The Laplacian polynomial of graphs derived from regular graphs and applications*

Jia-Bao Liu[a,b], Xiang-Feng Pan[a,†] and Fu-Tao Hu[a]
[a]School of Mathematical Sciences, Anhui University,
Hefei, Anhui, 230601, China
[b]Department of Public Courses, Anhui Xinhua University,
Hefei, Anhui, 230088, China

Abstract

Let $R(G)$ be the graph obtained from G by adding a new vertex corresponding to each edge of G and by joining each new vertex to the end vertices of the corresponding edge. Let $RT(G)$ be the graph obtained from $R(G)$ by adding a new edge corresponding to every vertex of G, and by joining the end vertices of each new edge to the corresponding vertex of G. In this paper, we determine the Laplacian polynomials of $RT(G)$ of a regular graph G. Moreover, we derive formulae and lower bounds of Kirchhoff indices of the graphs. Finally we also present the formulae for calculating the Kirchhoff indices of some special graphs as applications, which show the correction and efficiency of the proposed results.

Keywords: Kirchhoff index; Resistance distance; Laplacian polynomial; Laplacian spectrum.

AMS Classification: 05C35, 92E10.

*Partially supported by NNSFC (Nos.11471016, 11401004, 11171097 and 11371028), Anhui Provincial Natural Science Foundation (No.1408085QA03), Natural Science Foundation of Anhui Province of China (No.KJ2013B105).

†Corresponding author.
E-mail addresses: liujiabaoad@163.com (J.-B. Liu), xfpan@ustc.edu (X.-F. Pan), hufu@mail.ustc.edu.cn (F.-T. Hu).

1 Introduction

All graphs considered in this paper are simple and undirected. Let $G = (V(G), E(G))$ be a graph with vertex set $V(E) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. The adjacency matrix of G, denoted by $A(G)$, is the $n \times n$ matrix whose (i, j)-entry is 1 if v_i and v_j are adjacent in G and 0 otherwise. Let $B(G)$ denote the adjacency matrix and vertex-edge incidence matrix of G, which is the $n \times m$ matrix whose (i, j)-entry is 1 if v_i is incident to e_j and 0 otherwise. Denote $D(G)$ to be the diagonal matrix with diagonal entries $d_G(v_1), d_G(v_2), \ldots, d_G(v_n)$. The Laplacian matrix of G defined as $L(G) = D(G) - A(G)$. The Laplacian characteristic polynomial of $L(G)$, is defined as $\phi(L(G); x) = det(xI_n - L(G))$, or simply $\phi(L)$, where I_n is the identity matrix of size n, and its roots, denoted by $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G) = 0$ are called the Laplacian eigenvalues of G. The collection of eigenvalues of $L(G)$ together with their multiplicities are called the L-spectrum of G. Similar terminology will be used for $A(G)$. The adjacency characteristic polynomial of G, denoted by $\varphi(A(G); x)$, is defined as $\varphi(A(G); x) = det(xI_n - A(G))$, the eigenvalues of $A(G)$ are $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$. The collection of eigenvalues of $A(G)$ together with their multiplicities are called the A-spectrum of G. For other undefined notations and terminology from graph theory, the readers may refer to [1, 2, 27] and the references therein.

Klein and Randić [3] introduced a new distance function named resistance distance based on electrical network theory. The resistance distance between vertices i and j, denoted by r_{ij}, is defined to be the effective electrical resistance between them if each edge of G is replaced by a unit resistor [3]. The resistance distances attracted extensive attention due to its wide applications in physics, chemistry, etc. [4–7, 25]. For more information on resistance distances of graphs, the readers are referred to the recent papers [8, 9, 26]. A large amount of graph operations such as the Cartesian product, the Kronecker product, the corona and neighborhood corona graphs have been introduced in [17–21]. The following definition comes from [1] (See the definition in p. 63 in [1]).

Definition 1.1. (See [1]) Let $R(G) = (V(R(G)), E(R(G)))$ be the graph obtained from G by adding a new vertex e' corresponding to each edge $e = (a, b)$ of G and by joining each new vertex e' to the end vertices a and b of the corresponding edge $e = (a, b)$. (See Fig. 1(a) and (b) for example).

From the above definition, it is obvious that $R(G)$ is obtained from G by "changing each edge $e = (a, b)$ of G into a triangle $ae'b$". Thus, $V(R(G)) = V(G) \cup \{e' \mid e \in E(G)\}$ and $E(R(G)) = E(G) \cup \{(v_i, e'), (v_j, e') \mid e = (v_i, v_j) \in E(G)\}$. A very elementary and natural question is what it would be like if we change each edge and each vertex of G into a triangle, which is stated as the following definiton.

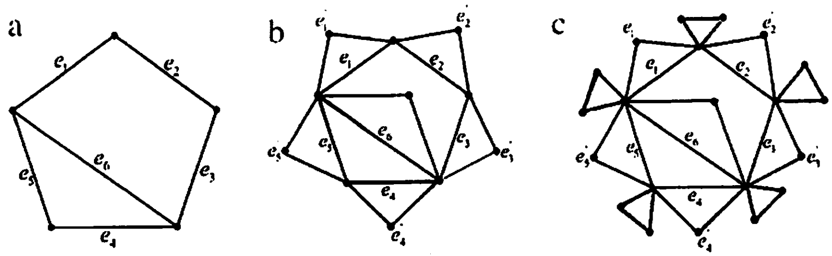

Figure 1: (a) The graph G. (b) The graph $R(G)$. (c) The graph $RT(G)$.

Definition 1.2. Let $RT(G) = (V(RT(G)), E(RT(G)))$ be the graph obtained from RG by adding a new edge $e''_i = (w^i_1, w^i_2)$ corresponding to each vertex v_i of G and by joining the two vertices of each new edge to each vertex v_i of G, $RT(G)$ is obtained from G by "changing each edge and each vertex of G into a triangle". Thus, $V(RT(G)) = \{e' \mid e \in E(G)\} \cup V(G) \cup \{w^i_1 \mid i = 1, 2, \dots, n\} \cup \{w^i_2 \mid i = 1, 2, \dots, n\}$ and $E(RT(G)) = \{(v_i, e'), (v_j, e') \mid e = (v_i, v_j) \in E(G)\} \cup E(G) \cup \{(v_i, w^i_1), (v_i, w^i_2) \mid v_i \in V(G), e''_i = (w^i_1, w^i_2) \in E(RT(G)), i = 1, 2, \dots, n\}$. (See Fig. 1(a), (b) and (c) for example).

As the authors of [11] pointed out, it is an interesting problem to study the Kirchhoff index of graphs derived from a single graph. In [12], the authors obtained formulae and lower bounds of the Kirchhoff index of some graphs. In [13], Wang et al. determined the Laplacian polynomials of $R(G)$ and $Q(G)$ of a regular graph G, they also derived formulae and lower bounds of the Kirchhoff index of those graphs. Motivated by the results, in this paper we further explore the Laplacian polynomials of $RT(G)$ of a regular graph G. Moreover, we derive the formulae and lower bounds of Kirchhoff index of the graphs. In particular, special formulae are proposed for the Kirchhoff index of $RT(G)$, where G is a complete graph K_n, a cycle C_n and a regular complete bipartite graph $K_{n,n}$.

2 Preliminaries

At the beginning of this section, we review some concepts in matrix theory. The Kronecker product $A \otimes B$ of two matrices $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{p \times q}$ is the $mp \times nq$ matrix obtained from A by replacing each element a_{ij} by $a_{ij}B$. The readers are referred to [22] for other properties of the Kronecker product not mentioned here.

The symbols $\mathbf{0}_n$ and $\mathbf{1}_n$ (resp., $\mathbf{0}_{mn}$ and $\mathbf{1}_{mn}$) will stand for the length-n column vectors (resp. $m \times n$ matrices) consisting entirely of 0's and 1's.

Lemma 2.1. (See [10]) Let M_1, M_2, M_3 and M_4 be respectively $p \times p$,

$p \times q$, $q \times p$ and $q \times q$ matrices with M_1 and M_4 invertible, then

$$det\begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix} = det(M_4) \cdot det(M_1 - M_2M_4^{-1}M_3) \quad (1)$$

$$= det(M_1) \cdot det(M_4 - M_3M_1^{-1}M_2), \quad (2)$$

where $M_1 - M_2M_4^{-1}M_3$ and $M_4 - M_3M_1^{-1}M_2$ are called the Schur complements of M_4 and M_1, respectively.

3 The Laplacian polynomials of RT(G)

For a regular graph G, the following theorem gives the representation of the Laplacian polynomial of $RT(G)$ by means of the characteristic polynomial and the Laplacian polynomial of G, respectively.

Theorem 3.1. Let G be an r-regular graph with n vertices and m edges, then (i) $\phi\Big(RT(G);\mu\Big) = (\mu-1)^n(\mu-2)^{m-n}(\mu-3)^n(3-\mu)^n$

$$\cdot\varphi\Big(G; \tfrac{(\mu-2)^2}{3-\mu} + \tfrac{r(2\mu-3)}{\mu-3} + \tfrac{2(\mu-2)}{(\mu-1)(\mu-3)}\Big).$$

(ii) $\phi\Big(RT(G);\mu\Big) = (\mu-1)^n(\mu-2)^{m-n}(\mu-3)^{2n}$

$$\cdot\phi\Big(G; \tfrac{(\mu-2)^2}{\mu-3} - \tfrac{r\mu}{\mu-3} - \tfrac{2(\mu-2)}{(\mu-1)(\mu-3)}\Big).$$

Proof. (i) Let G be an arbitrary r-regular graph with n vertices and m edges. Label the vertices of $RT(G)$ as follows. Let $I(G) = \{e_1, e_2, \dots, e_m\}$, $V(G) = \{v_1, v_2, \dots, v_n\}$ and $V(e'') = \{w_1, w_2\}$, and let w_1^i, w_2^i denote the vertices of the i-th copy of e'' for $i = 1, 2, \dots, n$, with the understanding that w_j^i is the copy of w_j for each j. Denote $W_j = \{w_j^1, w_j^2, \dots, w_j^n\}$, for $j = 1, 2$, then

$$I(G)\bigcup V(G)\bigcup[W_1\bigcup W_2] \quad (3)$$

is a partition of $V(RT(G))$. Obviously, the degrees of the vertices of $RT(G)$ are: $d_{RT(G)}(e_i) = 2$, for $i = 1, 2, \dots, m$, $d_{RT(G)}(v_i) = 2d_G(v_i) + 2$, for $i = 1, 2, \dots, n$, and $d_{RT(G)}(w_j^i) = 2$, for $i = 1, 2, \dots, n, j = 1, 2$.

Let B denotes vertex-edge incidence matrix of G. Since G is an r-regular graph, we have $D(G) = rI_n$. With respect to the partition (3), then the Laplacian matrix of $RT(G)$ can be written as

$$L(RT(G)) = \begin{bmatrix} 2I_m & -B^T & 0_{m\times 2n} \\ -B & L(G)+(r+2)I_n & -I_n\otimes \mathbf{1}_2^T \\ 0_{2n\times m} & -I_n\otimes\mathbf{1}_2 & I_n\otimes\begin{bmatrix} 2 & -1 \\ -1 & 2\end{bmatrix}\end{bmatrix},$$

where $\mathbf{1}_n$ denotes the all-one column vector with size n.

By Lemma 2.1, we have $\phi\Big(RT(G);\mu\Big)$

$$= det\left[\begin{array}{cc|c} (\mu-2)I_m & B^T & 0_{m\times 2n} \\ B & (\mu-r-2)I_n - L(G) & I_n\otimes \mathbf{1}_2^T \\ \hline 0_{2n\times m} & I_n\otimes \mathbf{1}_2 & I_n\otimes \begin{bmatrix} \mu-2 & 1 \\ 1 & \mu-2 \end{bmatrix} \end{array}\right]$$

$$= det\left[I_n\otimes \begin{bmatrix} \mu-2 & 1 \\ 1 & \mu-2 \end{bmatrix}\right]\cdot det\ S$$

$$= (\mu-1)^n(\mu-3)^n\cdot det\ S, \qquad (4)$$

where $S = \begin{bmatrix} (\mu-2)I_m & B^T \\ B & (\mu-r-2)I_n - L(G) \end{bmatrix} - \begin{bmatrix} 0_{m\times 2n} \\ I_n\otimes \mathbf{1}_2^T \end{bmatrix}$

$$\cdot\left[I_n\otimes \begin{bmatrix} \mu-2 & 1 \\ 1 & \mu-2 \end{bmatrix}\right]^{-1}\begin{bmatrix} 0_{2n\times m} & I_n\otimes \mathbf{1}_2 \end{bmatrix}$$

$$= \begin{bmatrix} (\mu-2)I_m & B^T \\ B & (\mu-r-2)I_n - L(G) \end{bmatrix} - \begin{bmatrix} 0_{m\times 2n} \\ I_n\otimes \mathbf{1}_2^T \end{bmatrix}$$

$$\cdot\left[I_n\otimes \begin{bmatrix} \mu-2 & 1 \\ 1 & \mu-2 \end{bmatrix}^{-1}\right]\begin{bmatrix} 0_{2n\times m} & I_n\otimes \mathbf{1}_2 \end{bmatrix}$$

$$= \begin{bmatrix} (\mu-2)I_m & B^T \\ B & (\mu-r-2)I_n - L(G) \end{bmatrix}$$

$$- \begin{bmatrix} 0_{m\times m} & 0_{m\times n} \\ 0_{n\times m} & I_n\otimes \mathbf{1}_2^T \begin{bmatrix} \mu-2 & 1 \\ 1 & \mu-2 \end{bmatrix}^{-1}\mathbf{1}_2 \end{bmatrix}$$

$$= \begin{bmatrix} (\mu-2)I_m & B^T \\ B & (\mu-r-2)I_n - L(G) \end{bmatrix} - \begin{bmatrix} 0_{m\times m} & 0_{m\times n} \\ 0_{n\times m} & I_n\otimes \frac{2}{\mu-1} \end{bmatrix}$$

$$= \begin{bmatrix} (\mu-2)I_m & B^T \\ B & (\mu-r-2-\frac{2}{\mu-1})I_n - L(G) \end{bmatrix}.$$

Let $l(G)$ be the line graph of G, it is well-known [23] that for a graph G,

$$BB^T = D(G) + A(G), B^TB = 2I_m + A(l(G)).$$

Consequently, $det\ S$

$$
\begin{aligned}
&= det\Big[(\mu-2)I_m\Big]\cdot det\Big[(\mu-r-2-\frac{2}{\mu-1})I_n-L(G)-\frac{1}{\mu-2}BB^T)\Big]\\
&= (\mu-2)^m\cdot det\Big[(\mu-r-2-\frac{2}{\mu-1}-\frac{(\mu-1)r}{\mu-2})I_n+\frac{\mu-3}{\mu-2}A(G)\Big] \quad (5)\\
&= (\mu-2)^{m-n}(3-\mu)^n\\
&\quad\cdot det\Big[(\frac{(\mu-2)^2}{3-\mu}+\frac{r(2\mu-3)}{\mu-3}+\frac{2(\mu-2)}{(\mu-1)(\mu-3)})I_n-A(G)\Big]\\
&= (\mu-2)^{m-n}(3-\mu)^n\\
&\quad\cdot\varphi\Big(G;\frac{(\mu-2)^2}{3-\mu}+\frac{r(2\mu-3)}{\mu-3}+\frac{2(\mu-2)}{(\mu-1)(\mu-3)}\Big). \quad (6)
\end{aligned}
$$

Actually, by virtue of (4) and (6) we have already established the statement (i) in Theorem 3.1.

(ii) Recall that $L(G)=rI_n-A(G)$. It follows from (5) that

$$
\begin{aligned}
&det\ S\\
&= (\mu-2)^m\cdot det\Big[(\mu-r-2-\frac{2}{\mu-1}-\frac{(\mu-1)r}{\mu-2})I_n+\frac{\mu-3}{\mu-2}A(G)\Big]\\
&= (\mu-2)^{m-n}(\mu-3)^n\\
&\quad\cdot det\Big[(\frac{(\mu-2)^2}{\mu-3}-\frac{r\mu}{\mu-3}-\frac{2(\mu-2)}{(\mu-1)(\mu-3)})I_n-D(G)+A(G)\Big]\\
&= (\mu-2)^{m-n}(\mu-3)^n\\
&\quad\cdot det\Big[(\frac{(\mu-2)^2}{\mu-3}-\frac{r\mu}{\mu-3}-\frac{2(\mu-2)}{(\mu-1)(\mu-3)})I_n-L(G)\Big]\\
&= (\mu-2)^{m-n}(\mu-3)^n\phi\Big(G;\frac{(\mu-2)^2}{\mu-3}-\frac{r\mu}{\mu-3}-\frac{2(\mu-2)}{(\mu-1)(\mu-3)}\Big). \quad (7)
\end{aligned}
$$

By combining (4) and (7), we get

$$
\begin{aligned}
\phi\Big(RT(G);\mu\Big) &= (\mu-1)^n(\mu-2)^{m-n}(\mu-3)^{2n}\\
&\quad\cdot\phi\Big(G;\tfrac{(\mu-2)^2}{\mu-3}-\tfrac{r\mu}{\mu-3}-\tfrac{2(\mu-2)}{(\mu-1)(\mu-3)}\Big).
\end{aligned}
$$

Thus the statement (ii) in Theorem 3.1 is proved. □

4 The Kirchhoff index of $RT(G)$

In this section, we will explore the Kirchhoff index of the $RT(G)$ of a regular graph G.

Zhu [14], Gutman and Mohar [15] proved that the relationship between Kirchhoff index of a graph and Laplacian eigenvalues of the graph as follows.

Lemma 4.1. ([14, 15]) Let G be a connected graph with $n \geq 2$ vertices, then

$$Kf(G) = n\sum_{i=1}^{n-1} \frac{1}{\mu_i}. \tag{8}$$

Denote by δ_i the degree of vertex $v_i \in V(G)$. Zhou and Trinajstić [16] proved that

Lemma 4.2. ([16]) Let G be a connected graph with $n \geq 2$ vertices, then

$$Kf(G) \geq -1 + (n-1)\sum_{v_i \in V(G)} \frac{1}{\delta_i}, \tag{9}$$

with equality attained if and only if $G = K_n$ or $G = K_{t,n-t}$ for $1 \leq t \leq \lfloor \frac{n}{2} \rfloor$.

The following lemma will be used later on.

Lemma 4.3. ([12]) Let G be a connected graph with $n \geq 2$ vertices and $\phi(G;\mu) = \mu^n + a_1\mu^{n-1} + a_2\mu^{n-2} + \cdots + a_{n-1}\mu$, then $\frac{Kf(G)}{n} = -\frac{a_{n-2}}{a_{n-1}}$, ($a_{n-2} =$ 1 *whenever* $n = 2$), where a_{n-1}, a_{n-2} are the coefficients of μ and μ^2 in the Laplacian characteristic polynomial, respectively.

Let K_n be the complete graph with n ($n \geq 2$) vertices. The following theorem shows that $Kf(RT(G))$ can be completely determined by the Kirchhoff index $Kf(G)$, the number of vertices and the vertex degree of regular graph G.

Theorem 4.4. Let G be a connected r-regular graph with n vertices, then $Kf(RT(G)) = \frac{(r+6)^2}{6}Kf(G) + \frac{(r+5)n}{2} + \frac{(r+6)(5n-4)n}{6} + \frac{(r-2)(r+6)n^2}{8}$.

Proof. Suppose first that $r = 1$, i.e. $G \cong K_2$. Since $Kf(RT(K_2)) = \frac{74}{3}$. It is easy to check that the result holds in this case. Suppose now that $r \geq 2$. Let

$$\phi(G;\mu) = \mu^n + a_1\mu^{n-1} + a_2\mu^{n-2} + \cdots + a_{n-1}\mu. \tag{10}$$

It follows from Theorem 3.1 (ii) that

$$\begin{aligned} &\phi\Big(RT(G);\mu\Big) \\ &= (\mu-1)^n(\mu-2)^{m-n}(\mu-3)^{2n}\phi\Big(G; \frac{(\mu-2)^2}{\mu-3} - \frac{r\mu}{\mu-3} - \frac{2(\mu-2)}{(\mu-1)(\mu-3)}\Big) \\ &= (\mu-1)^n(\mu-2)^{m-n}(\mu-3)^{2n}\phi\Big(G; \frac{\mu[(\mu^2-(r+5)\mu+(r+6)]}{(\mu-1)(\mu-3)}\Big). \end{aligned} \tag{11}$$

Combining (10) with (11), one can obtain that

$$\begin{aligned}
&\phi\Big(RT(G);\mu\Big)\\
&= (\mu-1)^n(\mu-2)^{m-n}(\mu-3)^{2n}\left\{\frac{\mu^n\big[\mu^2-(r+5)\mu+(r+6)\big]^n}{(\mu-1)^n(\mu-3)^n}+\cdots\right.\\
&\quad + a_{n-2}\frac{\mu^2\big[\mu^2-(r+5)\mu+(r+6)\big]^2}{(\mu-1)^2(\mu-3)^2}\\
&\quad \left.+ a_{n-1}\frac{\mu\big[\mu^2-(r+5)\mu+(r+6)\big]}{(\mu-1)(\mu-3)}\right\}\\
&= (\mu-2)^{m-n}(\mu-3)^n\Big\{\mu^n\big[\mu^2-(r+5)\mu+(r+6)\big]^n+\cdots\\
&\quad + a_{n-2}\mu^2(\mu-1)^{n-2}(\mu-3)^{n-2}\big[\mu^2-(r+5)\mu+(r+6)\big]^2\\
&\quad + a_{n-1}\mu(\mu-1)^{n-1}(\mu-3)^{n-1}\big[\mu^2-(r+5)\mu+(r+6)\big]\Big\},
\end{aligned}$$

where $\mu \neq 1, 3$. So the coefficient of μ^2 in $\phi(RT(G);\mu)$ is

$$\begin{aligned}
&(-2)^{m-n}(-3)^n\Big[a_{n-2}(r+6)^2(-1)^{n-2}(-3)^{n-2}+a_{n-1}(-r-5)(-1)^{n-1}\\
&\cdot(-3)^{n-1}+a_{n-1}(r+6)(n-1)(-1)^{n-2}(-3)^{n-1}+a_{n-1}(r+6)(-1)^{n-1}\\
&\cdot(n-1)(-3)^{n-2}\Big]+(m-n)(-2)^{m-n-1}(-3)^n a_{n-1}(r+6)(-1)^{n-1}\\
&\cdot(-3)^{n-1}+n(-3)^{n-1}(-2)^{m-n}a_{n-1}(r+6)(-1)^{n-1}(-3)^{n-1}, \qquad (12)
\end{aligned}$$

and the coefficient of μ in $\phi(RT(G);\mu)$ is

$$(-2)^{m-n}(-3)^n a_{n-1}(r+6)(-1)^{n-1}(-3)^{n-1}. \qquad (13)$$

Notice that $RT(G)$ has $3n+m$ vertices. It follows from Lemma 4.3, (12) and (13) that

$$\frac{Kf(RT(G))}{3n+m} = -\frac{a_{n-2}}{a_{n-1}}\frac{r+6}{3}+\frac{r+5}{r+6}+\frac{5n-4}{3}+\frac{m-n}{2}.$$

Substituting the result of Lemma 4.3 and $m=\frac{nr}{2}$ into the above equation.

$$\frac{Kf(RT(G))}{3n+\frac{nr}{2}} = \frac{r+6}{3}\frac{Kf(G)}{n}+\frac{r+5}{r+6}+\frac{5n-4}{3}+\frac{\frac{nr}{2}-n}{2}.$$

Simplifying the above result, one can obtain that
$Kf(RT(G)) = \frac{(r+6)^2}{6}Kf(G)+\frac{(r+5)n}{2}+\frac{(r+6)(5n-4)n}{6}+\frac{(r-2)(r+6)n^2}{8}.$

Summing up, we complete the proof. □

Remark 4.5. Comparison to the Laplacian polynomials and its Kirchhoff indices of $R(G)$ and $Q(G)$ in [13], the graph $RT(G)$ has more vertices and edges. It is clear that handling the problems of Laplacian polynomial and Kirchhoff index are more difficult and complex, but we deduce those with a simple approach.

In what follows, we propose a lower bound for the Kirchhoff index for $RT(G)$ in terms of the number of vertices and the vertex degree of a connected regular graph.

Corollary 4.6. Let G be a connected r-regular graph with n vertices, then $Kf(RT(G)) \geq \frac{(r+6)^2(n^2-n-r)}{6r} + \frac{(r+5)n}{2} + \frac{(r+6)(5n-4)n}{6} + \frac{(r-2)(r+6)n^2}{8}$, and the equality holds if and only if $G \cong K_n$ or $G \cong K_{\frac{n}{2},\frac{n}{2}}$ and n is even.

Proof. It follows from Lemma 4.2 and Theorem 4.4 that

$$Kf(RT(G)) \geq \frac{(r+6)^2}{6}\left(\frac{(n-1)n}{r} - 1\right) + \frac{(r+5)n}{2} + \frac{(r+6)(5n-4)n}{6} + \frac{(r-2)(r+6)n^2}{8}$$
$$= \frac{(r+6)^2(n^2-n-r)}{6r} + \frac{(r+5)n}{2} + \frac{(r+6)(5n-4)n}{6} + \frac{(r-2)(r+6)n^2}{8}.$$

Clearly, the equality holds if and only if $G \cong K_n$ or $G \cong K_{\frac{n}{2},\frac{n}{2}}$ and n is even. □

5 Some applications

In this section, we discuss some special graphs and give formulae for their Kirchhoff indices.

5.1 Complete graph K_n $(n \geq 2)$

It is well known that K_n is $(n-1)$-regular and $Kf(K_n) = n-1$.

It follows from Theorem 4.4 that

$$Kf(RT(K_n)) = \frac{(r+6)^2}{6}Kf(K_n) + \frac{(r+5)n}{2} + \frac{(r+6)(5n-4)n}{6} + \frac{(r-2)(r+6)n^2}{8}$$
$$= \frac{(r+6)^2(n-1)}{6} + \frac{(r+5)n}{2} + \frac{(r+6)(5n-4)n}{6} + \frac{(r-2)(r+6)n^2}{8}.$$

Particularly, if $G \cong K_2$, one can obtain $Kf(RT(K_2)) = \frac{74}{3}$ by substituting $n=2, r=1$ into above formula. In order to illustrate the correction and efficiency of the above results, one can check $Kf(RT(K_2))$ for simplicity, see Figure 2 (a). It is easy to obtain $r_{12} = r_{13} = \frac{2}{3}, r_{14} = r_{15} = r_{16} = r_{17} = \frac{4}{3}; r_{23} = r_{24} = r_{26} = \frac{2}{3}, r_{25} = r_{27} = \frac{4}{3};\ r_{34} = r_{36} = \frac{4}{3}, r_{35} = r_{37} = \frac{2}{3}; r_{45} = r_{47} = \frac{6}{3}, r_{46} = \frac{2}{3}; r_{56} = \frac{6}{3}, r_{57} = \frac{2}{3}; r_{67} = \frac{6}{3}$.

Hence, $Kf(RT(K_2)) = \frac{74}{3}$, which coincides with the above result.

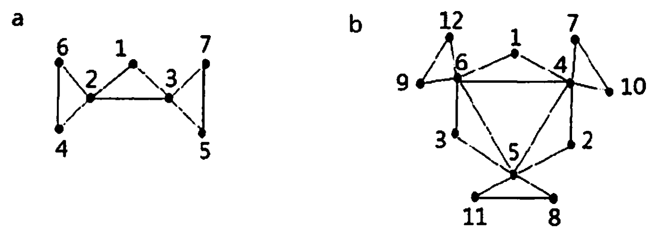

Figure 2: (a) The graph $RT(K_2)$. (b) The graph $RT(C_3)$.

5.2 Cycle C_n $(n \geq 3)$

It was reported in [24] that $Kf(C_n) = \frac{n^3-n}{12}$. It follows from Theorem 4.4 that

$$\begin{aligned} & Kf(RT(C_n)) \\ = & \frac{(r+6)^2}{6} Kf(C_n) + \frac{(r+5)n}{2} + \frac{(r+6)(5n-4)n}{6} + \frac{(r-2)(r+6)n^2}{8} \\ = & \frac{(r+6)^2(n^3-n)}{72} + \frac{(r+5)n}{2} + \frac{(r+6)(5n-4)n}{6} + \frac{(r-2)(r+6)n^2}{8}. \end{aligned}$$

Similarly, for graph $RT(C_3)$, see Figure 2 (b). One can obtain $Kf(RT(C_3)) = \frac{455}{6}$, which also coincides with the above formula.

5.3 Complete bipartite graph $K_{n,n}$

Note that $K_{n,n}$ is n-regular with $2n$ vertices. Recall from [12] that

$$Kf(K_{n,n}) = 4n - 3. \tag{14}$$

It follows from (14) and Theorem 4.4 that

$$\begin{aligned} Kf(RT(K_{n,n})) &= \tfrac{(r+6)^2}{6} Kf(K_{n,n}) + \tfrac{(r+5)\cdot 2n}{2} + \tfrac{(r+6)(10n-4)n}{3} + \tfrac{(r-2)(r+6)n^2}{2} \\ &= \tfrac{(r+6)^2(4n-3)}{6} + (r+5)n + \tfrac{(r+6)(10n-4)n}{3} + \tfrac{(r-2)(r+6)n^2}{2}. \end{aligned}$$

6 Conclusions

In this paper, based on the earlier definition $R(G)$, we introduce a novel graph operation $RT(G)$, and explore its Laplacian polynomial and Kirchhoff index. By utilizing the spectral graph theory, we establish the explicit formula for $Kf(RT(G))$ in terms of $Kf(G)$, the number of vertices and the vertex degree of regular graph G, based on which we propose a lower bound for the Kirchhoff index for $RT(G)$ with respect to the number of vertices and the vertex degree.

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