & n=4k+3, k \geq 1 \\ n^2+2n+3-2j+(d-1)(n^2-2) & \text{if } \frac{3n+3}{4}+(n-2)(d-1) \\ & < j \leq 1+(n-2)d, n \geq 5, \\ & n=4k+3, k \geq 1 \\ \frac{2n^2+5n+5-4j}{2}+(d-1)(n^2-2) & \text{if } \frac{n+3}{2}+(n-2)(d-1) \\ & < j \leq \frac{3n+1}{4} \\ & +(n-2)(d-1), n \geq 5, \\ & n=4k+5, k \geq 1 \\ n^2+2n+2-2j+(d-1)(n^2-2) & \text{if } \frac{3n+1}{4}+(n-2)(d-1) \\ & < j \leq 1+(n-2)d, n \geq 5, \\ & n=4k+5, k \geq 1 \end{cases}$$

$$f(e_{n,j}) = \frac{(n^2+5n+2)d-4(n+1)(d-1)-2j}{2} \quad \text{if } 1 \leq j \leq 1+(n-1)d.$$

For this labeling f, the constant $c_f = \frac{2n^2+9n-1}{2}+(d-1)(n^2-2)$.

If $n=3$, then $f(c_1)+f(x_{1,1})+f(e_{1,1}) = \frac{3n-1}{2}+(d-1)(n-2)+d+4+n^2+3n-5+(n^2-n-1)(d-1) = c_f$.

If $n \geq 5$ and $n \equiv 1 (mod\ 4)$, then $f(c_1)+f(x_{1,1})+f(e_{1,1}) = \frac{3n-1}{2}+(d-1)(n-2)-d+2+(n-1)(d+1)+n^2+n+1+(n^2-2n+2)(d-1) = cf$.

If $n \geq 5$ and $n \equiv 3 (mod\ 4)$, then $f(c_1)+f(x_{1,1})+f(e_{1,1}) = \frac{3n-1}{2}+(d-1)(n-2)-d+1+(n-1)(d+1)+n^2+n+2+(n^2-2n+2)(d-1) = c_f$.

If $n \geq 5$ and $i = 2$, then $f(c_i) + f(x_{i,1}) + f(e_{i,1}) = \frac{3n+1}{2} + (d-1)(n-2) - d + 1 + (n-1)\left(d + \frac{3}{2}\right) + \frac{2n^2+n+3}{2} + (d-1)(n^2 - 2n + 2) = c_f$.

If $n \geq 5$ and $i \equiv 1 (mod\, 2), 3 \leq i \leq n-2$, then $f(c_i) + f(x_{i,1}) + f(e_{i,1}) = \frac{3n+1-2i}{2} + (d-1)(n-2) - d + 2 + (n-1)\left(d + \frac{3}{2}\right) + \frac{i-1}{2} + \frac{2n^2+n+2+i}{2} + (d-1)(n^2 - 2n + 2) = c_f$.

If $n \geq 7$ and $i \equiv 0 (mod\, 4), 4 \leq i \leq n-2$, then $f(c_i) + f(x_{i,1}) + f(e_{i,1}) = \frac{3n-3+2i}{2} + (d-1)(n-2) - d + \frac{3}{2} + (n-1)\left(d + \frac{3}{2}\right) - \frac{i-1}{2} + \frac{2n^2+n+5-i}{2} + (d-1)(n^2 - 2n + 2) = c_f$.

If $n \geq 9$ and $i \equiv 2 (mod\, 4), 6 \leq i \leq n-2$, then $f(c_i) + f(x_{i,1}) + f(e_{i,1}) = \frac{3n-3+2i}{2} + (d-1)(n-2) - d + \frac{3}{2} + (n-1)\left(d + \frac{1}{2}\right) - \frac{i-1}{2} + \frac{2n^2+3n+3-i}{2} + (d-1)(n^2 - 2n + 2) = c_f$.

If $n \geq 5$ and $i \equiv 1 (mod\, 2), 1 \leq i \leq n-2, 1+(i-1)(t-1) < j \leq 1+(i-1)t$, $1 \leq t \leq d$, then $f(c_i) + f(x_{i,j}) + f(e_{i,j}) = \frac{3n+1-2i}{2} + (d-1)(n-2) - t - (i-1)(d-t) + (n-1)(d+2) + j + \frac{d(i-2)(i-1)}{2} + n^2 + 2 + i + t + (i-1)(d-t) + (n^2 - 2n + 1)(d-1) - j - \frac{d(i-2)(i-1)}{2} = c_f$.

If $n \geq 5$ and $i \equiv 0 (mod\, 2), 1 < i \leq n-2, 1+(i-1)(t-1) < j \leq 1+(i-1)t$, $1 \leq t \leq d$, then $f(c_i)+f(x_{i,j})+f(e_{i,j}) = \frac{3n-3+2i}{2}+(d-1)(n-2)-d+(t-1)i+(n-1)(d+2)+j+\frac{d(i-2)(i-1)}{2}+n^2+5-it+(d-1)(n^2-2n+2)-j-\frac{d(i-2)(i-1)}{2} = c_f$.

If $n = 3$ and $1 \leq j \leq 1+(n-2)d$, then $f(c_{n-1})+f(x_{n-1,j})+f(e_{n-1,j}) = \frac{5(n-1)+2}{2} + (d-1)(n-2) + j + (n+1)^2 - j + (d-1)(n^2 - n) = c_f$.

If $n \geq 5$ and $1 \leq j \leq \frac{n+3}{2} + (n-2)(d-1)$, then $f(c_{n-1}) + f(x_{n-1,j}) + f(e_{n-1,j}) = \frac{5(n-1)+2}{2} + (d-1)(n-2) + j + (n+1)^2 - j + (d-1)(n^2 - n) = c_f$.

If $n \geq 5$ and $n \equiv 3 (mod\, 4)$, $\frac{n+3}{2}+(n-2)(d-1) < j \leq \frac{3n+3}{4}+(n-2)(d-1)$, then $f(c_{n-1})+f(x_{n-1,j})+f(e_{n-1,j}) = \frac{5(n-1)+2}{2}+(d-1)(n-2)+\frac{4j-n-3}{2} - (d-1)(n-2) + \frac{2n^2+5n+5-4j}{2} + (d-1)(n^2-2) = c_f$.

If $n \geq 5$ and $n \equiv 3 (mod\, 4)$, $\frac{3n+3}{4} + (n-2)(d-1) < j \leq 1 + (n-2)d$, then $f(c_{n-1}) + f(x_{n-1,j}) + f(e_{n-1,j}) = \frac{5(n-1)+2}{2} + (d-1)(n-2) + 2j - 2 - (d-1)(n-2) + n^2 + 2n + 3 - 2j + (d-1)(n^2 - 2) = c_f$.

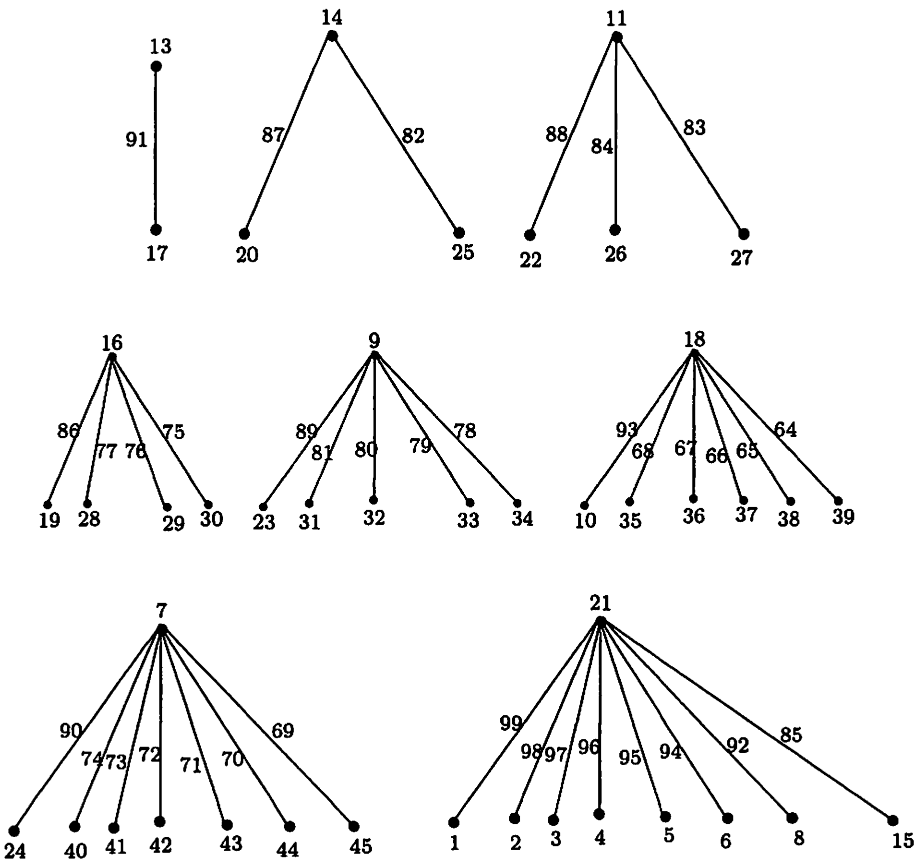
If $n \geq 5$ and $n \equiv 1 (mod\, 4)$, $\frac{n+3}{2}+(n-2)(d-1) < j \leq \frac{3n+1}{4}+(n-2)(d-1)$, then $f(c_{n-1})+f(x_{n-1,j})+f(e_{n-1,j}) = \frac{5(n-1)+2}{2}+(d-1)(n-2)+\frac{4j-n-3}{2} - (d-1)(n-2) + \frac{2n^2+5n+5-4j}{2} + (d-1)(n^2-2) = c_f$.

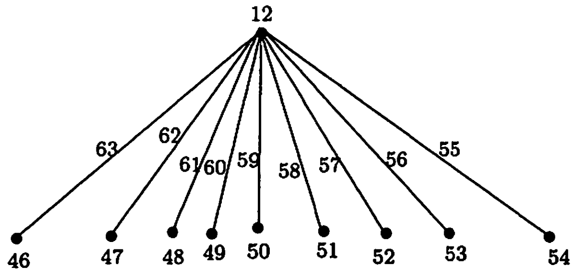
If $n \geq 5$ and $n \equiv 1(mod\ 4)$, $\frac{3n+1}{4} + (n-2)(d-1) < j \leq 1 + (n-2)d$, then $f(c_{n-1}) + f(x_{n-1,j}) + f(e_{n-1,j}) = \frac{5(n-1)+2}{2} + (d-1)(n-2) + 2j - 1 - (d-1)(n-2) + n^2 + 2n + 2 - 2j + (d-1)(n^2-2) = c_f$.

If $1 \leq j \leq 1 + (n-1)d$, then $f(c_n) + f(x_{n,j}) + f(e_{n,j}) = \frac{3n-3}{2} + (d-1)(n-2) + \frac{n^2+n+2j}{2} + \frac{(n-2)(n-1)}{2}(d-1) + \frac{(n^2+5n+2)d-4(n+1)(d-1)-2j}{2} = c_f$.

Hence f is a super edge magic labeling. □

Illustration. $St(1,2,3,4,5,6,7,8,9)$ is super edge magic.





The f defined in the proof of Result 2.1 provides a labeling for $St(1,2,3,4,5,6,7,8,9)$ with $c_f = 121$.

Figure 2

Result 2.2. *For any odd integer $n \geq 3$, if $a_1, a_2, a_3 \ldots, a_n$ is an increasing sequence of nonnegative integers, then the n-star $St(a_1, a_2, a_3, \ldots, a_n)$ is not super edge magic when $a_1 = 0$.*

Proof. Let G be the n-star $St(a_1, a_2, a_3, \ldots, a_n)$. When $a_1 = 0$, the graph G has an isolated vertex. By the definition of super edge magic the vertex label of an isolated vertex is not equal to the sum of the vertex labels and edge label of any edge in G. Hence G is not super edge magic. □

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