Total-Kernel in Oriented Circular Ladder and Mobius Ladder

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Abstract. A kernel in a directed graph $D(V, E)$ is a set S of vertices of D such that no two vertices in S are adjacent and for every vertex u in $V \setminus S$ there is a vertex v in S, such that (u, v) is an arc of D. The definition of kernel implies that the vertices in the kernel form an independent set. If the vertices of the kernel induce an independent set of edges we obtain a variation of the definition of the kernel, namely a total-kernel. The problem of existence of a kernel is itself a $NP-$complete problem for a general digraph. But in this paper, we solve the strong total-kernel problem of an oriented Circular Ladder and Mobius Ladder.in polynomial time.

Keywords: oriented graph, kernel, strong kernel number, NP-complete, strong orientation

1 Introduction

The concept of kernel is widespread and appears in diverse fields such as logic, computational complexity, artificial intelligence, graph theory, game theory, combinatorics and coding theory [3, 4]. Efficient routing among a set of mobile hosts is one of the most important functions in ad hoc wireless networks. Dominating-set-based routing to networks with unidirectional links is proposed in [1, 9]. A few years ago a new interest for these studies arose due to their applications in finite model theory. Indeed variants of kernel are the best properties to provide counter examples of $0-1$ laws in fragments of monadic second order logic [8].

A *kernel* [6] in a directed graph $D(V, E)$ is a set S of vertices of D such that no two vertices in S are adjacent and for every vertex u in $V \setminus S$ there is a vertex v in S, such that (u, v) is an arc of D. The minimum cardinality of all possible kernels in a directed graph D is denoted by $\kappa(D)$ and is called the kernel number. Whereas an independent dominating set in an undirected graph $G(V, E)$ is a set S of vertices of G such that no two vertices of

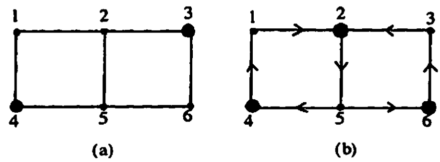


Figure 1: (a): $\gamma_i = 2$;(b): Kernel number =3

S are adjacent in G and for every vertex u in $V \setminus S$, there is a vertex v in S, such that there exist an edge between u and v in G. An independent domination number is the minimum cardinality of all independent dominating sets of G and is denoted by $\gamma_i(G)$.The concept of kernels in digraphs was introduced in different ways [10, 15]. Von Neumann and Morgenstern [15] were the first to introduce kernels when describing winning positions in 2 person games. They proved that any directed acyclic graph has a unique kernel. Not every digraph has a kernel and if a digraph has a kernel, this kernel is not necessarily unique. All odd length directed cycles and most tournaments have no kernels [3, 4]. If D is finite, the decision problem of the existence of a kernel is NP-complete for a general digraph [5, 14], and for a planar digraph with indegrees ≤ 2, outdegrees ≤ 2 and degrees ≤ 3 [7]. It is further known that a finite digraph all of whose cycles have even length has a kernel [12], and that the question of the number of kernels is NP-complete even for this restricted class of digraphs [13].

In this paper we view the kernel problem from a different perspective. In the literature, only the existence of kernel of a digraph G and its applications are extensively studied. Our aim in this paper is to investigate all strong orientations of a graph G and to determine the strong kernel number of G. This number is different from the independent domination number γ_i for undirected graphs where γ_i is the cardinality of a minimum independent dominating set [2]. For the graph in Figure 1 (a), $\Gamma = \{3, 4\}$ is an independent dominating set. Thus $\gamma_i = 2$ where as it is easy to verify that the kernel number is 3.

An orientation of an undirected graph G is an assignment of exactly one direction to each of the edges of G. There are $2^{|E|}$ orientations for G. An orientation O of an undirected graph G is said to be *strong* if for any two vertices x, y of $G(O)$, there are both (x, y)-path and (y, x)-path in $G(O)$ [16].

Let G be an undirected graph. Let $O_x(G)$ denote all possible orientations of a graph G and $O_s(G)$ denote the set of all strong orientations of G. For an orientation $O \in O_x$, let $G(O)$ denote the directed graph with orientation O and whose underlying graph is G. The kernel number of $G(O)$ is denoted by $\kappa(G(O))$.For convenience we write as $\kappa(O)$. We define the kernel number of G as follows.The kernel number of G is defined as $\kappa_x(G) = \min\{\kappa(O) : O \in O_x(G)\}$. Similarly we define the strong kernel number of G as $\kappa_s(G) = \min\{\kappa(O) : O \in O_s(G)\}$. When there is no ambiguity we refer to $\kappa_s(G)$ as κ_s.

The *strong kernel problem* of an undirected graph G is to find a kernel K of $G(O)$ for some strong orientation O of G such that $|K| = \kappa_s$. An optimal lower bound for $\kappa_s(G)$ when G is a regular graph has been obtained in [11].

2 Total-Kernel in Oriented Circular Ladder and Mobius Ladder

The definition of kernel implies that the vertices in the kernel form an independent set. If the vertices of the kernel induce an independent set of edges we obtain a variation of the definition of the kernel, namely a total-kernel. Here, we prove that the strong total-kernel problem is polynomially solvable for Circular Ladder and Mobius Ladder.

3 Lower Bound for Strong Total-Kernel Number for r-regular Graphs

A *total-kernel* of a digraph $D(V, E)$ is a non-empty subset K of V such that K induces an independent set of edges and for every vertex u in $V \setminus K$, there is a vertex v in K such that (u, v) is an arc of D. See Figure 2. The minimum cardinality of all possible total-kernels in a directed graph D is denoted by $\xi(D)$ and is called the *total-kernel number*.

Let G be an undirected graph. For an orientation $O \in O_x$, let the total-kernel number of $G(O)$ be denoted by $\xi(G(O))$. For convenience we write $\xi(G(O))$ as $\xi(O)$. We define the total-kernel number ξ_x of G as $\xi_x(G) = \min\{\xi(O) : O \in O_x(G)\}$. Similarly we define the strong total-kernel number of G as $\xi_s(G) = \min\{\xi(O) : O \in O_s(G)\}$. When there is no ambiguity we refer to $\xi_s(G)$ as ξ_s. A strong total-kernel problem of an undirected graph G is to find a total-kernel K of $G(O)$ for some strong orientation O of G such that $|K| = \xi_s$.

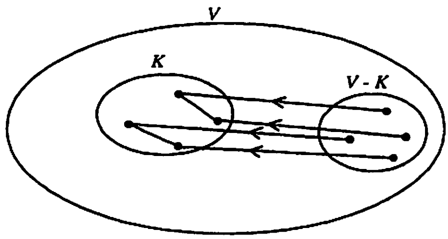


Figure 2: Diagramatic representation of a total-kernel

Notation 1 *For $S \subseteq V$, let $N^+[S]$ denote the set of all vertices in S together with vertices on the outgoing edges incident with vertices of S. Similarly let $N^-[S]$ denote the set of all vertices in S together with vertices on the incoming edges incident with vertices of S.*

A major break through in the study of total-kernel of graphs is the following theorem which derives a lower bound for the strong total-kernel number for r-regular graphs.

Theorem 2 *Let G be an r-regular graph on n vertices. Then $\xi_s \geq \lceil n/(2r-1) \rceil$.*

Proof. Let $O \in O_s$ and K be a total-kernel of $G(O)$. By definition, the vertices of K induce an independent set of edges in $G[K]$. Consider an edge $\overrightarrow{e} = (\overrightarrow{u,v})$ in this set. Since $G(O)$ is strongly connected, there are at most $r-1$ incoming edges and at least one outgoing edge at every vertex of $G(O)$. But an outgoing edge at u may be an incoming edge at v. See Figure 3. Hence $|N^-[\{u,v\}]| \leq 2r-1$. As $|V| = n$, $\xi_s \geq \lceil n/(2r-1) \rceil$. ■

4 Strong Total-Kernel Problem in Oriented Circular Ladder

Definition 1 *A circular ladder $CL(n)$ is the union of an outer cycle Γ_O : $u_1u_2...u_nu_1$ and an inner cycle Γ_I : $v_1v_2...v_nv_1$ with additional edges u_iv_i, $i = 1, 2, ..., n$ called spokes. For convenience $u_1, u_2, ..., u_n$ are represented by $1, 2, \ldots, n$ and $v_1, v_2, ..., v_n$ by $n+1, n+2, \ldots, 2n$ respectively. For $1 \leq i, j \leq n$, we call the oriented spoke $\overrightarrow{(i, n+i)}$, an inward spoke and the oriented spoke $\overrightarrow{(n+j, j)}$ an outward spoke. See Figure 4.*

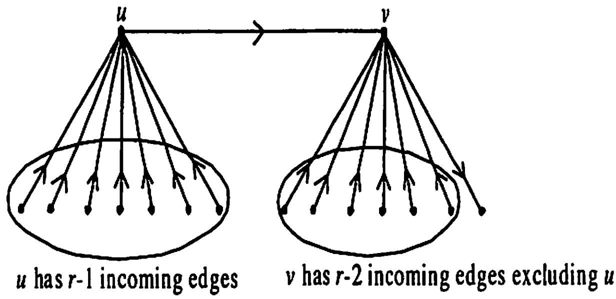


Figure 3: $|N^{-}[\{u, v\}]| = 2r - 1$

Lemma 1 *Let G be the circular ladder $CL(n), n \geq 4$ with an inward spoke and an outward spoke. Let the outer cycle Γ_O and the inner cycle Γ_I be oriented in the clockwise and anticlockwise direction respectively. All other spokes are oriented arbitrarily. Then G is strongly connected.*

Proof. Let $\vec{e_1} = (\overrightarrow{i, n+i})$ and $\vec{e_2} = (\overrightarrow{n+j, j})$ for some $i, j, 1 \leq i, j \leq n$ be an inward spoke and an outward spoke respectively. For $u, v \in V$, we claim that there exist directed paths from u to v and from v to u. Suppose both u, v lie on Γ_O or Γ_I, then our claim is true since Γ_O is oriented in the clockwise direction and Γ_I is oriented in the anticlockwise direction.

Suppose u lies on Γ_O and v lies on Γ_I. See Figure 5. The directed (u, i)-path on Γ_O in the clockwise direction followed by $\vec{e_1}$, followed by the directed $(n+i, v)$-path on Γ_I in the anticlockwise direction is a path from u to v. In the same way we trace out a directed path from v to u. The directed $(v, n+j)$-path on Γ_I in the anticlockwise direction followed by $\vec{e_2}$, followed by the directed (j, u)-path on Γ_O in the clockwise direction is a path from v to u. Thus G is strongly connected. ■

Lemma 2 *For $m \not\equiv 0 \pmod 5$, let t be the least positive integer such that $5|(m-4t)$. Then $t + \frac{(m-4t)}{5} = \lceil m/5 \rceil$.*

Proof. Let $m = 5k + s, 1 \leq s \leq 4$. Then $\lceil m/5 \rceil = k + 1$.

It is enough to prove that $t + \frac{(m-4t)}{5} = k + 1$.

Case 1 ($m \equiv 1 \pmod 5$):

Let $m = 5k+1$ for some integer k. Now $5|(m-4t)$ implies $5|(1-4t)$ which inturn implies $t = 4$. Thus $t + \frac{(m-4t)}{5} = 4 + \frac{(5k-15)}{5} = 4 + (k-3) = k + 1$.

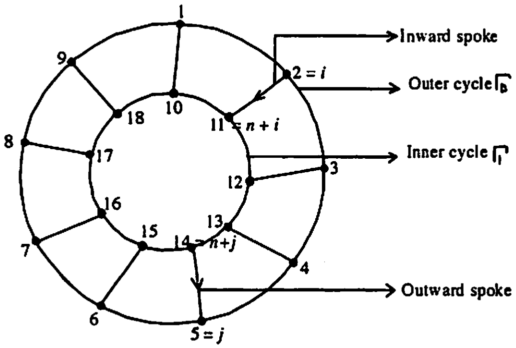


Figure 4: $CL(9)$, $(\overrightarrow{2,11})$ and $(\overrightarrow{14,5})$ represent inward spoke and outward spoke respectively.

Case 2 ($m \equiv 2 \pmod 5$)**:**

Let $m = 5k+2$ for some integer k. Now $5|(m-4t)$ implies $5|(2-4t)$ which inturn implies $t = 3$. Thus $t + \frac{(m-4t)}{5} = 3 + \frac{(5k-10)}{5} = 3 + (k-2) = k+1$.

Case 3 ($m \equiv 3 \pmod 5$)**:**

Let $m = 5k+3$ for some integer k. Now $5|(m-4t)$ implies $5|(3-4t)$ which inturn implies $t = 2$. Thus $t+\frac{(m-4t)}{5} = 2+\frac{(5k-5)}{5} = 2+(k-1) = k+1$.

Case 4 ($m \equiv 4 \pmod 5$)**:**

Let $m = 5k+4$ for some integer k. Now $5|(m-4t)$ implies $5|(4-4t)$ which inturn implies $t = 1$. Thus $t + \frac{(m-4t)}{5} = 1 + \frac{5k}{5} = k+1$.

Thus the lemma is true in all cases. ∎

Theorem 3 *Let G be $CL(n), n \geq 4$. Then $\xi_s = \lceil 2n/5 \rceil$.*

Proof. Orient G as in Lemma 1. Since G is strongly connected, there is at least one incoming edge and at least one outgoing edge at every vertex of G. We note that two edges $\overrightarrow{e_1} = (\overrightarrow{p,q})$, $\overrightarrow{e_2} = (\overrightarrow{r,s})$ of G are adjacent iff

(i) $|p-r| = 1$ and $|q-s| = 1, 1 \leq p,q,r,s \leq n$ or $n+1 \leq p,q,r,s \leq 2n$. See Figure 6.

(ii)$|p-r| = 1$ and $|q-s| = n, 1 \leq p,q,r \leq n$ and $n+1 \leq s \leq 2n$. See Figure 7 (a).

(iii) $|p-r| = 0$ and $|q-s| = n-1, 1 \leq p,q,r \leq n$ and $n+1 \leq s \leq 2n$. See Figure 7 (b).

(iv). $|p-r| = n-1$ and $|q-s| = 0, 1 \leq r \leq n$ and $n+1 \leq p,q,s \leq 2n$. See Figure 8 (a).

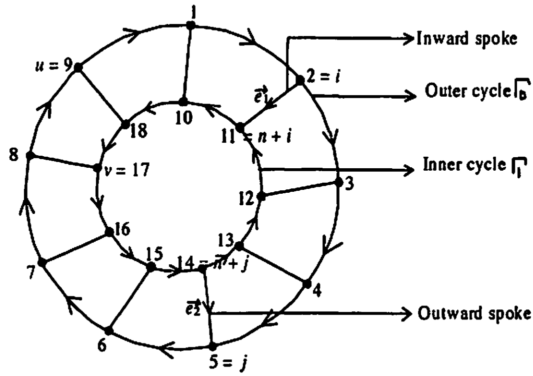


Figure 5: Oriented Circular Ladder $CL(9)$

(v). $|p-r| = n$ and $|q-s| = 1, 1 \le r \le n$ and $n+1 \le p, q, s \le 2n$. See Figure 8 (b).

(vi) $|p-r| = n+1$ and $|q-s| = 0, 1 \le p, q, s \le n$ and $n+1 \le r \le 2n$.

(vii) $|p-r| = 1$ and $|q-s| = 1, 1 \le p, q, s \le n$ or $n+1 \le r \le 2n$.

(viii) $|p-r| = 1$ and $|q-s| = n, n+1 \le p, q, r \le 2n$ and $1 \le s \le n$.

(ix) $|p-r| = 0$ and $|q-s| = n+1, 1 \le s \le n$ and $n+1 \le p, q, r \le 2n$.

We consider two cases.

Case 1 ($2n \equiv 0 \pmod 5$):

Let $K_i = \{5i-4, 5i-3, n+5i-2, n+5i-1\}, 1 \le i \le n/5$. Clearly $\overrightarrow{e_i} = (5i-4, 5i-3)$ is an edge on Γ_O and $\overrightarrow{f_i} = (n+5i-2, n+5i-1)$ is an edge on Γ_I. Since $|(n+5i-2)-(5i-4)| = n+2$ and $|(n+5i-1)-(5i-3)| = n+2$, $\overrightarrow{e_i}$ and $\overrightarrow{f_i}$ are non-adjacent.

Now let $K = \cup K_i, 1 \le i \le n/5$. We claim that K induces an independent set of edges. Consider $\overrightarrow{e_i}, \overrightarrow{e_j} \in K$. Then $\overrightarrow{e_i} = (5i-4, 5i-3)$ and $\overrightarrow{e_j} = (5j-4, 5j-3)$. Clearly $|(5j-4)-(5i-4)| = |5(j-i)| = 5\,|j-i| \ne 1$. Similarly $|(5j-3)-(5i-3)| = |5(j-i)| = 5\,|j-i| \ne 1$. Therefore $\overrightarrow{e_i}$ and $\overrightarrow{e_j}$ are not adjacent.

Consider $\overrightarrow{f_i}$ and $\overrightarrow{f_j} \in K$. Then $\overrightarrow{f_i} = (n+5i-2, n+5i-1)$ and $\overrightarrow{f_j} = (n+5j-2, n+5j-1)$. Since $|(n+5j-2)-(n+5i-2)| = 5\,|j-i| \ne 1$ and $|(n+5j-1)-(n+5i-1)| = 5\,|j-i| \ne 1$, $\overrightarrow{f_i}$ and $\overrightarrow{f_j}$ are non-adjacent.

Consider $\overrightarrow{e_i}$ and $\overrightarrow{f_j} \in K$. Then $\overrightarrow{e_i} = (5i-4, 5i-3)$ and $\overrightarrow{f_j} = (n+5j-2, n+5j-1)$. Since $|(n+5j-2)-(5i-4)| = |n+5(j-i)+2| \ne n+1$ and $|(n+5j-1)-(5i-3)| = |n+5(j-i)+2| \ne n+1$, $\overrightarrow{e_i}$ and

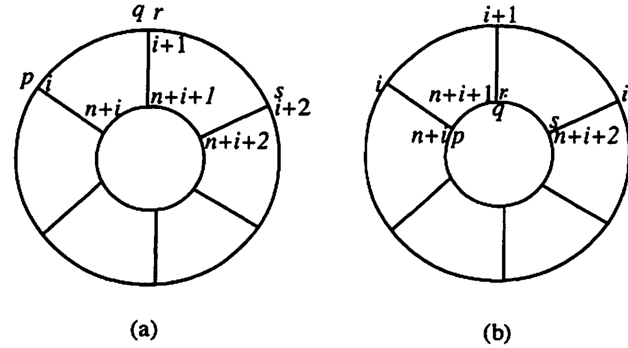


Figure 6: $(\overrightarrow{p,q}) = (\overrightarrow{i,i+1}), (\overrightarrow{r,s}) = (\overrightarrow{i+1,i+2})$. For both cases (a) and (b), $|p-r| = 1$ and $|q-s| = 1$

$\overrightarrow{f_j}$ are non-adjacent. In the same way, consider $\overrightarrow{f_i}$ and $\overrightarrow{e_j} \in K$. Then $\overrightarrow{f_i} = (n+5i-2, n+5i-1)$ and $\overrightarrow{e_j} = (5j-4, 5j-3)$. Clearly $\overrightarrow{f_i}$ and $\overrightarrow{e_j}$ are not adjacent, since $|(5j-4)-(n+5i-2)| = |n-5(j-i)+2| \neq n+1; |(5j-3)-(n+5i-1)| = |n-5(j-i)+2| \neq n+1$. Hence $\overrightarrow{f_i}$ and $\overrightarrow{e_j}$ are non-adjacent. This proves our claim.

The orientation of G yields one incoming and one outgoing arc at every vertex of K_i. As G is 3 regular, orient the third edge incident at every vertex of K_i as an incoming arc. See Figure 9 (a). Since $2n \equiv 0$ mod 5, there exist an integer k such that $k = 2n/5$. Since K induces an independent set of edges $|K| = 2n/5$.

Case 2 ($2n \not\equiv 0$ **(mod** 5)):

By lemma 1 we find the least positive integer t such that $5|(2n-4t)$. Let $(2n-4t)/5 = r$.

The set $K' = \cup K_i$ where $K_i = \{2i-1, n+2i-1\}, 1 \leq i \leq t$ induce non-adjacent edges in G as the edges are spokes in G.

We claim that $K_j = \{2t+5j-4, 2t+5j-3, n+2t+5j-2, n+2t+5j-1\}, 1 \leq j \leq r/2$ induce two non-adjacent edges in G. Clearly $\overrightarrow{e_j} = (2t+5j-4, 2t+5j-3)$ and $\overrightarrow{f_j} = (n+2t+5j-2, n+2t+5j-1), 1 \leq j \leq r/2$ are edges in the outer cycle Γ_O and in the inner cycle Γ_I respectively. Since $|(n+2t+5j-2)-(2t+5j-4)| = n+2$ and $|(n+2t+5j-1)-(2t+5j-3)| = n+2$, $\overrightarrow{e_i}$ and $\overrightarrow{f_j}$ are non-adjacent.

Now let $K" = \cup K_j, 1 \leq j \leq r/2$. We claim that $K"$ induces an inde-

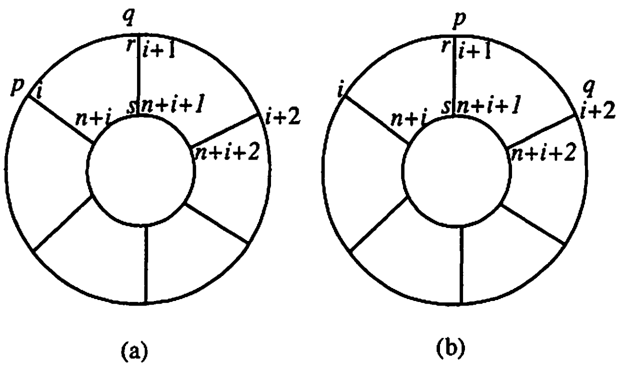


Figure 7: (a):$(\overrightarrow{p,q}) = (\overrightarrow{i, i+1}), (\overrightarrow{r,s}) = (\overrightarrow{i+1, n+i+1})$ where $|p-r| = 1, |q-s| = n$; (b): $(\overrightarrow{p,q}) = (\overrightarrow{i+1, i+2}), (\overrightarrow{r,s}) = (\overrightarrow{i+1, n+i+1})$ where $|p-r| = 0$ and $|q-s| = n-1$

pendent set of edges. Let $\overrightarrow{e} \in K_j$ and $\overrightarrow{f} \in K_k$, without loss of generality let $j < k$.

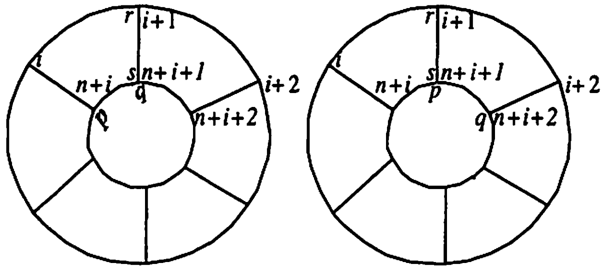
Suppose $\overrightarrow{e}$ and $\overrightarrow{f}$ are on the outer cycle Γ_O. Then $\overrightarrow{e} = (2t+5j-4, 2t+5j-3)$ and $\overrightarrow{f} = (2t+5k-4, 2t+5k-3)$. Clearly $|(2t+5k-4) - (2t+5j-4)| = |5(k-j)| = 5\,|k-j| \neq 1$. Similarly $|(2t+5k-3) - (2t+5j-3)| = |5(k-j)| = 5\,|k-j| \neq 1$. Therefore $\overrightarrow{e}$ and $\overrightarrow{f}$ are not adjacent.

Suppose $\overrightarrow{e}$ and $\overrightarrow{f}$ are on the inner cycle Γ_I. Then $\overrightarrow{e} = (n+2t+5j-2, n+2t+5j-1)$ and $\overrightarrow{f} = (n+2t+5k-2, n+2t+5k-1)$. So we have $|(n+2t+5k-2) - (n+2t+5j-2)| = 5\,|k-j| \neq 1$.Similarly $|(n+2t+5k-1) - (n+2t+5j-1)| = 5\,|k-j| \neq 1$. Hence $\overrightarrow{e}$ and $\overrightarrow{f}$ are non-adjacent.

Suppose $\overrightarrow{e}$ lies on the outer cycle Γ_O and $\overrightarrow{f}$ lies on the inner cycle Γ_I. Then $\overrightarrow{e} = (2t+5j-4, 2t+5j-3)$ and $\overrightarrow{f} = (n+2t+5k-2, n+2t+5k-1)$. The conditions $|(n+2t+5k-2) - (2t+5j-4)| = |n+5(k-j)+2| \neq n+1$ and $|(n+2t+5k-1) - (2t+5j-3)| = |n+5(k-j)+2| \neq n+1$, imply that $\overrightarrow{e}$ and $\overrightarrow{f}$ are non-adjacent. In the same way, if $\overrightarrow{e}$ lies on the inner cycle Γ_I and $\overrightarrow{f}$ on the outer cycle Γ_O, then $\overrightarrow{e}$ and $\overrightarrow{f}$ are non-adjacent.

Let $K = K' \cup K"$. We claim that K induces an independent set of edges. Let $\overrightarrow{e} \in K'$ and $\overrightarrow{f} \in K"$.

Without loss of generality let $\overrightarrow{e} \in K_i$ and $\overrightarrow{f} \in K_j, i < j$. Then



(a) (b)

Figure 8: (a):$(\overrightarrow{p,q}) = (\overrightarrow{n+i,n+i+1}), (\overrightarrow{r,s}) = (\overrightarrow{i+1,n+i+1})$ where $|p-r| = n-1, |q-s| = 0$; (b): $(\overrightarrow{p,q}) = (\overrightarrow{n+i+1,n+i+2}), (\overrightarrow{r,s}) = (\overrightarrow{i+1,n+i+1})$ where $|p-r| = n$, $|q-s| = 1$

$\overrightarrow{e} = (2i-1, n+2i-1)$ and $\overrightarrow{f} = (2t+5j-4, 2t+5j-3)$. Clearly $|(2t+5j-4)-(2i-1)| = |2t+5j-2i-3| \neq 0$.

Similarly $|(2t+5j-3)-(n+2i-1)| = |2t+5j-n-2i-2| \neq n-1$. Therefore $\overrightarrow{e}$ and $\overrightarrow{f}$ are not adjacent.

Suppose $\overrightarrow{e} = (2i-1, n+2i-1)$ and $\overrightarrow{f} = (n+2t+5j-2, n+2t+5j-1)$. Clearly $|(n+2t+5j-2)-(2i-1)| = |n+2t+5j-2i-1| \neq n-1$. Similarly $|(n+2t+5j-1)-(n+2i-1)| = |2t+5j-2i| \neq 0$. Therefore $\overrightarrow{e}$ and $\overrightarrow{f}$ are not adjacent.

The orientation of G yields one incoming and one outgoing arc at every vertex of K_i and K_j. As G is 3 regular, orient the third edge incident at every vertex of K_j as an incoming arc. See Figure 9 (b). K induces an independent set of edges and $|K| = \lceil 2n/5 \rceil$, since $2n \not\equiv 0 \bmod 5$. ■

5 Strong Total - Kernel Problem in Oriented Mobius Ladder

Definition 2 *The Mobius Ladder M_n is the graph obtained from the ladder $P_n \times P_2$ by joining the opposite end points of the two copies of P_n. Labeling the vertices of one copy of P_n as $1, 2, ..., n$ and the other copy of P_n as $n+1, n+2, ..., 2n$. See Figure 10.*

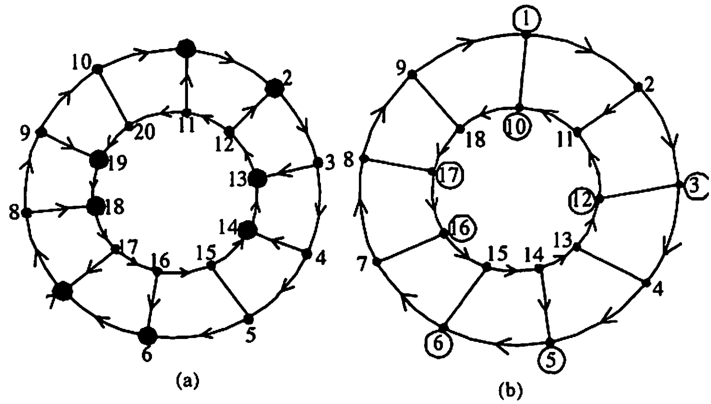


Figure 9: Encircled vertices induce an independent set of edges, which form a total - kernel in $CL(10)$ and $CL(9)$

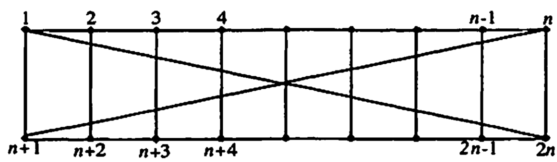


Figure 10: Labeling of M_n

Theorem 4 *Let G be the Mobius Ladder M_n and Orient the cycle* $1, 2, 3, ...,$ $n-1, n, 2n, 2n-1, ..., n+2, n+1, 1$ *in the clockwise direction. Then O is a strong orientation of G.*

Proof. For $u, v \in V(G), 1 \leq u, v \leq 2n$. Cycle $1, 2, ..., n-1, n, 2n, 2n-1, ..., n+2, n+1, 1$ is oriented in clockwise direction. Hence there exist a directed path from u to v and v to u. Thus G is strongly connected. ■

In the case of mobius ladder, we have the similar results to that of circular ladder on strong total-kernel number. The result is true for $n \geq 9$.Therefore we manually verify the results for M_n, $n < 9$.

For $n = 4, \xi_s = 2$. See Figure 11.

Theorem 5 *Let G be* $M_n, n \geq 4$. *Then* $\xi_s = \lceil 2n/5 \rceil$.

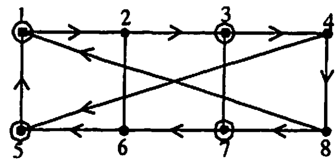


Figure 11: $\xi_s = 2$ in M_4

Theorem 6 *The total-kernel problem is polynomially solvable for oriented circular ladder and mobius ladder.*

6 Conclusion

In this paper we have determined the lower bound for the strong total-kernel number for regular graphs and also provided the strong orientation of circular ladder and mobius ladder. We have also determined their strong total-kernel number and proved that the strong kernel problem is polynomially solvable. It would be interesting to consider some more regular graphs.

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