Some new results on k-cordial labeling

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Abstract

Hovey [11] called a graph G is $A-$cordial where A is an additive Abelian group and $f: V(G) \to A$ is a labeling of the vertices of G with elements of A such that when the edges of G are labeled by the induced labeling $f: E(G) \to A$ by $f^*(xy) = f(x) + f(y)$ then the number of vertices (resp. edges) labeled with a and the number of vertices (resp. edges) labeled with b differ by at most one for all $a, b \in A$. When $A = \mathbb{Z}_k$, we call a graph G is $k-$cordial instead of $\mathbb{Z}_k-$cordial. In this paper, we give a sufficient condition for the join of two $k-$cordial graphs to be $k-$cordial and we give also a necessary condition for certain Eulerian graphs to be $k-$cordial when k is even and finally we complete the characterization of the $4-$cordiality of the complete tripartite graph.

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1. Introduction

All graphs considered in this paper are finite, simple and undirected. For most of the graph theory terminology and notation, we follow [6] and especially of graph labeling [8].

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, and let A be an Abelian group (with addition). A vertex labeling $f : V(G) \longrightarrow A$ induces an edge labeling $f^* : E(G) \longrightarrow A$, defined by $f^*(xy) = f(x) + f(y)$, for all edges $xy \in E(G)$. For $a \in A$, let $n_a(f) = \left| f^{-1}(a) \right|$ and $m_a(f) = \left| f^{*-1}(a) \right|$. A labeling f of a graph G is said to be $A-$cordial labeling if $\left| n_a(f) - n_b(f) \right| \leq 1$ and $\left| m_a(f) - m_b(f) \right| \leq 1$ for all $a, b \in A$. A graph G is called $A-$*cordial* if it admits an $A-$cordial labeling. When $A = \mathbb{Z}_k$, we use the term $k-$cordial labeling instead of $\mathbb{Z}_k$ $-$cordial labeling, and we say that G is $k-$cordial if G is $\mathbb{Z}_k$ $-$cordial.

The notion of $A-$cordial labeling was first introduced by Hovey [11] who introduced a simultaneous generalization of harmonious [10] and cordial [3] labelings and he observed that a (p,q) graph G is harmonious [10] if and only if G is $q-$cordial. However, Youssef [14] observed that one direction of this result is not strictly true. The result may be restated as every harmonious (p,q) graph is $q-$cordial and the converse is true if $q \geq p-1$. Hovey also observed that a (p,q) graph G is elegant [5] if and only if G is $p-$cordial and finally a graph G is cordial [3] if and only if G is $2-$cordial. See [1-2, 4-5, 7, 9, 12] for other related topics.

In the previously cited paper by Hovey [11], he has obtained the following results: caterpillars are $k-$cordial for all k; all trees are $k-$cordial for $k = 2, 3, 4$ and 5; odd cycles with pendant edges attached are $k-$cordial for all k ; cycles are k-cordial for all odd k; for k even C_{2mk+j} is k-cordial when $0 \leq j < \frac{k}{2}+2$ and when $k < j < 2k$; $C_{(2m+1)k}$ is not $k-$cordial for even k; K_n is 3-cordial; and K_{nk} is $k-$cordial if and only if $n=1$ and he also advanced the following conjectures: all trees are $k-$cordial for all k; all connected graphs are $3-$cordial; and C_{2mk+j} is k-cordial if and only if $j \neq k$, where k and j are even and $0 \leq j < 2k$. The last conjecture was verified by Tao [13]. Youssef [14] proved some necessary conditions for a graph to be $k-$cordial and gave some new families of $k-$cordial graphs. The reference [8] surveys the current state of knowledge for $k-$cordial labeling and many other labelings.

In this section we mention some results which are referred to throughout the paper. In [11] Hovey showed that

Lemma 1. If f is a $k-$cordial labeling of G so is $af+b$ for any unit $a \in \mathbb{Z}_k$ and $b \in \mathbb{Z}_k$.

In [14] Youssef proved the following results

Lemma 2. If G is a (p,q) $k-$cordial graph with $p \equiv 0 \pmod{k}$, then $G+\overline{K}_n$ is $k-$cordial for all positive integer n.

Theorem 1. $K_{m,n}$ is $4-$cordial if and only if m or $n \not\equiv 2 \pmod{4}$.

In the next section of this paper we give a sufficient condition for the join of two $k-$cordial graphs to be $k-$cordial and we give another sufficient condition in case of $4-$cordial graphs . In Section 3, we give a necessary condition for certain Eulerian graphs to be $k-$cordial for even k and we give the complete characterization of the $4-$cordiality of complete tripartite graphs.

2. Further results of k-cordial graphs

In this section, we prove that the join of two $k-$cordial graphs is $k-$cordial under certain conditions

Theorem 2.1 Let G and H be (p_1,q_1) and (p_2,q_2) $k-$cordial graphs. If (p_1 or $p_2 \equiv 0 \pmod{k}$) and (q_1 or $q_2 \equiv 0 \pmod{k}$), then $G+H$ is $k-$cordial.

Proof Let g (resp. h) be a $k-$cordial labeling of G (resp. H). Define the labeling $f : V(G+H) \to \mathbb{Z}_k$ as $f\big|_{V(G)} = g$ and $f\big|_{V(H)} = h$. Since, p_1 or $p_2 \equiv 0(\bmod k)$, then $n_i(g) = \frac{p_1}{k}$ or $n_i(h) = \frac{p_2}{k}$ for all $0 \le i \le k-1$. Similarly, Since, q_1 or $q_2 \equiv 0(\bmod k)$, $m_j(g) = \frac{q_1}{k}$ or $m_j(h) = \frac{q_2}{k}$ for all $0 \le j \le k-1$.

Now, $\left|n_i(f) - n_j(f)\right| = \left|(n_i(g) + n_i(h)) - (n_j(g) + n_j(h))\right|$

$$= \left|(n_i(g) - n_j(g)) + (n_i(h) - n_j(h))\right| \le 1$$

Also,

$\left|m_i(f) - m_j(f)\right|$

$$= \left|(m_i(g) + m_i(h) + \frac{p_1}{k}\sum_{i=0}^{k-1} n_i(h)) - (m_j(g) + m_j(h) + \frac{p_1}{k}\sum_{j=0}^{k-1} n_j(h))\right|$$

$= \left|(m_i(g) - m_j(g)) + (m_i(h) - m_j(h)) + \frac{p_1}{k}0\right| \le 1$. This completes the proof. □

Lemma 2 above is a straightforward corollary of Theorem 2.1. The following is another straightforward corollary.

Corollary 1 If G is $k-$cordial, then $G + \overline{K}_n$ is k-cordial for every $n \equiv 0 \pmod{k}$.

Theorem 2.2 Let G and H be (p_1, q_1) and (p_2, q_2) $4-$cordial graphs with q_1 and $q_2 \equiv 0 \pmod 4$. If p_1 or $p_2 \not\equiv 2 \pmod 4$, then $G+H$ is $4-$cordial.

Proof Without any loss of generality, we take the case $p_1 \not\equiv 2 \pmod 4$. If $p_1 \equiv 0 \pmod 4$, then $G+H$ is $4-$cordial by Theorem 2.1. Now, let g (resp. h) be a $4-$cordial labeling of G (resp. H). We have two other cases to consider:

Case 1: $p_1 \equiv 1 \pmod 4$

Since G is 4-cordial, then $n_i(g) = n_j(g) = n_k(g) = n_t(g) - 1$, where $\{i, j, k, t\} = \mathbb{Z}_4$ and since H is 4-cordial, then by Lemma 1, there exist $a \in \mathbb{Z}_4$ such that $h + a$ is a $4-$cordial labeling with $n_t(h+a) \le n_s(h+a)$ for every $s \in \{i, j, k\}$. Hence, $G+H$ is $4-$cordial with the labeling $f : V(G+H) \to \mathbb{Z}_4$ which is defined as $f|_{V(G)} = g$ and $f|_{V(H)} = h + a$.

Case 2: $p_1 \equiv 3 \pmod 4$

Similarly, since G is 4-cordial, then $n_i(g) = n_j(g) = n_k(g) = n_t(g) + 1$, where $\{i, j, k, t\} = \mathbb{Z}_4$, again since H is $4-$cordial, then by Lemma 1, there exist $b \in \mathbb{Z}_4$ such that $h + b$ is a $4-$cordial labeling with $n_t(h+b) \ge n_s(h+b)$ for every $s \in \{i, j, k\}$. As in case 1, $G+H$ is $4-$cordial. □

Note that, in the above theorem, if p_1 and $p_2 \equiv 2 \pmod 4$, then the graph $G+H$ may be not $4-$cordial for example the graph $\bar{K}_2 + \bar{K}_2 = K_{2,2}$ is not $4-$cordial by Theorem 1.

3. 4-cordiality of complete multipartite graphs

The following lemma shows that adding a number of isolated vertices congruent to $0 \pmod{k}$ to one partition (then to many partitions) of complete multipartite $k-$cordial graph produces another complete multipartite $k-$cordial graph.

Lemma 3.1 If $K_{m_1,m_2,\cdots,m_r}$ is $k-$cordial so is $K_{m_1+t,m_2,\cdots,m_r}$ for all $t \equiv 0(\bmod k)$.

Proof Label the t vertices by $0,1,\cdots,k-1;0,1,\cdots, \quad k-1 ;\cdots;0,1,\cdots,k-1$ .So every vertex of the partitions of $m_2,m_3,\cdots m_r$ vertices join all the t vertices which gives an equal number of edges labeled $0,1,\cdots,k-1$.□

Lemma 3.2 Let $k=4(2t-1)$ be an integer where $t \geq 1$. If $r \equiv 2(\bmod 4)$ and $m_i \equiv 2(2t-1)(\bmod k)$ for $1 \leq i \leq r$, then the graph $K_{m_1,m_2,\cdots m_r}$ is not $k-$cordial.

Proof Let $G=K_{m_1,m_2,\cdots m_r}$, $V(G)=\bigcup_{i=1}^{r}\left\{u_{i,j} : 1 \leq j \leq m_i\right\}$, $p=|V(G)|=\sum_{i=1}^{r} m_i$,$q=|E(G)|=\sum_{1\leq i<j\leq r} m_i m_j$ and suppose that G is $k-$cordial with labeling f, then $\sum_{e\in E(G)} f^*(e) \equiv$ ($\sum_{v\in V(G)} \deg(v)f(v)$)$(\bmod k)$ and as $p,q \equiv 0(\bmod k)$, (all next congruencies are taken $(\bmod k)$)

$$\sum_{i=0}^{k-1}\frac{q}{k}i \equiv \sum_{i\neq 1} m_i \sum_{j=1}^{m_1} f(u_{1,j}) + \sum_{i\neq 2} m_i \sum_{j=1}^{m_2} f(u_{2,j}) + \cdots + \sum_{i\neq r} m_i \sum_{j=1}^{m_r} f(u_{r,j})$$

,

$$\frac{q}{2}(k-1) \equiv \sum_{i\neq 1} m_i \sum_{j=1}^{m_1} f(u_{1,j}) + \sum_{i\neq 2} m_i \sum_{j=1}^{m_2} f(u_{2,j}) + \cdots +$$

$$(\sum_{i\neq r} m_i)(\sum_{i=0}^{k-1}\frac{p}{k}i - \sum_{i=1}^{r-1}\sum_{j=1}^{m_i} f(u_{i,j})),$$

$$\frac{q}{2}(k-1) \equiv (m_r - m_1)\sum_{j=1}^{m_1} f(u_{1,j}) + (m_r - m_2)\sum_{j=1}^{m_2} f(u_{2,j}) + \cdots$$

$$(m_r - m_{r-1})\sum_{j=1}^{m_{r-1}} f(u_{r-1,j}) + \frac{p}{2}(k-1)\sum_{i\neq r} m_i$$ .Thus

$\frac{q}{2}(k-1) \equiv 0 (\bmod k)$, so $q \equiv 0 (\bmod 2k)$ a contradiction.□

Note that if G is Eulerian graph with number of edges $q \equiv k \pmod{2k}$, then the graph may or may not be $k-$cordial when k is even . For example C_n is not $k-$cordial when $n \equiv k (\bmod 2k)$ [11,14] also C^2_{4n+2} is not $4-$cordial for all n [14] but on the other hand the friendship graph $C_3^{(4)} = K_1 + 4K_2$, the graph consisting of 4 copies of the cycle C_3 with one vertex in common, is $4-$cordial (by labeling the vertex of K_1 by 0, the first component of K_2 by $1, 1$, the second by $2, 2$, the third by $1, 3$ and the fourth by $0, 3$). The following proposition determines a necessary condition for certain Eulerian graphs to be $k-$cordial for even k.

Proposition 3.3 Let $k = 2^s(2t-1)$ be an integer where $s,t \geq 1$ and G be a graph of $q \equiv 0 \pmod{k}$ edges such that 2^s divides the degree of every vertex of G. If $q \equiv k \pmod{2k}$, then G is not $k-$cordial.

Proof Let $q \equiv k \pmod{2k}$ and suppose that G is $k-$cordial with labeling f, then $\sum_{e\in E(G)} f^*(e) \equiv (\sum_{v\in V(G)} \deg(v)f(v)) \pmod{k}$ and hence $\sum_{i=0}^{k-1} \frac{q}{k} i \equiv \sum_{v\in V(G)} \deg(v)f(v)$, and as $2^s \Big| \sum_{v\in V(G)} \deg(v)f(v)$, then $\frac{k}{2} \equiv 2^s x \pmod{k}$ where $\sum_{v\in V(G)} \deg(v)f(v) = 2^s x$. That is, $2^s x \equiv \frac{k}{2} \pmod{k}$ a contradiction since $\gcd(2^s, k)$ does not divide $\frac{k}{2}$. □

The following corollary is straightforward.

Corollary 2 If $k = 4(2t-1)$, $r \equiv 3 \pmod{4}$ and $m_i \equiv 2(2t-1) \pmod{k}$, then the graph $K_{m_1,m_2,\cdots m_r}$ is not $k-$cordial.

Let $K_{m,n,p}$ be the complete tripartite graph with $V(K_{m,n,p}) = \{u_i : 1 \leq i \leq m\} \cup \{v_j : 1 \leq j \leq n\} \{w_k : 1 \leq k \leq p\} \cup$ and $E(K_{m,n}) = \{u_i v_j : 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{u_i w_j : 1 \leq i \leq m, 1 \leq k \leq p\} \cup \{v_j w_k : 1 \leq j \leq n, 1 \leq k \leq p\}$.

Youssef [14] completed the characterization of the $4-$cordiality of the complete bipartite graphs. However, the following theorem extends this result to complete tripartite graphs. Using the symmetry of $K_{m,n,p}$, we may assume that $m \pmod{4} \leq n \pmod{4} \leq p \pmod{4}$.

Theorem 3.4 $K_{m,n,p}$ is $4-$cordial if and only if $(m,n,p)(\bmod 4) \not\equiv (0,2,2),(2,2,2)$.

Proof Necessity, if $m,n,p \equiv 2(\bmod 4)$, then $K_{m,n,p}$ is not $4-$cordial by Corollary 2 . If $m \equiv 0(\bmod 4)$ and $n,p \equiv 2(\bmod 4)$, let $q = \left|E(K_{m,n,p})\right|$ and suppose that $K_{m,n,p}$ is $4-$cordial with labeling f, then

$$\sum_{e \in E(K_{m,n,p})} f^{*}(e) \equiv \Big(\sum_{v \in V(K_{m,n,p})} \deg(v) f(v) \Big)(\bmod 4)$$

$$\Rightarrow \frac{6q}{4} \equiv \Big((n+p)\sum_{i=1}^{m} f(u_i) + (m+p)\sum_{j=1}^{n} f(v_j) + (m+n)\sum_{k=1}^{p} f(w_k)\Big)$$
$(\bmod 4)$

$$\Rightarrow \frac{3q}{2} \equiv \Big((n+p)\sum_{i=1}^{m} f(u_i) + m\Big(\sum_{j=1}^{n} f(v_j) + \sum_{k=1}^{p} f(w_k)\Big) + p\sum_{j=1}^{n} f(v_j)$$
$$+n\sum_{k=1}^{p} f(w_k)\Big)(\bmod 4)$$

$$\Rightarrow \frac{3q}{2} = \Big(p\sum_{j=1}^{n} f(v_j) + n\sum_{k=1}^{p} f(w_k)\Big)(\bmod 4)$$

$$\Rightarrow \frac{3q}{2} = p\Big((0+1+2+3)\frac{m+n+p}{4} - \sum_{i=1}^{m} f(u_i) - \sum_{k=1}^{p} f(w_k)\Big)$$
$$+n\sum_{k=1}^{p} f(w_k)\ (\bmod 4)$$

$$\Rightarrow \frac{3q}{2} = \frac{6p}{4}(m+n+p) + (n-p)\sum_{k=1}^{p} f(w_k) - p\sum_{i=1}^{m} f(u_i)\ (\bmod 4).$$

Then $\sum_{i=1}^{m} f(u_i) \equiv 1(mod2)$ and hence $\sum_{j=1}^{n} f(v_j) + \sum_{k=1}^{p} f(w_k) \equiv 1(\text{mod}\,2)$.

Without any loss of generality we may assume that $\sum_{k=1}^{p} f(w_k) \equiv 1(\text{mod}\,2)$.

Now, let O_1 (resp. O_2 ,resp. O_3) be the number of vertices whose label is odd in the m-set (resp n -set, resp. p-set) and E_1 (resp. E_2, resp. E_3) be the number of vertices whose label is even in the m-set (resp n -set, resp. p-set). Since $\sum_{i=1}^{m} f(u_i) \equiv 1(mod2)$, then O_1 is odd and hence E_1 is odd too. Also , as $\sum_{k=1}^{p} f(w_k) \equiv 1(\text{mod}\,2)$, then O_3 is odd and hence E_3 is odd too. So, we can deduce that both of O_2 and E_2 are even. We calculate $m_0(f) + m_2(f)$ and $m_1(f) + m_3(f)$.

$m_0(f) + m_2(f) = O_1(O_2 + O_3) + E_1(E_2 + E_3) + E_2E_3 + O_2O_3$,

$m_1(f) + m_3(f) = O_1(E_2 + E_3) + E_1(O_2 + O_3) + O_2E_3 + E_2O_3$

Put $x = O_2 + O_3$, $y = E_2 + E_3$ and subtract $m_0(f) + m_2(f)$ from $m_1(f) + m_3(f)$, we get:

$m_1(f) + m_3(f) - (m_0(f) + m_2(f)) = (O_1 - E_1)(y - x) + (E_3 - O_3)(O_2 - E_2)$. As $x + y = n + p \equiv 0(\text{mod}\,4)$ and since both x and y are odd, then $y - x \equiv \pm 2(\text{mod}\,8)$. Similarly both of O_1 and E_1 are odd and $O_1 + E_1 \equiv 0(\text{mod}\,4)$, then $O_1 - E_1 \equiv \pm 2(\text{mod}\,8)$. Note also, $O_2 - E_2 \equiv \pm 2(\text{mod}\,8)$ and $E_3 - O_3 \equiv 0(\text{mod}\,4)$.

Hence, $m_1(f) + m_3(f) - (m_0(f) + m_2(f)) \equiv 4(\text{mod}\,8)$ which a contradiction, since $m_i(f) = \frac{q}{4}$ for every $i \in \mathbb{Z}_4$.

Sufficiency, if two of m, n or p, say m and n, satisfy that $m+n\equiv 0(\bmod 4)$ and m or $n\not\equiv 2(\bmod 4)$, then $K_{m,n}$ is 4-cordial by Theorem 1 and hence $K_{m,n,p}=K_{m,n}+\overline{K}_p$ is 4-cordial by Lemma 1. This case showed that $K_{m,n,p}$ is 4-cordial when $(m,n,p)(\bmod 4)\equiv(0,0,0)$, $(0,0,1)$, $(0,0,2)$, $(0,0,3)$, $(0,1,3)$, $(1,1,3)$, $(1,2,3)$ and $(1,3,3)$. If one of m, n or $p\equiv 0(\bmod 4)$, say m, and another one $\not\equiv 2(\bmod 4)$, say p, then $K_{m,n,p}=K_{m,n}+\overline{K}_p$ is $4-$cordial by Theorem 2.2. This case covers the more cases of $(m,n,p)(\bmod 4)\equiv$ $(0,1,1)$, $(0,1,2)$, $(0,2,3)$ and $(0,3,3)$. Finally, as $K_{1,1,1}$, $K_{1,1,2}$, $K_{1,2,2}$, $K_{2,2,3}$, $K_{2,3,3}$ and $K_{3,3,3}$ are $4-$cordial by the labeling f where $f(V(K_{m,n,p}))$ for each graph is given as the following respectively $:\{0;1;2\}$, $\{0;1;2,3\}$, $\{0;1,2;2,3\}$, $\{0,1;0,3;1,2,3\}$, $\{0,1;0,2,3;1,2,3\}$, and $\{0,1,2;0,1,3;0,2,3\}$, then $K_{m,n,p}$ is $4-$cordial in the remaining cases by Lemma 3.1. □

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