On the forcing connected geodetic number and the connected geodetic number of a graph

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Abstract

For two vertices u and v of a nontrivial connected graph G, the set $I[u,v]$ consists of all vertices lying on some $u-v$ geodesic in G, including u and v. For $S \subseteq V(G)$, the set $I[S]$ is the union of all sets $I[u,v]$ for $u,v \in S$. A set $S \subseteq V(G)$ is a connected geodetic set of G if $I[S] = V(G)$ and the subgraph in G induced by S is connected. The minimum cardinality of a connected geodetic set of G is the connected geodetic number $g_c(G)$ of G and a connected geodetic set of G whose cardinality equals $g_c(G)$ is a minimum connected geodetic set of G. A subset T of a minimum connected geodetic set S is a forcing subset for S if S is the unique minimum connected geodetic set of G containing T. The forcing connected geodetic number $f_c(S)$ of S is the minimum cardinality of a forcing subset of S and the forcing connected geodetic number $f_c(G)$ of G is the minimum forcing connected geodetic number among all minimum connected geodetic sets of G. Therefore, $0 \leq f_c(G) \leq g_c(G)$. We determine all pairs a, b of integers such that $f_c(G) = a$ and $g_c(G) = b$ for some nontrivial connected graph G. We also consider a problem of realizable triples of integers.

Keywords: connected geodetic set, connected geodetic number, forcing connected geodetic number.

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1 Introduction

All graphs considered in this paper are finite, undirected, loopless, and without multiple edges. We refer the reader to the book [14] for graph theory notation and terminology not described in this paper. For vertices u and v in a connected graph G, the *distance* $d_G(u,v)$ (or simply $d(u,v)$) is the length of a shortest $u-v$ path in G. An $u-v$ path of length $d(u,v)$ is called a $u-v$ *geodesic*. The *geodetic interval* $I[u,v]$ is the set consisting of u, v, and all vertices lying on some $u-v$ geodesic in G, while for a set $S \subseteq V(G)$, the *geodetic closure of* S is the set $I[S] = \bigcup_{u,v \in S} I[u,v]$. A set S of vertices is called a *geodetic set* of G if $I[S] = V(G)$ and the minimum cardinality of a geodetic set of G is the *geodetic number* $g(G)$ of G. The geodetic sets of connected graphs were introduced in [7] by Harary et al. as a tool for studying metric properties of connected graphs. The complexity of the problem of finding the geodetic number has been studied in [1, 5]. Dourado et al. [5] have shown that the corresponding GEODETIC SET decision problem, namely *"given a nontrivial connected graph G and an integer $k \leq |V(G)|$, is there a set $S \subseteq V(G)$ with $|S| = k$ such that $I[S] = V(G)$?"* remains NP-complete even when restricted to chordal and chordal bipartite graphs.

For a nontrivial connected graph G, a set $S \subseteq V(G)$ is a *connected geodetic set* of G if S is a geodetic set and the subgraph of G induced by S is connected. The *connected geodetic number* $g_c(G)$ of G is the minimum cardinality of a connected geodetic set of G. This concept was introduced and studied independently by Mojdeh and Rad [8] and Santhakumaran et al. [9, 10]. A connected geodetic set whose cardinality equals $g_c(G)$ is called a *minimum connected geodetic set.* A subset T of a minimum connected geodetic set S is a *forcing subset for* S if S is the unique minimum connected geodetic set of G containing T. The *forcing connected geodetic number* $f_c(S)$ of S is the minimum cardinality among all forcing subsets of S and the *forcing connected geodetic number* $f_c(G)$ of G is the minimum forcing connected geodetic number among all minimum connected geodetic sets of G. In particular, $f_c(G) = 0$ if and only if G has exactly one minimum connected geodetic set. Forcing concepts have been widely investigated in graph theory, such as forcing convexity numbers [4], forcing geodetic numbers [3, 11, 12, 13, 15], and forcing connected geodetic numbers [10]. The forcing geodetic sets and the forcing geodetic number of a graph were introduced by Chartrand and Zhang [3].

If G is a nontrivial connected graph, then $V(G)$ is certainly a connected geodetic set of G and so

$$2 \leq g_c(G) \leq |V(G)|. \tag{1}$$

Furthermore, by the definition of $f_c(G)$, it follows that

$$0 \leq f_c(G) \leq g_c(G). \tag{2}$$

It is then natural to ask which pairs of integers are realizable as

A : the connected geodetic number and the order, or
B : the forcing connected geodetic number and the connected geodetic number,

respectively, of some graph. Problem A can be answered fairly easily with an additional definition and result. A vertex in a graph G is *simplicial* if the subgraph in G induced by its neighborhood is complete. In particular, every end-vertex is simplicial.

Lemma 1.1 [9] *Let G be a nontrivial connected graph.* (a) *If v is either a cut-vertex or a simplicial vertex of G, then v belongs to every connected geodetic set of G.* (b) *$g_c(G) = 2$ if and only if $G = K_2$.*

Theorem 1.2 *A pair a, b of integers is realizable as the connected geodetic number and the order, respectively, of some nontrivial connected graph if and only if either $2 \leq a = b$ or $3 \leq a < b$.*

Proof. By Lemma 1.1(b), there is a connected graph of order b whose connected geodetic number equals 2 if and only if $b = 2$. For $a \geq 3$, let $G = K_b$ if $a = b$ while $G = \overline{K}_{a-1} \vee K_{b-a+1}$ if $a < b$. Then G is a connected graph of order b with $g_c(G) = a$. Thus the result follows by (1). ∎

In contrast, only a partial result has been obtained for Problem B.

Theorem 1.3 [10] *For every pair a, b of integers with $0 \leq a \leq b-4$, there exists a connected graph G such that $f_c(G) = a$ and $g_c(G) = b$.*

In this note we present in Section 2 a complete answer to Problem B by determining all pairs a, b of integers for which there exists a nontrivial connected graph G with $f_c(G) = a$ and $g_c(G) = b$. We also investigate another realization problem in Section 3.

2 The result on realizable pairs

Recall that a nontrivial connected graph G has a unique minimum connected geodetic set if and only if $f_c(G) = 0$. Thus the following is an immediate consequence of Lemma 1.1(a).

Corollary 2.1 *For each integer $n \geq 2$, $f_c(K_n) = 0$ and $g_c(K_n) = n$.*

Suppose that G is a nontrivial connected graph and $(f_c(G), g_c(G)) = (a, b)$. Then by (1) and (2) it follows that $b \geq 2$ and $0 \leq a \leq b$. Furthermore, Lemma 1.1(b) implies that either $(a, b) = (0, 2)$ or $b \geq 3$. It turns out that the converse is also true. Thus the following is the complete description of pairs a, b of integers such that $f_c(G) = a$ and $g_c(G) = a$ for some graph G.

Theorem 2.2 *A pair a, b of integers with $0 \leq a \leq b$ and $b \geq 2$ is realizable as the forcing connected geodetic number and the connected geodetic number, respectively, of some nontrivial connected graph if and only if either $(a, b) = (0, 2)$ or $b \geq 3$.*

Proof. We consider the following four cases.

Case 1. $a = 0$. Then for each $b \geq 2$, the complete graph of order b has the desired property.

Case 2. $1 \leq a \leq b - 2$. Let $G_{a,b}$ be the connected graph of order $a + b$ with $V(G_{a,b}) = V_1 \cup V_2 \cup \cdots \cup V_{a+2}$, where $|V_1| = b - a - 1$, $|V_{a+2}| = 1$, and $V_i = \{u_i, v_i\}$ for $2 \leq i \leq a + 1$, such that two vertices $x \in V_i$ and $y \in V_j$ $(1 \leq i, j \leq a + 2)$ are adjacent if and only if $|i - j| \leq 1$. If S is a connected geodetic set of $G_{a,b}$, then $V_1 \cup V_{a+2} \subseteq S$ by Lemma 1.1(a). Furthermore, $|S \cap V_i| \geq 1$ for $2 \leq i \leq a+1$ since $G_{a,b}[S]$ is connected. Thus $g_c(G_{a,b}) \geq b$. Since $S^* = V_1 \cup V_{a+2} \cup \{u_2, u_3, \ldots, u_{a+1}\}$ is clearly a connected geodetic set, it follows that $g_c(G_{a,b}) = b$. Next let S be an arbitrary minimum connected geodetic set. If $T \subseteq S$ and $T \cap V_i = \emptyset$ for some i $(2 \leq i \leq a + 1)$, then there exist at least two minimum connected geodetic sets containing T as a subset and so T is not a forcing subset for S. Therefore, if T is a forcing subset for S, then T contains one of u_i and v_i for $2 \leq i \leq a + 1$, that is, $|T| \geq a$. Since $T^* = \{u_2, u_3, \ldots, u_{a+1}\}$ is a forcing subset for S^*, it follows that $f_c(G_{a,b}) = a$.

Case 3. *b is odd and $a \in \{b - 1, b\}$*. First suppose that $b = 3$. Then one can verify that $(f_c(W_{1,4}), g_c(W_{1,4})) = (2, 3)$ and $(f_c(C_4), g_c(C_4)) = (3, 3)$, where $W_{1,4} = C_4 \vee K_1$ is the wheel of order 5.

Therefore, assume that $b = 2n + 1$ for some integer $n \geq 2$. We first construct the connected graph $G_{2n,2n+1}$ of order $3n + 1$ from n copies $P_1, P_2, \ldots, P_n$ of paths of order 3, where $P_i = (u_i, v_i, w_i)$ for $1 \leq i \leq n$, by adding a new vertex x and joining x to u_i and w_i for $1 \leq i \leq n$. Since $S^* = V(G_{2n,2n+1}) - \{v_1, v_2, \ldots, v_n\}$ is a connected geodetic set, $g_c(G_{2n,2n+1}) \leq |S^*| = 2n + 1$. On the other hand, if S is an arbitrary connected geodetic set, then $x \in S$ and at least two of the three vertices u_i, v_i, and w_i belong to S for $1 \leq i \leq n$. Therefore, $|S| \geq 2n+1$, implying that $g_c(G_{2n,2n+1}) = 2n+1$. Furthermore, we see that S is a minimum connected geodetic set of $G_{2n,2n+1}$ if and only if (i) $x \in S$ and (ii) $|\{u_i, v_i, w_i\} \cap S| = 2$ for $1 \leq i \leq n$. If T is a forcing subset for a minimum connected geodetic set S, then $|\{u_i, v_i, w_i\} \cap T| = 2$ for $1 \leq i \leq n$. Thus $|T| \geq 2n$. Since $T^* = S^* - \{x\}$ is a forcing subset for S^*, it follows that $f_c(G_{2n,2n+1}) = 2n$.

Now obtain the graph $G_{2n+1,2n+1}$ from $G_{2n,2n+1}$ by adding a new vertex y and joining y to u_i and w_i for $1 \leq i \leq n$. Then every connected geodetic set S contains one of x and y and at least two of the three vertices u_i, v_i, and w_i for $1 \leq i \leq n$. Since S^* is a connected geodetic set, $g_c(G_{2n+1,2n+1}) = 2n + 1$. In fact, S is a minimum connected geodetic set if and only if (i) $|\{x, y\} \cap S| = 1$ and (ii) $|\{u_i, v_i, w_i\} \cap S| = 2$ for $1 \leq i \leq n$. Therefore, for an arbitrary minimum connected geodetic set S, a subset $T \subseteq S$ is a forcing subset for S if and only if $T = S$. Thus, $f_c(G_{2n+1,2n+1}) = g_c(G_{2n+1,2n+1}) = 2n + 1$.

Case 4. *b is even and $a \in \{b - 1, b\}$.* Suppose first that $b = 4$. Then we have $(f_c(K_2 \square P_3), g_c(K_2 \square P_3)) = (3, 4)$ and $(f_c(C_5), g_c(C_5)) = (4, 4)$, where $K_2 \square P_3$ is the cartesian product of K_2 and P_3.

Hence, assume finally that $b = 2n$ for some integer $n \geq 3$. We first construct the connected graph $G_{2n-1,2n}$ of order $3n - 1$ from a path $P = (u_0, v_0, w_0, x_0)$ of order 4 and $n-2$ copies $P_1, P_2, \ldots, P_{n-2}$ of paths of order 3, where $P_i = (u_i, v_i, w_i)$ for $1 \leq i \leq n - 2$, by adding a new vertex x and joining x to (i) u_0 and x_0 and (ii) u_i and w_i for $1 \leq i \leq n-2$. Also, construct the graph $G_{2n,2n}$ from $G_{2n-1,2n}$ by adding a new vertex y and joining y to (i) u_0 and x_0 and (ii) u_i and w_i for $1 \leq i \leq n-2$. Then one can verify in a similar manner as in Case 3 that $(f_c(G_{2n-1,2n}), g_c(G_{2n-1,2n})) = (2n-1, 2n)$ and $(f_c(G_{2n,2n}), g_c(G_{2n,2n})) = (2n, 2n)$. This completes the proof. ∎

3 On realizable triples

Recall the inequalities (1) and (2) in Section 1. Combining the two, we see that if G is a nontrivial connected graph, then $0 \le f_c(G) \le g_c(G) \le |V(G)|$. This suggests another realization problem.

A triple (a, b, c) of nonnegative integers is said to be *realizable* if there exists a connected graph G for which $f_c(G) = a$, $g_c(G) = b$, and $|V(G)| = c$. Therefore, if (a, b, c) is a realizable triple, then either (i) $(a, b, c) = (0, 2, 2)$ or (ii) $0 \le a \le b \le c$ and $b \ge 3$ by Theorems 1.2 and 2.2.

As an example, we show that those triples (a, b, c) of positive integers with $b \ge a + 2$ and $c \ge a + b$ are realizable.

Proposition 3.1 *The triple (a, b, c) of positive integers is realizable if $b \ge a + 2$ and $c \ge a + b$.*

Proof. Consider the connected graph G of order c with $V(G) = V_1 \cup V_2 \cup \cdots \cup V_{a+2}$, where $|V_1| = b - a - 1$, $|V_2| = c - a - b + 2$, $|V_i| = 2$ for $3 \le i \le a + 1$ (if $a \ge 2$), and $|V_{a+2}| = 1$, such that two vertices $x \in V_i$ and $y \in V_j$ $(1 \le i, j \le a + 2)$ are adjacent if and only if $|i - j| \le 1$. Then one can verify that $f_c(G) = a$ while $g_c(G) = b$. ■

Before continuing our discussion, let us present several additional results.

Lemma 3.2 *Let G be a nontrivial connected graph. Then $g_c(G) = |V(G)|$ if and only if every vertex is either a cut-vertex or a simplicial vertex. Furthermore, if $g_c(G) = |V(G)|$, then $f_c(G) = 0$.*

Proof. Theorem 1.1(a) implies that $g_c(G) = |V(G)|$ is a necessary condition. Also, if $g_c(G) = |V(G)|$, then $f_c(G) = 0$ since $V(G)$ is the unique minimum connected geodetic set. For the converse, suppose that G contains a vertex v that is neither a cut-vertex nor a simplicial vertex. Then there are two nonadjacent vertices x and y such that $vx, vy \in E(G)$. Hence v belongs to an $x - y$ geodetic and so $V(G) - \{v\}$ is a connected geodetic set of G. Therefore, $g_c(G) \le |V(G)| - 1$. ■

Lemma 3.3 *The connected geodetic number of a graph G equals 3 if and only if either $G = K_3$ or $G = \overline{K}_2 \vee H$ for some graph H.*

Proof. The necessity of $g_c(G) = 3$ immediately follows. Suppose that $g_c(G) = 3$ and let S be a minimum connected geodetic set. If $G[S] = K_3$, then clearly $G = K_3$. Otherwise, $G[S]$ is a path of order 3, say $G[S] = (v_1, v_2, v_3)$. Therefore, each vertex in $V(G) - \{v_1, v_3\}$ must be adjacent to both v_1 and v_3, which is the desired result. ■

Corollary 3.4 *If G is a connected graph of order at least 4 with $g_c(G) = 3$, then $f_c(G) \geq 1$.*

We are prepared to determine all realizable triples (a, b, c) where $0 \leq a \leq 2$.

Theorem 3.5 *Let a, b, and c be integers with $0 \leq a \leq b \leq c$ and $b \geq 2$. If (a, b, c) is realizable, then either $(a, b, c) = (0, 2, 2)$ or $b \geq 3$. Furthermore,*

(a) *(a, b, b) is realizable if and only if $a = 0$.*

(b) *$(0, b, c)$ is realizable if and only if $b = c$ or $4 \leq b < c$.*

(c) *$(1, b, c)$ is realizable if and only if $3 \leq b < c$.*

(d) *$(2, b, c)$ is realizable if and only if $3 \leq b < c$ and $c \neq 4$.*

Proof. We may assume that $b \geq 3$. We first verify (a). By Lemma 3.2, if (a, b, b) is realizable, then $a = 0$. Also, Corollary 2.1 shows that $(0, b, b)$ is realizable for each integer b greater than 1. Furthermore, (c) is an immediate consequence of (a) and Proposition 3.1.

For (b), we may assume that $3 \leq b < c$ by (a). If $4 \leq b < c$, then consider the graph G obtained from $H \cong K_{c-b+1}$ by adding $b - 1$ new vertices $v_1, v_2, \ldots, v_{b-1}$ and joining (i) $v_1, v_2, \ldots, v_{b-2}$ to every vertex of H and (ii) v_{b-1} to exactly one vertex of H. Then one can verify that G is a connected graph of order c with $f_c(G) = 0$ and $g_c(G) = b$. For the converse, observe that the triple $(0, 3, c)$ is realizable if and only if $c = 3$ by Corollary 3.4.

For (d), first one can verify that $(2, b, 4)$ is not realizable by inspecting all connected graphs of order 4. We therefore show that $(2, b, c)$ is realizable if $3 \leq b < c$ and $c \geq 5$. If $b = 3$, then let $G = K_{2,2,c-4}$ if $c \neq 6$ and $G = K_{1,1,2,2}$ if $c = 6$. If $b \geq 4$ and $c = b + 1$, then let G be the graph obtained from the 4-cycle $(v_1, v_2, v_3, v_4, v_1)$ by adding $b - 3$ pendant edges at v_1. If $b \geq 4$ and $c \geq b + 2$, then the result follows by Proposition 3.1. ■

Next we determine all realizable triples $(3, b, c)$.

Lemma 3.6 *Let G be a connected graph of order $n \geq 4$. Then $f_c(G) = g_c(G) = 3$ if and only if G is $(n-2)$-regular, that is, $G = K_{2,2,\ldots,2}$.*

Proof. It is straightforward to verify that $f_c(K_{2,2,\dots,2}) = g_c(K_{2,2,\dots,2}) = 3$. Now assume that G is a connected graph of order $n \geq 4$ with $f_c(G) = g_c(G) = 3$. By Lemma 3.3, there are two nonadjacent vertices x and y with $\deg x = \deg y = n-2$ and every 3-set containing x and y is a minimum connected geodetic set of G. If $\Delta(G) = n-1$, then let z be a vertex whose degree equals $n-1$. Then $\{x, z\}$ is a forcing subset for $\{x, y, z\}$ and so $f_c(G) \leq 2$. Therefore, $\Delta(G) = n-2$. If $\delta(G) \leq n-3$, then let z be a vertex with $\deg z = \delta(G)$. We claim that $\{x, z\}$ is a forcing subset for $\{x, y, z\}$. Assume, to the contrary, that there exists a minimum connected geodetic set $S = \{x, z, y'\}$ where $y' \neq y$. Since $xy' \in E(G)$ and $G[S]$ cannot be a triangle, it follows that $zy' \notin E(G)$. Furthermore, every vertex $v \in V(G) - \{z, y'\}$ must be adjacent to both z and y'. However, this is impossible since $\deg z \leq n-3$. Thus, we conclude that $\Delta(G) = \delta(G) = n-2$. ■

Corollary 3.7 *If G is a connected graph of odd order and $g_c(G) = 3$, then $0 \leq f_c(G) \leq 2$.*

Theorem 3.8 *Let b and c be integers with $3 \leq b < c$. The triple $(3, b, c)$ is realizable if and only if $b \geq 4$ or c is even.*

Proof. By Lemma 3.6 and Corollary 3.7, the triple $(3, 3, c)$ is realizable if and only if c is even.

For $b = 4$, first observe that the graphs $C_5 + e$ and C_6 show that the triples $(3, 4, 5)$ and $(3, 4, 6)$ are realizable, respectively. For $c \geq 7$, consider the graph G of order c obtained from the 6-cycle $(v_1, v_2, \dots, v_6, v_1)$ by adding $c-6$ new vertices and joining each of these new vertices to the three vertices v_1, v_3, and v_5. One can then verify that $f_c(G) = 3$ and $g_c(G) = 4$.

Next suppose that $b \geq 5$. By Proposition 3.1, we may assume that $c \in \{b+1, b+2\}$. If $c = b+1$, then let G be the graph obtained from the 5-cycle $(v_1, v_2, \dots, v_5, v_1)$ by adding $b-4$ pendant edges at v_1. If $c = b+2$, then let G be the connected graph with $V(G) = V_1 \cup V_2 \cup \cdots \cup V_5$, where $|V_1| = |V_3| = 1$, $|V_2| = |V_4| = 2$, and $|V_5| = b-4$, such that two vertices $x \in V_i$ and $y \in V_j$ $(1 \leq i, j \leq 5)$ are adjacent if and only if either $i = j = 4$ or $|i - j| = 1$. Then in each case, $f_c(G) = 3$ while $g_c(G) = b$. ■

Hence, it remains to investigate those triples (a, b, c), where $4 \leq a \leq b < c$. For $a = 4$, we have the following.

Proposition 3.9 *Let b and c be integers with $4 \leq b < c$. The triple $(4, b, c)$ is realizable if one of the following occurs:* (a) $b = 4$, (b) $b \geq 9$ *and* $c = b + 1$, (c) $b \geq 5$ *and* $c \geq b + 2$.

Proof. For (a), observe that C_5 has the desired property for the triple $(4, 4, 5)$ while $K_{3,c-3}$ shows that $(4, 4, c)$ is realizable for $c \geq 6$.

For (b), let G be the connected graph of order $b + 1$ obtained from the 5-cycle $(v_1, v_2, \ldots, v_5, v_1)$ by adding $b - 4$ new vertices $u_1, u_2, \ldots, u_{b-4}$ and joining (i) u_i to both v_i and v_{i+1} for $1 \leq i \leq 4$ and (ii) u_i to both v_1 and v_5 for $5 \leq i \leq b - 4$.

Finally, we consider (c). If $b \geq 6$ and $c \geq b + 4$, then the result holds by Proposition 3.1. If $b \geq 5$ and $c = b + 2$, then let G be the connected graph of order $b + 2$ obtained from the 7-path $(v_1, v_2, \ldots, v_7)$ by (i) joining each of v_1 and v_7 to v_4 and (ii) adding $b - 5$ pendant edges at v_4 (if $b \geq 6$). Similarly, if $b \geq 6$ and $c = b + 3$, then let G be the connected graph of order $b + 3$ obtained from the 9-path $(v_1, v_2, \ldots, v_9)$ by (i) joining each of v_1 and v_9 to v_4 and (ii) adding $b - 6$ pendant edges at v_4 (if $b \geq 7$).

Thus it remains to consider the triples $(4, 5, c)$ where $c \geq 8$. For the triple $(4, 5, 8)$, let G be the connected graph of order 8 obtained from two disjoint copies $(v_1, v_2, v_3, v_4, v_1)$ and $(u_1, u_2, u_3, u_4, u_1)$ of C_4 by joining each of v_1 and v_2 to u_i for $1 \leq i \leq 4$. For $c \geq 9$, let G be the connected graph of order c obtained from the 4-cycle $(v_1, v_2, v_3, v_4, v_1)$ and $H \cong K_{c-4}$ with $V(H) = \{u_1, u_2, \ldots, u_{c-4}\}$ by joining (i) each of v_1 and v_3 to both u_1 and u_2 and (ii) each of v_2 and v_4 to the $c - 7$ vertices $u_3, u_4, \ldots, u_{c-5}$.

In each case, one can verify that G shows the realizability of the corresponding triple. ■

By inspecting all connected graphs of order 6 (see [6] pp.218–224), we see that neither $(4, 5, 6)$ nor $(5, 5, 6)$ is realizable. Thus, we have only three triples $(4, b, c)$ whose realizability remain unknown, namely $(4, b, b + 1)$ for $6 \leq b \leq 8$. In fact, for each integer $a \geq 4$, the triple $(a, b, b + 1)$ is realizable with at most finitely many exceptions.

Proposition 3.10 *The triple $(a, b, b + 1)$ of positive integers is realizable if $b \geq 2a + 1$.*

Proof. Since the result has been verified for $1 \leq a \leq 4$, assume that $a \geq 5$. Let G be the connected graph of order $b + 1$ obtained from the $(a + 1)$-cycle $(v_1, v_2, \ldots, v_{a+1}, v_1)$ by adding $b - a$ new vertices $u_1, u_2, \ldots, u_{b-a}$ and joining (i) u_i to both v_i and v_{i+1} for $1 \leq i \leq a$ and (ii) u_i to both v_1 and v_{a+1} for $a + 1 \leq i \leq b - a$. Then $f_c(G) = a$ while $g_c(G) = b$. ■

In general, it is unknown which triples $(a, b, b+1)$ with $a \leq b \leq 2a$ are realizable for $a \geq 4$. However, we are able to show that each triple $(a, a, a+1)$ is not realizable if $a \geq 5$. In order to do this, we first present an additional result. Let $\kappa(G)$ denote the *connectivity* of G.

Lemma 3.11 *If G is a nontrivial connected graph that is not complete, then $g_c(G) \leq |V(G)| - \kappa(G) + 1$.*

Proof. Since the result is immediate if G contains cut-vertices, assume that $\kappa = \kappa(G) \geq 2$. Let u and v be nonadjacent vertices in G. Thus by Whitney's Theorem, there are at least κ internally disjoint $u - v$ paths each of which has length at least 2. Let $P_1, P_2, \ldots, P_\kappa$ be internally disjoint $u - v$ paths such that $\sum_{i=1}^{\kappa} |V(P_i)|$ is minimum. Then for $1 \leq i \leq \kappa$, if the first three vertices in P_i are u, x_i, y_i, then $v \neq x_i$ and $uy_i \notin E(G)$. Thus $V(G) - \{x_1, x_2, \ldots, x_{\kappa-1}\}$ is a connected geodetic set of G. ■

Theorem 3.12 *If G is a connected graph and $g_c(G) = |V(G)| - 1 \geq 5$, then $f_c(G) \leq g_c(G) - 1$. Furthermore, if $\kappa(G) = 1$, then $f_c(G) \leq g_c(G) - 2$.*

Proof. By Lemma 1.1(a), if G is a nontrivial connected graph containing ℓ_1 cut-vertices and ℓ_2 simplicial vertices, then $f_c(G) \leq g_c(G) - (\ell_1 + \ell_2)$. Therefore, it suffices to show that if G is a connected graph with $g_c(G) = |V(G)| - 1 \geq 5$, then G contains a simplicial vertex. By Lemma 3.11, we may assume that $\kappa(G) \in \{1, 2\}$.

If $\kappa(G) = 1$, then let x be a cut-vertex and assume that deleting x from G results in k components $G_1, G_2, \ldots, G_k$. Let $S = V(G) - \{v\}$ be a minimum connected geodetic set. Without loss of generality, we may assume that $v \in V(G_1)$. Note that each of $G_2, G_3, \ldots, G_k$ must contain vertices that are not cut-vertices. Furthermore, each of these vertices must be a simplicial vertex since no proper subset of S is a geodetic set. Thus $f_c(G) \leq |V(G_1)| - 1 \leq |V(G)| - k - 1 = g_c(G) - k \leq g_c(G) - 2$.

We next assume that $\kappa(G) = 2$. Thus $\delta(G) \geq 2$. If there exists a 2-subset $\{x_1, x_2\}$ of nonadjacent vertices that is not a vertex cut of G, then at least one of x_1 and x_2 must be a simplicial vertex since $V(G) - \{x_1, x_2\}$ is not a geodetic set. Therefore, assume that if $\{x_1, x_2\}$ is a 2-subset of $V(G)$ that is not a vertex cut, then $x_1x_2 \in E(G)$. Since $g_c(G) = |V(G)| - 1 \geq 5$, we may assume that $\Delta(G) \geq 3$. Let v_0 be a vertex whose degree equals $\Delta(G)$. Then there is a spanning tree T of G such that $\deg_T v_0 = \Delta(T) = \Delta(G)$. Let $N_G(v_0) = N_T(v_0) = \{v_1, v_2, \ldots, v_\Delta\}$, where $\Delta = \Delta(G) \geq 3$. Furthermore, let $u_1, u_2, \ldots, u_\Delta$ be end-vertices in T such that v_i lies on the unique $v_0 - u_i$ path in T for $1 \leq i \leq \Delta$. Since no 2-subset of $U = \{u_1, u_2, \ldots, u_\Delta\}$ is a vertex cut of G, it follows that $G[U]$ is complete by the assumption. This in turn implies that no two vertices in $N_G(v_0)$ form a vertex cut of G and so every two vertices in $N_G(v_0)$ are adjacent in G again by the assumption. Therefore, v_0 is a simplicial vertex in G. ■

Corollary 3.13 *The triple $(a, a, a+1)$ is realizable if and only if $a \in \{3, 4\}$.*

For $a \geq 5$, we obtain the following by modifying the graphs constructed in the proof of Theorem 2.2.

Proposition 3.14 *The triple (a, b, c) of integers is realizable if $5 \leq a \leq b$ and $c \geq \lceil a/2 \rceil + b$.*

Proof. Let $H_1 \cong \lfloor a/2 \rfloor P_3$ if a is odd and $H_1 \cong (a/2 - 2)P_3 + P_4$ if a is even. (Hence H_1 is a disconnected linear forest.) Obtain G from H_1 by (i) adding $H_2 \cong K_{c-\lceil a/2 \rceil - b+2}$ and joining each vertex in H_2 to every end-vertex in H_1 and (ii) adding $H_3 \cong \overline{K}_{b-a}$ and joining each vertex in H_3 to every vertex in H_2 (if $a < b$). Then G is a connected graph of order c with $f_c(G) = a$ and $g_c(G) = b$. ■

It appears that characterizing realizable triples is much more challenging than characterizing realizable pairs done in Theorems 1.2 and 2.2. We conclude this section by summarizing the results obtained thus far on realizable triples.

Theorem 3.15 *Let a, b, and c be nonnegative integers. If (a, b, c) is realizable, then $0 \leq a \leq b \leq c$ and $b \geq 2$. Furthermore,*

- *(a, b, b) is realizable if and only if $a = 0$.*
- *$(0, b, c)$ is realizable if and only if $b = c$ or $4 \leq b < c$.*
- *$(1, b, c)$ is realizable if and only if $3 \leq b < c$.*
- *$(2, b, c)$ is realizable if and only if $3 \leq b < c$ and $c \neq 4$.*
- *$(3, b, c)$ is realizable if and only if* (a) *$4 \leq b < c$ or* (b) *$b = 3$ and c is even.*
- *$(4, b, c)$ is realizable if one of the following occurs:* (a) *$4 = b < c$,* (b) *$b \geq 9$ and $c = b + 1$,* (c) *$b \geq 5$ and $c \geq b + 2$. Also, $(4, 5, 6)$ is not realizable.*

- $(a, a, a+1)$ *is realizable if and only if* $a \in \{3, 4\}$.
- $(a, b, b+1)$ *is realizable if* $a \geq 1$ *and* $b \geq 2a + 1$.
- (a, b, c) *is realizable if* $a \geq 1$, $b \geq a + 2$, *and* $c \geq a + b$.
- (a, b, c) *is realizable if* $a \geq 5$ *and* $c \geq \lceil a/2 \rceil + b$.

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