CAYLEY GRAPHS ISOMORPHIC TO THE PRODUCT OF TWO CAYLEY GRAPHS

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ABSTRACT. Let $\star$ be a binary graph operation. We call $\star$ a Cayley operation if $\Gamma_1 \star \Gamma_2$ is a Cayley graph for any two Cayley graphs Γ_1 and Γ_2. In this paper, we prove that the cartesian, (categorical or tensor) direct and lexicographic products are Cayley operations. We also investigate the following question:
Under what conditions on a binary graph operation $\star$ and Cayley graphs Γ_1 and Γ_2, the graph product $\Gamma_1 \star \Gamma_2$ is again a Cayley graph.
The latter question is studied for the union, join (sum), replacement and zig-zag products of graphs.

1. INTRODUCTION

Let G be a non-trivial group and let S be a non-empty subset of G such that $1 \notin S$ and $S = S^{-1} := \{s^{-1} \mid s \in S\}$. The Cayley graph $\Gamma = Cay(G,S)$ is the graph whose vertex set $V(\Gamma)$ is G, and the edge set $E(\Gamma)$ is $\{\{g, gs\} \mid g \in G, s \in S\}$. Also Cayley graph $Cay(G,S)$ is connected if and only if S generates G, see [3, 6].

In [2] it is proved that the Cayley graph of the standard semi-direct product $A \rtimes B$ of finite groups A, B with certain choices of generators for these three groups is essentially the zig-zag product of the Cayley graphs of A and B. This invention used to construct expander graphs (see [2, Theorem 4.2]).

We are motivated by the latter result to study the following question.

Question 1.1. *Let $\star$ be a binary graph operation and Γ_1, Γ_2 be two Cayley graphs. Under what conditions on $\star$ and Cayley graphs Γ_1, Γ_2, the graph product $\Gamma_1 \star \Gamma_2$ is again a Cayley graph?*

We prove that the cartesian product of any two Cayley graphs is again a Cayley graph (see Theorem 3.1, below). The latter conclusion is also valid for lexicographic and tensor products (see Theorems 4.1 and 6.1, below). These such binary operations are called Cayley operations. For a binary

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graph operation $\star$, we call $\star$ a Cayley operation if $\Gamma_1 \star \Gamma_2$ is a Cayley graph for any two Cayley graphs Γ_1 and Γ_2. Of course, not all graph products are Cayley: we find necessary and sufficient conditions on two Cayley graphs Γ_1 and Γ_2 such that $\Gamma_1 \star \Gamma_2$ is again a Cayley graph, where $\star$ is either the union or sum products of graphs (see Theorems 7.2 and 8.1, below). We also study Question 1.1, where $\star$ is the replacement or zig-zag operation.

All graphs considered in this paper are finite and simple. All graph operations considered here are well known and we refer the reader to [4] for more information on composite graphs.

In Section 2 we recall and fix some notation and results which we will use in the sequel. In Sections 3 to 9 we study Question 1.1 for cartesian product, categorical product, strong product, lexicographic product, union, sum and replacement and zig-zag of two graphs.

2. Preliminaries

In this section, we give some necessary notation and results to define ceratin graph products.

For any graph Γ, we denote by $V(\Gamma)$ and $E(\Gamma)$ the set of vertices and edges of Γ, respectively.

Let A and B be two permutation groups acting on sets X and Y, respectively.

(a) The direct product $A + B$ is a permutation group on the disjoint union $X \cup Y$ whose elements are ordered $a + b$ for $a \in A$ and $b \in B$, the action is given by

$$(a+b)t = \begin{cases} a(t) & t \in X \\ b(t) & t \in Y \end{cases}$$

(b) The Cartesian product $A \times B$ is a permutation group on $X \times Y$ such that for any $x \in X$, $y \in Y$ we have

$$(a,b)(x,y) = (a(x), b(y))$$

(c) For any $a \in A$, $b \in B$ and $x \in X$, define $a^*(x',y') = (a(x'),y')$ and

$$b^*_x(x',y') = \begin{cases} (x',b(y')) & x = x' \\ (x',y') & x \neq x' \end{cases}$$

for all $(x',y') \in X \times Y$. Then the wreath product $A \wr B$ is defined as follows:

$$A \wr B = \langle a^*, b^*_x \mid a \in A,\ b \in B,\ x \in X \rangle.$$

A permutation group A on a set X is called transitive if for each pair $x, y \in X$ there exists $a \in A$ such that $a(x) = y$. If, in addition, such a is unique for each (ordered) pair (x,y), then the permutation group A is called regular.

For $g \in G$, the mapping $\rho_g : G \to G$, defined by $\rho_g(x) = gx$, is called the (left) translation of $\Gamma = Cay(G, S)$ by g. The set

$$G_L = \{\rho_g \mid g \in G\}$$

is a subgroup of the full automorphism group $Aut(\Gamma)$ acting regularly on $V(\Gamma)$. We may identify G with G_L, in particular, $Aut(\Gamma)$ is transitive on $V(\Gamma)$. The following characterization of Cayley graph is well-known (see [8]).

Theorem 2.1. *Let Γ be a graph. The automorphism group $Aut(\Gamma)$ has a subgroup G which acts regularly on $V(\Gamma)$ if and only if Γ is a Cayley graph $Cay(G, S)$ for some subset S of G.*

3. Cartesian product

For given graphs G_1 and G_2, we define their cartesian product $G_1 \Box G_2$ as the graph whose vertex set is $V(G_1) \times V(G_2)$ and two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are connected by an edge if and only if $u_1 = v_1$ and $\{u_2, v_2\} \in E(G_2)$ or $u_2 = v_2$ and $\{u_1, v_1\} \in E(G_1)$. Obviously, $|E(G_1 \Box G_2)| = n_1 m_2 + n_2 m_1$, where n_i and m_i denote the number of vertices and edges of G_i, respectively. The cartesian product of two graphs is connected if and only if both components are connected. We now prove that $\Box$ is a Cayley operation.

Theorem 3.1. *Let $G = Cay(A, S_A)$ and $H = Cay(B, S_B)$ be two Cayley graphs. Then the Cartesian product $G \Box H$ is the Cayley graph of the direct product $A \times B$ and with the generating subset $(S_A, 1) \cup (1, S_B)$.*

Proof. By Theorem 2.1, it is enough to show that there exists a subgroup Y of $Aut(G \Box H)$ isomorphic to $A \times B$ acting regularly on $V(G \Box H) = V(G) \times V(H)$. According to the definition of A_L and B_L, we have $A \leqslant Aut(G)$ and $B \leqslant Aut(H)$. It is easily to see that for $(a, b) \in A \times B$ we have $(\rho_a, \rho_b) \in Aut(G \Box H)$. Let $(x, y), (x', y') \in V(G) \times V(H)$; since A and B acts regularly on $V(G)$ and $V(H)$, respectively. Then there exist unique ρ_a and ρ_b such that $\rho_a(x) = x'$ and $\rho_b(y) = y'$. Thus $A \times B$ acts regularly on $V(G \Box H)$ and by Theorem 2.1 we have

$$Cay(A, S_A) \Box Cay(B, S_B) \cong Cay(A \times B, T)$$

for some subset T of $A \times B$. It is easy to see that T may be taken as $\{(s, 1), (1, t) | s \in S_A, t \in S_B\}$. This completes the proof. $\Box$

4. Direct product

For any two graphs G_1 and G_2, we define their direct (or categorical) product $G_1 \times G_2$ as the graph on the vertex set $V(G_1) \times V(G_2)$ and two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ connected by an edge if and only if $\{u_1, v_1\} \in E(G_1)$ and $\{u_2, v_2\} \in E(G_2)$. Obviously, $|E(G_1 \times G_2)| = 2m_1m_2$, where m_i denote the number of edges of G_i. The direct product of two graphs is sometimes also called tensor product or categorical product and is denoted by $G_1 \dot{\times} G_2$.

Similar cartesian product, direct product is Cayley operation and can preserve Cayley graphs. In this product $A_L \times B_L$ is subgroup of $Aut(G \times H)$ that acts regular on $V(G \times H)$ and we have

Theorem 4.1. *Let $G = Cay(A, S_A)$ and $H = Cay(B, S_B)$ be two Cayley graphs, then the direct product $G \times H$ is the Cayley graph of the direct product $A \times B$ and with the generating subset $S_A \times S_B$.*

Proof. The proof is similar to 3.1. □

5. Strong product

The strong product $G_1 \boxtimes G_2$ of two graphs G_1 and G_2 is the graph with vertex set $V(G_1) \times V(G_2)$ in which $u = (u_1, u_2)$ is adjacent with $v = (v_1, v_2)$ whenever $(\{u_1, v_1\} \in E(G_1)$ and $u_2 = v_2)$ or $(u_1 = v_1$ and $\{u_2, v_2\} \in E(G_2))$ or $(\{u_1, v_1\} \in E(G_1)$ and $\{u_2, v_2\} \in E(G_2))$. In this product the number of edges is equal to $n_1m_2 + n_2m_1 + 2m_1m_2$.

Definition 5.1. Let G and H be two graphs such that $V(G) = V(H)$. Then we define graphs $G\dot{\cup}H$ and $G\dot{\cap}H$ as follows.

$V(G\dot{\cup}H) = V(G\dot{\cap}H) = V(G)$ and $E(G\dot{\cup}H) = E(G) \cup E(H)$, $E(G\dot{\cap}H) = E(G) \cap E(H)$.

Lemma 5.2. *If $G = Cay(A, S)$ and $H = Cay(A, S')$, then $G\dot{\cup}H = Cay(A, S \cup S')$ and if $S \cap S' \neq \emptyset$, then $G\dot{\cap}H = Cay(A, S \cap S')$.*

Proof. Let e be an edge of $Cay(A, S \cup S')$. Then there exist $a \in A$ and $s \in S \cup S'$ such that e connects two vertices a and as. Since $s \in S \cup S'$, then $s \in S$ or $s \in S'$, which means that $e \in E(G)$ or $e \in E(H)$. Therefore, by definition, $e \in E(G\dot{\cup}H)$. Conversely, in the similar way, any edge of $E(G\dot{\cup}H)$ is an edge of $Cay(A, S \cup S')$. This result together with $V(G\dot{\cup}H) = V(Cay(A, S \cup S')) = A$ means that $G\dot{\cup}H = Cay(A, S \cup S')$. Similarly we can see that $G\dot{\cap}H$ is Cayley graph and $G\dot{\cap}H = Cay(A, S \cap S')$. □

Theorem 5.3. *Strong product is a Cayley operation.*

Proof. Let $G = Cay(A, S_A)$ and $H = Cay(B, S_B)$ be two Cayley graphs; then by Theorems 3.1 and 4.1 we have

$$\begin{aligned} G\Box H &= Cay(A \times B, (S_A, 1_B) \cup (1_A, S_B)) \\ G \times H &= Cay(A \times B, (S_A, S_B)) \end{aligned}$$

Therefore we have $G \boxtimes H = (G\Box H)\dot{\cup}(G \times H)$. Now Lemma 5.2, implies that; $G \boxtimes H$ is a Cayley graph. □

6. Composition

The composition $G_1[G_2]$ of graphs G_1 and G_2 with disjoint vertex sets and edge sets is again a graph on vertex set $V(G_1) \times V(G_2)$ in which $u = (u_1, u_2)$ is adjacent to $v = (v_1, v_2)$ whenever u_1 is adjacent to v_1 or $u_1 = v_1$ and u_2 is adjacent to v_2. We have $G_1[G_2] = G_2[G_1]$ if G_1 and G_2 are complete graph or totally disconnected graph. The easiest way to visualize the composition $G_1[G_2]$ is to expand each vertex of G_1 into a copy of G_2, with each edge of G_1 replaced by the set of all possible edges between the corresponding copies of G_2. Hence the number of edges in $G_1[G_2]$ is given by $|E(G_1[G_2])| = n_1m_2 + m_1n_2^2$. The composition of two graphs is sometimes also called lexicographic products and it is denoted by $G_1 \circ G_2$, see [1].

Theorem 6.1. *Let $G = Cay(A, S)$ and $H = Cay(B, T)$ be two Cayley graphs; then the composition $G[H] = Cay(A \times B, S \times B \cup \{1_A\} \times T)$.*

Proof. It is an easy exercise to show that $G[H] = \underbrace{(G \Box H)}_{X} \dot\cup \underbrace{(G \times K_{|V(H)|})}_{Y}$, where $K_{|V(H)|}$ is the complete graph with $|V(H)|$ vertex. If we show that X and Y are Cayley graphs then Lemma 5.2 completes the proof. Assume $K_{|V(H)|} = Cay(B, B - 1_B)$ then by Theorems 3.1, 4.1 we have $G[H]$ is a Cayley graph and a similar proof to Theorem 3.1, implies that $G[H] = Cay(A \times B, S \times B \cup \{1_A\} \times T)$. □

7. Union

The simplest operation we consider here is a union of two graphs. A union $G_1 \cup G_2$ of two graphs G_1 and G_2 is the graph with the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2)$. Here we assume that $V(G_1)$ and $V(G_2)$ are disjoint. If G_1 and G_2 are two connected graphs then, it is well-known that

$$Aut(G_1 \cup G_2) = \begin{cases} S_2 \wr Aut(G_1) & G_1 \cong G_2 \\ Aut(G_1) + Aut(G_2) & else \end{cases}$$

Lemma 7.1. *If two permutation groups A and B acting transitively on the sets X and Y, then the wreath product $A \wr B$ acts transitively on $X \times Y$.*

Proof. Let $(x, y), (x', y')$ be two arbitrary elements of $X \times Y$; we show there is $t \in A \wr B$ that $t(x, y) = (x', y')$. Since $x, x' \in X$ $(y, y' \in Y)$ and A acts transitively on X (B acts transitively on Y), then there is $a \in A$ $(b \in B)$ that $a(x) = x'$ $(b(y) = y')$. Thus for $t = b^*_{x'}a^*$ we have

$$t(x, y) = b^*_{x'}a^*(x, y) = b^*_{x'}(a(x), y) = b^*_{x'}(x', y) = (x', b(y)) = (x', y')$$

□

Theorem 7.2. *Let G and H be two graphs. Then $Aut(G \cup H)$ acts transitively on $V(G \cup H)$ if and only if $G \cong H$*

Proof. If $G \cong H$ then $Aut(G \cup H) = S_2 \wr Aut(G)$ and by Lemma 7.1, $Aut(G \cup H)$ acts transitively on $V(G \cup H)$.

For the converse, let $Aut(G \cup H)$ act transitively on $V(G \cup H)$. If $G \ncong H$ then $Aut(G \cup H) = Aut(G) + Aut(H)$ and for $x \in X$, $y \in Y$ there is no $t \in Aut(G) + Aut(H)$ such that $t(x) = y$; a contradiction. This completes the proof. □

Lemma 7.3. *If permutation groups H and K acts regularly on sets X and Y, respectively. Then the wreath product $H \wr K$ contains a subgroup T acting regularly as a permutation group on $X \times Y$.*

$$T = \{\ k^*_{x_1} k^*_{x_2} \dots k^*_{x_n} h^* \mid h \in H,\ k \in K,\ |X| = n\ \}$$

Proof. we show now that T is subgroup of $H \wr K$. To see this, assume $t = k^*_{x_1} k^*_{x_2} \dots k^*_{x_n} h^*$ and $t' = k'^*_{x_1} k'^*_{x_2} \dots k'^*_{x_n} h'^*$ be two arbitrary elements of T; thus we have:

$$\begin{aligned}
tt' &= k^*_{x_1} k^*_{x_2} \dots k^*_{x_n} h^* k'^*_{x_1} k'^*_{x_2} \dots k'^*_{x_n} h'^* \\
&= k^*_{x_1} k^*_{x_2} \dots k^*_{x_n} (h^* k'^*_{x_1} k'^*_{x_2} \dots k'^*_{x_n} h^{*-1}) h^* h'^* \\
&= k^*_{x_1} k^*_{x_2} \dots k^*_{x_n} (k'^*_{x_1} k'^*_{x_2} \dots k'^*_{x_n})^{h^{*-1}} h^* h'^* \\
&= k^*_{x_1} k^*_{x_2} \dots k^*_{x_n} {k'^*_{x_1}}^{h^{*-1}} {k'^*_{x_2}}^{h^{*-1}} \dots {k'^*_{x_n}}^{h^{*-1}} h^* h'^* \\
&= k^*_{x_1} k^*_{x_2} \dots k^*_{x_n} k'^*_{h^{*-1}(x_1)} k'^*_{h^{*-1}(x_2)} \dots k'^*_{h^{*-1}(x_n)} (hh')^* \\
&\ \ \vdots \\
&= (kk')^*_{x_1} (kk')^*_{x_2} \dots \underbrace{(kk')}_{\in K}{}^*_{x_n} \underbrace{(hh')}_{\in H}{}^* \in T
\end{aligned}$$

If $(a, b), (a', b')$ be two arbitrary elements of $X \times Y$ then $a, a' \in X$ and H acts regularly on the set X. Thus there is a unique $h \in H$ such that $h(a) = a'$; similarly there is a unique $k \in K$ such that $k(b) = b'$. Therefore $a' \in X$ and for some $j \in [n]$ we have $a' = x_j$. It is easy to see that $k^*_{x_1} k^*_{x_2} \dots k^*_{x_j} \dots k^*_{x_n} h^*$ is a unique in T and

$$\begin{aligned}
k^*_{x_1} k^*_{x_2} \dots k^*_{x_j} \dots k^*_{x_n} h^*(a, b) &= k^*_{x_1} k^*_{x_2} \dots k^*_{x_j} \dots k^*_{x_n} (\underbrace{h(a)}_{a'}, b) \\
&= k^*_{a'}(a', b) = (a', \underbrace{k(b)}_{b'})
\end{aligned}$$

□

Note that if $G = Cay(A, S)$ then A_L acts regularly on $V(G) = A$. By Lemma 7.3 we have the following corollary.

Corollary 7.4. *If $G = Cay(A, S)$ then group $S_n \wr Aut(G)$ acts transitively on $[n] \times \{a_1, a_2, \dots a_m\}$, where $[n] = \{1, 2 \dots n\}$ and $A = \{a_1, a_2, \dots a_m\}$. Therefore by Lemma 7.3, the wreath product $\langle(123\dots n)\rangle \wr A_L$ contains a subgroup T acting regularly as a permutation group on $[n] \times A$, where*

$$T = \{\ \varphi_{a_1}\varphi_{a_2}\cdots\varphi_{a_m}\psi \mid \psi \in \langle(123\dots n)\rangle,\ \varphi_{a_i} \in A_L\ \}$$

In the next theorem we show that the union is a Cayley operation whenever all component are isomorphic.

Theorem 7.5. *If $G_i = Cay(A_i, S_i)$, $1 \leq i \leq t$. Then $G = \bigcup_{i=1}^{t} G_i$ is Cayley graph if and only if $G_i \cong G_j$, for any $i, j \in [t]$.*

Proof. If for i and j we have $G_i \ncong G_j$. Then by Theorem 7.2, $Aut(G)$ has not subgroup that acts transitive on $\bigcup_{i=1}^{t} A_i$. If all component are isomorphic with graph H then $Aut(G) = S_n \wr Aut(H)$. By before corollary there is subgroup of $Aut(G)$ that acts regular on $V(G)$. In this case union is cayely operation. □

8. Sum (join)

A sum $G_1 + G_2$ of two graphs G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$ is the graph on the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2) \cup \{\ \{u_1, u_2\} \mid u_1 \in V(G_1), u_2 \in V(G_2)\}$. Hence, the sum of two graphs is obtained by connecting each vertex of one graph to each vertex of the other graph, while keeping all edges of both graphs. The sum of two graphs is sometimes also called a join, and is denoted by $G_1 \nabla G_2$. Therefore $G_1 + G_2 = \overline{\overline{G_1} \cup \overline{G_2}}$.

If G_1 and G_2 be two graphs then $Aut(G_1 + G_2) \equiv Aut(\overline{G_1 + G_2}) \equiv Aut(\overline{G_1} \cup \overline{G_2})$, on the other hand $Aut(G_1) + Aut(G_2) \equiv Aut(\overline{G_1}) + Aut(\overline{G_2})$. Then if no component of $\overline{G_1}$ is isomorphic with a component of $\overline{G_2}$, then $Aut(G_1 + G_2) \equiv Aut(G_1) + Aut(G_2)$ and we have next theorem:

Theorem 8.1. *Let $G = Cay(A, S)$ and $H = Cay(B, T)$, then $G + H$ is Cayley graph if and only if one of the following two conditions is satisfied:*
(i) G and H are two complete graphs.
(ii) $\overline{G} \cong \overline{H}$.

Proof. If $S = A - 1_A$ and $T = B - 1_B$ then $G \cong K_{|A|}$ and $H \cong K_{|B|}$. Thus $G + H \cong K_{|A|+|B|}$, that it is a Cayley graph. Also we know $Aut(G + H) = Aut(\overline{G} \cup \overline{H})$ and by Theorem 7.5, $Aut(\overline{G} \cup \overline{H})$ has regular subgroup if and only if $\overline{G} \cong \overline{H}$. □

9. Replacement and Zig-Zag

In this section we describe the replacement and Zig-zag product and investigate Cayley operation for zig-zag and replacement product.

Let G be any (n,k)-graph, G is k-regular with n vertex. By a randomly numbering of G we mean a randomly numbering of the edges around each vertex of G by the numbers in $\{1,\ldots,k\}$. More precisely, a randomly numbering of G is a set φ_G consisting of bijection maps $\varphi_G^x : N_G(x) \to [k]$ for any $x \in V(G)$, where $N_G(x) = \{y \in V(G) \mid xy \in E(G)\}$.

Example 9.1. Suppose $G = Cay(A,S)$, then the edges around each vertex of G are naturally labeled by the elements of S and we have: if $xy \in E(G)$ then $\varphi_G^x(y) = f(x^{-1}y)$ where f is a bijection map from S to $[|S|]$.

Definition 9.2. Let G be an (n,k)-graph and let H be a (k,k')-graph with $V(H) = [k] = \{1,\ldots,k\}$ and fix a randomly numbering φ_G of G. The replacement product $G \circledR_{\varphi_G} H$ is the graph whose vertex set is $V(G)\times V(H)$ and there is an edge between vertices (v,k) and (w,l) whenever $v = w$ and $kl \in E(H)$ or $vw \in E(G)$, $\varphi_G^v(w) = k$ and $\varphi_G^w(v) = l$. Also the zig-zag product of two graphs G and H with above properties show with $G Z_{\varphi_G} H$ and define as follow:

The vertex set $G Z_{\varphi_G} H$ is $V(G) \times [k]$ and two vertices (u,i) and (v,j) are adjacent if $\{u,v\} \in E(G)$ and there are $i', j' \in V(H) = [k]$ such that $\{i,i'\}, \{j,j'\} \in E(H)$ and $\varphi_G^u(v) = i'$, $\varphi_G^v(u) = j'$

Note that the definition of $G \circledR_{\varphi_G} H$ and $G Z_{\varphi_G} H$ clearly depends on φ_G. It follows from the definition that $G \circledR_{\varphi_G} H$ and $G Z_{\varphi_G} H$ are a regular graph and in fact $G \circledR_{\varphi_G} H$ is a $(nk, k'+1)$-graph and $G Z_{\varphi_G} H$ is a (nk, k'^2)-graph (see [2, 5, 7]).

Definition 9.3. An action of a group B on a group A is a group homomorphism $\theta : B \to Aut(A)$. In other words, each element $b \in B$ corresponds to an automorphism θ_b of A. Let $A \rtimes B$ set of pairs (a,b) of elements $a \in A$ and $b \in B$, with the following operation for the product of two elements

$$(a,b)(a',b') = (a.\theta_b(a'), b.b')$$

Then $A \rtimes B$ forms a group of order $|A||B|$ with identity element $(1_A, 1_B)$ and inverse $(a,b)^{-1} = (\theta_{-b}(a^{-1}), b^{-1})$. This group is called the semidirect product of A and B with respect to the action θ.

Note the direct product of two groups $A \times B$ is a special case of a semidirect product where θ_b is the identity automorphism of A for all $b \in B$.

We now consider the case when the two components of the product graph be Cayley graphs of the type $G = Cay(A,S)$ and $H = Cay(B,T)$. Furthermore, suppose that B acts on A in such a way that $S = \alpha^B = \{\ \theta_b(\alpha) \mid b \in B\}$ for $\alpha \in S$. So the edges around each vertex of G are naturally labeled by the elements of B for example if $xx' \in E(G)$ then $x^{-1}x' \in S = \alpha^B$. This enables us to define the replacement and zig-zag products of G and H. In this case for any edge $(x,y)(x',y')$ of $G \circledR H$

we have $x = x'$ and $yy' \in E(H)$ or $xx' \in E(G)$ and $\varphi^x(x') = \theta_y(\alpha)$, $\varphi^{x'}(x) = \theta_{y'}(\alpha)$.

Theorem 9.4. *Let $G = Cay(A,S)$ and $H = Cay(B,T)$ be two Cayley graphs and $S = \alpha^B$, then $G®H = Cay(A \rtimes B, M)$ and $GZH = Cay(A \rtimes B, M')$. Where $M = \{(1_A, s) \cup (\alpha, 1_B) \mid s \in S\}$ and $M' = \{(1_A, s)(\alpha, 1_B)(1_A, s') \mid s, s' \in S\}$.*

Proof. The only important detail to prove is that we show $A \rtimes B$ is subgroup of $Aut(G®H)$ and acts regularly on $V(G®H) = A \times B$. For any $a \in A$ and $b \in B$ define $\varphi_{(a,b)}(x,y) = (a\theta_b(x), by)$, we first show $\varphi_{(a,b)} \in Aut(G®H)$ and after $A \rtimes B = \{\varphi_{(a,b)} \mid a \in A, b \in B\}$ is regular in $A \times B$. We prove if $(x,y)(x',y') \in E(G®H)$ then $(a\theta_b(x), by)(a\theta_b(x'), by') \in E(G®H)$. It is easily to see that if $x = x'$ and $y^{-1}y' \in T$ then $a\theta_b(x) = a\theta_b(x')$ and $by^{-1}by' = y^{-1}y' \in T$, also if $xx' \in E(G)$ then by $S = \alpha^B$ we have:

$$a\theta_b(x)^{-1}a\theta_b(x') = \theta_b(x)^{-1}\theta_b(x') = \theta_b(x^{-1}x') = \theta_b(\underbrace{x^{-1}x'}_{\in S}) \in S$$

Therefore from $\varphi^x(x') = \theta_y(\alpha)$ we have $x' = x\theta_y(\alpha)$,

$$a\theta_b(x') = a\theta_b(x\theta_y(\alpha)) = a(\theta_b(x)\theta_b\theta_y(\alpha)) = a\theta_b(x)\theta_{by}(\alpha)$$

Thus $\varphi^{ax}(ax') = \theta_{by}(\alpha)$ and $(a\theta_b(x), by)(a\theta_b(x'), by') \in E(G®H)$. Let $(x,y), (x',y') \in V(G®H)$, we show there is unique $\varphi(a,b)$ that $\varphi_{(a,b)}(x,y) = (x',y')$. Because $y, y' \in V(H)$ and B acts regularly in $V(H)$, then there is unique $\varphi_b \in B_L$ that $\varphi_b(y) = by = y'$. If $a = x'\theta_b(x^{-1})$ then

$$\varphi_{(a,b)}(x,y) = (a\theta_b(x), by) = (x'\theta_b(x^{-1})\theta_b(x), y') = (x', y')$$

Similar proof to Theorem 3.1, implies that generating subset is $(1_A, S) \cup (\alpha, 1_B)$. Similarly $GZH = Cay(A \rtimes B, M')$,where

$$M' = \{(1_A, s)(\alpha, 1_B)(1_A, s') \mid s, s' \in S\}.$$

□

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