Rooted HIST property on planar triangulations

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Abstract

A spanning tree with no vertices of degree two of a graph is called a *homeomorphically irreducible spanning tree* (or *HIST*) of the graph. It has been proved that every planar triangulation G with at least four vertices has a HIST H [1]. However, the previous result asserts nothing whether the degree of a fixed vertex v of G is at least three or not in H. In this paper, we prove that if a planar triangulation G has $2n$ ($n \geq 2$) vertices, then, for any vertex v, G has a HIST H such that the degree of v is at least three in H. We call such a spanning tree a *rooted HIST of G with root v*.

1 Introduction

Let G be a graph and let H be a subgraph of G. If H contains all vertices of G, then it is called a *spanning subgraph* of G. If a spanning subgraph H of G is a tree, then it is called a *spanning tree* of G. It is a fundamental problem deciding whether a graph has particular types of spanning subgraph in graph theory. For example, in the Hamiltonian path problem, we seek a spanning tree with all but two vertices of degree two. In this paper, we search "homeomorphically irreducible spanning trees", a class antithetical to Hamiltonian paths.

A graph is said to be *homeomorphically irreducible* if it has no vertices of degree two. Let T be a spanning tree of a graph G. If T has no vertices of degree two, then T is called a *homeomorphically irreducible spanning tree* (or *HIST*) of G. For example, the octahedron has a HIST with two vertices

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of degree three and four vertices of degree one. (See Figure 1.) Joffe has constructed infinite families of 4-regular, 3-connected planar graphs that have no HISTs [5]. Albertson, Berman, Hutchinson, and Thomassen have shown that it is an NP-complete problem deciding whether a graph contains a HIST [1]. So, we consider the problem restricted to "triangulations".

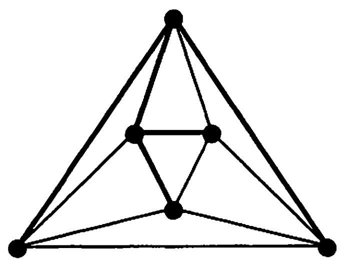

Figure 1: A HIST on the octahedral graph.

A *triangulation* on a surface is a simple graph embedded in the surface such that each face is triangular. A *near triangulation* R is a 2-connected simple graph on the plane with boundary cycle of length $k \geq 3$ such that each face of R is triangular other than the outer face. Hill conjectured that every planar triangulation other than the complete graph K_3 with three vertices has a HIST [4]. Malkevitch extended Hill's conjecture to near triangulations [6]. For their conjectures, Albertson, Berman, Hutchinson, and Thomassen have proved the following.

Theorem 1 [Albertson, Berman, Hutchinson, and Thomassen [1]] *Every near triangulation with at least four vertices has a HIST.*

Moreover, they extend Hill's conjecture to all triangulations on any surface, i.e., "*every triangulation on any surface with at least four vertices has a HIST* ". For this conjecture, Davidow, Hutchinson and Huneke have proved that every toroidal triangulation has a HIST [2]. In [3], Fiedler, Huneke, Richter and Robertson have proved that every projective planar triangulation has a near triangulation as a spanning subgraph. By their result and Theorem 1, it has already been proved that every projective planar triangulation has a HIST. However, the previous results assert nothing whether the degree of a fixed vertex of a graph is at least three or not in a HIST of the graph. So, in this paper, we consider "rooted HISTs".

Let G be a simple graph. Fix a vertex v of G such that the degree of v in G, denoted by $d_G(v)$, is at least three. If G has a HIST H such that $d_H(v) \geq 3$, then H is called a *rooted HIST of G with root v*. For any vertex v of G, where $d_G(v) \geq 3$, if G has a rooted HIST with root v, then we call that G satisfies the *rooted HIST property*.

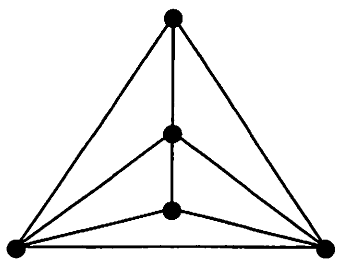

Figure 2: A triangulation on the plane not satisfying the rooted HIST property.

There exist planar triangulations not satisfying the rooted HIST property. For example, let G be a planar triangulation with five vertices. (See Figure 2). Since the complete graph K_5 with five vertices is not embeddable into the plane, G has a vertex v of degree three. It is obvious that G has no rooted HIST with root v. So, we also consider "rooted near 1-HISTs".

Let G be a simple graph. If G has a spanning tree T with at most k vertices of degree two, then T is called a *near k-HIST*. Fix a vertex v of G such that $d_G(v) \geq 2$. If G has a spanning tree H such that $d_H(v) \geq 2$ and $d_H(u) \neq 2$, where $u \in V(H) - \{v\}$, then H is called a *rooted near 1-HIST of G with root v*. For any vertex v of G, where $d_G(v) \geq 2$, if G has a rooted near 1-HIST with root v, then we call that G satisfies the *rooted near 1-HIST property*. Let $|G|$ denote the number of vertices of G. In this paper, we prove the following.

Theorem 2 *Let G be a near triangulation.*

(i) If $|G| = 2n - 1$ $(n \geq 2)$, then G satisfies the rooted near 1-HIST property.

(ii) If $|G| = 2n$, then G satisfies the rooted HIST property.

By the definitions of rooted HISTs and rooted near 1-HISTs, if a graph G has no vertices of degree at most two and satisfies the rooted HIST property, then G satisfies the rooted near 1-HIST property. Since every planar triangulation with at least four vertices has no vertex of degree at most two, Theorem 2 implies the following.

Corollary 3 *Every planar triangulation satisfies the rooted near* 1*-HIST property.*

In Section 2, we show infinite families of near triangulations not satisfying the rooted HIST property or the rooted near 1-HIST property. In Section 3, we prove Theorem 2.

2 Infinite family not satisfying rooted HIST property

In this section, we show two infinite families of near triangulations. One does not satisfy the rooted HIST property, and the other does not satisfy the rooted near 1-HIST property.

Proposition 4 *There exists an infinite family of near triangulation on odd number of vertices not satisfying the rooted HIST property.*

Proof. Let G be a near triangulation, whose all vertices are on the boundary cycle B. Suppose that G has exactly two vertices of degree two, denoted by a, b, such that the difference on the length of two paths on B from a to b is one. Let $X = ax_1x_2 \dots x_kb$ and $Y = ay_1y_2 \dots y_{k-1}b$ denote two paths on B from a to b, where $|X| \geq |Y|$. Suppose that, for each $i = 1, 2, \dots, k$, G has edges x_iy_i and x_iy_{i-1}. (See Figures 3). We prove that such G does not satisfy the rooted HIST property. Suppose that G has a rooted HIST H with root x_1. Since $d_G(x_1) = 3$, H contains all edges incident to x_1. If $d_H(y_1) \geq 3$, then H contains a cycle, and hence $d_H(y_1) = 1$ and $d_H(x_2) \geq 3$. So, H contains x_2x_3 and x_2y_2 since H does not contain x_2y_1. By the same arguments, for each i, H must contain x_ix_{i+1}, x_iy_i. Therefor, we have $d_H(y_i) = 1$. Then, for $i = k-1$, $x_{k-1}y_{k-1}$ and $x_{k-1}x_k$ must be contained in H and we have $d_H(y_{k-1}) = 1$. So x_kb must contained in H and we have $d_H(x_k) = 2$, a contradiction. ■

Proposition 5 *There exists an infinite family of near triangulation on even number of vertices not satisfying the rooted near* 1*-HIST property.*

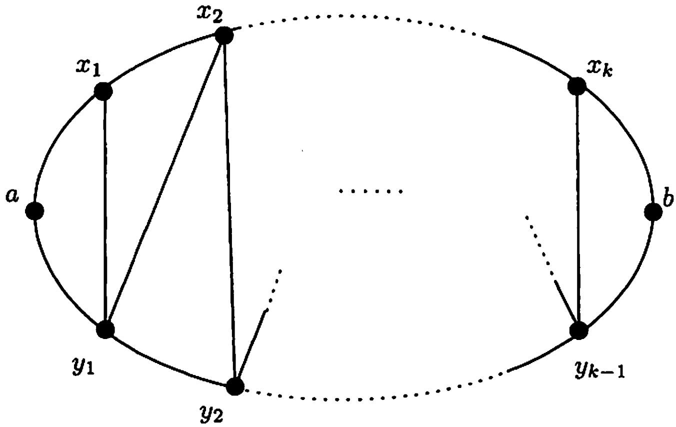

Figure 3: An infinite family not satisfying the rooted HIST property.

Proof. Let G be a near triangulation, whose all vertices are on the boundary cycle B. Suppose that G has exactly two vertices of degree two, denoted by a, b, such that the length of two paths on B from a to b is the same. Let $X = ax_1x_2 \ldots x_kb$ and $Y = ay_1y_2 \ldots y_{k-1}y_kb$ denote two paths on B from a to b. Suppose that, for each $i = 1, 2, \ldots, k$, G has edges x_iy_i and x_iy_{i-1}. (See Figures 4). We prove that such G does not satisfy the rooted HIST property. Suppose that G has a rooted near 1-HIST H with root a. Since $d_G(a) = 2$, H contains all edges incident to a. If $d_H(x_1) \geq 3$, then H contains a cycle, and hence $d_H(x_1) = 1$ and $d_H(y_1) \geq 3$. So, H contains y_1x_2 and y_1y_2 since H does not contain x_1y_1. By the same arguments, for each i, H must contain y_ix_{i+1}, y_iy_{i+1}. Therefor, we have $d_H(x_i) = 1$. Then, for $i = k-1$, $y_{k-1}x_k$ and $y_{k-1}y_k$ must be contained in H and we have $d_H(x_k) = 1$. So y_kb must contained in H and we have $d_H(y_k) = 2$, a contradiction. ■

3 Proof of the main result

Let G be a near triangulation with boundary cycle B and let $f = xyz$ be a face of G such that at least one edge of f is contained in B. Without loss of generality, we may suppose $xy \in E(B)$. If z is not contained in B, then f is called a *trivial face*. Suppose that f is not a trivial face, i.e., z is contained in B. If neither xz nor yz is contained in B, then f is called a

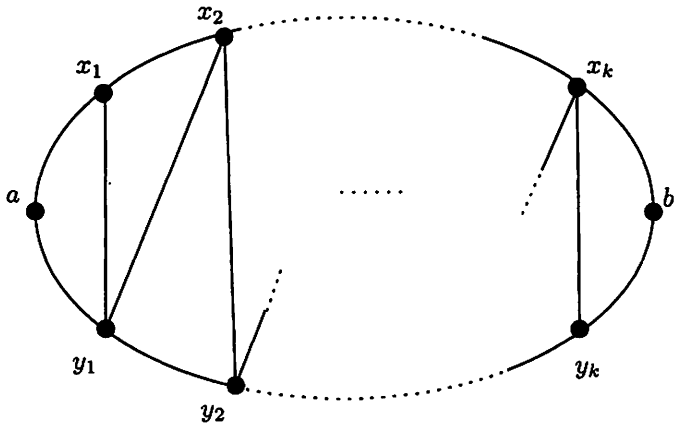

Figure 4: An infinite family not satisfying the rooted near 1-HIST property.

crossing face. Otherwise (i.e., either xz or yz is contained in B), f is called a *leaf face.* (See Figure 5.)

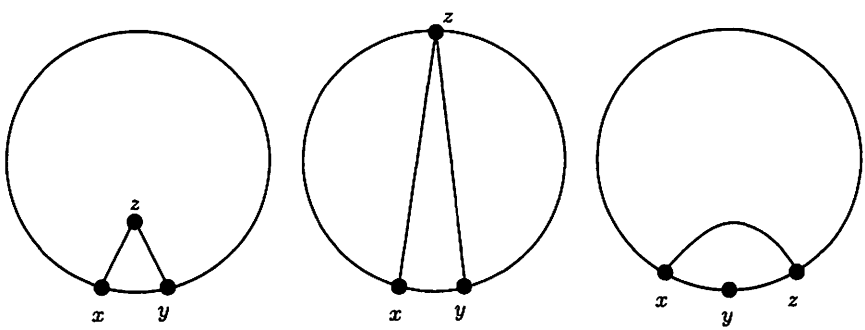

Figure 5: A trivial face (left), a crossing face (center) and a leaf face (right).

Suppose that $f = xyz$ is a crossing face of G, where $xy \in E(B)$. Let B' be a path from x to z on B containing y. Then we call the union of B' and xz a *separating cycle of f*. (Note that a crossing face has two separating cycles).

We prove Theorem 2 by induction on the number of edges of G. In order to do so, we consider three kinds of transformations, as follows. Let xyz be a trivial face, where $xy \in E(B)$. We call removing xy an *edge deletion* of xy. Let f_1 and f_2 be two *adjacent* leaf faces (i.e., f_1 and f_2 has at least one common vertex) and let x, y be vertices of f_1, f_2 with degree two, respectively. We call removing two vertices x and y a *(2,2)-deletion* of x, y. On the other hand, let $f_1 = xyz$ be a leaf face, where $d_G(y) = 2$ and let $f_2 = xy'z$ be a crossing face such that $xy' \in E(B)$. We call removing two vertices x and y a *(2,3)-deletion* of x, y. (We use a (2,2)-deletion and a (2,3)-deletion when $|G| \geq 5$). Note that these transformations do not change the parity of $|G|$. For these transformations, in [1], the following lemma has been proved. We state it adapted to our need.

Lemma 6 [Albertson, Berman, Hutchinson, and Thomassen [1]] *Let G be a near triangulation with at least five vertices. Then G has a face or faces such that we can apply an edge deletion, a (2,2)-deletion or a (2,3)-deletion.* ■

Proof of Theorem 2. First, we prove that if $|G| = 2n-1$, then G satisfies the condition (i) of the theorem. After that, we prove that if $|G| = 2n$, then G satisfies the condition (ii) of the theorem.

Case(a) $|G| = 2n - 1$.

In this case, we prove that G satisfies the rooted near 1-HIST property by induction on the number of edges of G. If $|G| = 3$ (i.e., G is a complete graph K_3), then for any vertex v of G, we can find a rooted near 1-HIST with root v, and hence G satisfies the rooted near 1-HIST property. So, we suppose $|G| \geq 5$. By Lemma 6, G has a face (or faces) such that we can apply an edge deletion, a (2,2)-deletion or a (2,3)-deletion. Let G' be a near triangulation obtained from G by one operation of an edge deletion, a (2,2)-deletion or a (2,3)-deletion. Observe that G' satisfies the rooted near 1-HIST property by the induction hypothesis. We shall prove that, for each operation, if G' satisfies the rooted near 1-HIST property, then G also satisfies the rooted near 1-HIST property.

First, we suppose that G' is obtained from G by an edge deletion. Since G' satisfies the rooted near 1-HIST property, for any vertex v of G', we can find a rooted near 1-HIST H' with root v. Moreover, each H' is also a rooted near 1-HIST of G with root v.

Secondly, we suppose that G' is obtained from G by a (2,3)-deletion of x, y. Let x^-xx^+ be a crossing face and xyx^+ be a leaf face of G such that $d_G(x) = 3$ and $d_G(y) = 2$. Since G' satisfies the rooted near 1-HIST property, for any vertex v of G', we can find a rooted near 1-HIST H' with root v. Moreover, for any vertex $v \neq x, y$ of G, H' extends to a rooted near 1-HIST H with root v by adding edges xx^+, yx^+. So, we must prove that G has a rooted near 1-HIST with root x and that with root y. Let H'' be a rooted near 1-HIST of G' with root x^+. By adding edges x^+x, xy (resp., x^+y, yx) to H'', we obtain a rooted near 1-HIST of G with root x (resp., y).

Thirdly, we suppose that G' is obtained from G by a (2,2)-deletion of x, y. Let x^-xx^+ and x^+yy^+ be two leaf faces of G such that $d_G(x) = d_G(y) = 2$. Since G' satisfies the rooted near 1-HIST property, for any vertex v of G', we can find a rooted near 1-HIST H' with root v. Moreover, for any vertex $v \neq x, y$ of G, H' extends to a rooted near 1-HIST H of G with root v by adding edges xx^+, yx^+. So, we must prove that G has a rooted near 1-HIST with root x (resp., y). By symmetry, it suffices to show that G has a rooted near 1-HIST with root x. We shall prove that we can apply an edge deletion, a (2,2)-deletion or a (2,3)-deletion to G so that the resulting near triangulation still contains the vertex x. Let y^+pq be a face of G such that y^+p is contained in the boundary cycle B of G. If y^+pq is a trivial face or a leaf face, we can apply an edge deletion or a (2,2)-deletion to G so that the resulting near triangulation still contains the vertex x. So, we may suppose that y^+pq is a crossing face.

Claim 1 *Let f be a crossing face of a near triangulation G and let C be a separating cycle of f. In the interior of C, G has a face or faces such that we can apply an edge deletion, a* (2,2)*-deletion or a* (2,3)*-deletion.*

Proof. We may suppose that G has no trivial face and two adjacent leaf faces in the interior of C. Otherwise, we can apply an edge deletion or a (2,2)-deletion to it or them. (Note that we must not consider the face f as a leaf face in the interior of C). If the interior of C contains no crossing face other than f, then G has exactly one leaf face and f in the interior of C. So, we can apply a (2,3)-deletion to the leaf face and f. If the interior of C contains crossing faces other than f, then let f' be a crossing face in the interior of C such that the interior of the separating cycle C' of f' does not contain f and that the interior of C' contains as few faces as possible. By the minimality of C', G has exactly one leaf face and f' in the interior of C'. So, we can apply a (2,3)-deletion to the leaf face and f'.

Let C be the separating cycle of y^+pq such that $x \notin V(C)$. By Claim 1, we can apply an edge deletion, a (2,2)-deletion or a (2,3)-deletion to G so that the resulting near triangulation still contains the vertex x. So, we can find a rooted near 1-HIST H of G with root x by above arguments. In the case when $|G| = 2n - 1$, the theorem is satisfied.

Case(b) $|G| = 2n$.

In this case, we prove that G satisfies the rooted HIST property by induction on the number of edges of G. If $|G| = 4$, then G is the complete graph K_4 or K_4 minus one edge. For any vertex v of G such that $d_G(v) \geq 3$, we can find a rooted HIST with root v, and hence G satisfies the rooted HIST property. So, we suppose $|G| \geq 6$. By Lemma 6, we can apply an edge deletion, a (2,2)-deletion or a (2,3)-deletion to G. Let G' be a near triangulation obtained from G by one operation of an edge deletion, a (2,2)-deletion or a (2,3)-deletion. Observe that G' satisfies the rooted HIST property by the induction hypothesis. We shall prove that, for each operation, if G' satisfies the rooted HIST property, then G also satisfies the rooted HIST property. (Note that we must pay attention to a vertex v such that $d_G(v) \geq 3$, but $d_{G'}(v) = 2$ since a vertex of degree two does not become a root of a rooted HIST).

First, we suppose that G' is obtained from G by an edge deletion of xy. Let xyz be a trivial face of G such that xy is contained in the boundary cycle B of G. Since G' satisfies the rooted HIST property, for any vertex v of G' such that $d_{G'}(v) \geq 3$, we can find a rooted HIST H' with root v. Moreover, H' is also a rooted HIST of G by the same arguments on the case(a). So, we must prove that we can find a rooted HIST of G with root x (resp., y) when $d_G(x) = 3$ (resp., $d_G(y) = 3$). By symmetry, it suffices to show that G has a rooted HIST of G with root x when $d_G(x) = 3$. (Note that we easily find such a rooted HIST if we can apply an edge deletion, a (2,2)-deletion or a (2,3)-deletion to G so that the resulting near triangulation still contains x of degree three). Since $d_G(x) = 3$, G has a trivial face xzx^-, where xx^- is contained in B. Let x^-pq be a face of G such that x^-p is contained in B, where $p \neq x$. If x^-pq is a trivial face, then we can apply an edge deletion to G so that the resulting near triangulation still contains x of degree three, by the same arguments on the case(a). If x^-pq a crossing face, then let C be the separating cycle of x^-pq such that $x \notin V(C)$. By Claim 1 on the case(a), we can apply an edge deletion, a (2,2)-deletion or a (2,3)-deletion to G so that the resulting near triangulation still contains x of degree three. So, we may suppose that x^-pq is a leaf face. If $q \neq y$, then G has a face qrs such that qr is contained in B, where $r \neq p$, and hence we can apply an edge deletion, a (2,2)-deletion or a (2,3)-deletion to G so that the resulting near triangulation still contains x of degree three by the same arguments on the case(a). So, we may suppose that $q = y$. Let x^-yt be a face of G such that $t \neq p$. Note that $t \neq z$, otherwise $|G| = 5$, contrary to $|G| = 2n$. We can obtain a near triangulation G'' from G by removing p, t and the edge x^-y. Since $d_{G''}(x) = 3$, we can find a rooted HIST H'' of G'' with root x by the induction hypothesis. Moreover, H'' extends a rooted HIST of G with root x by adding two edges yp, yt.

Secondly, we suppose that G' is obtained from G by a (2,3)-deletion of x, y. Let x^-xx^+ be a crossing face and xyx^+ be a leaf face of G such that $d_G(x) = 3$ and $d_G(y) = 2$. Since G' satisfies the rooted HIST property, for any vertex v of G' such that $d_{G'}(v) \geq 3$, we can find a rooted HIST H' of G' with root v. Moreover, H' extends to a rooted HIST of G with root v by adding edges xx^+ and yx^+. (This operation implies that G has a rooted HIST with root x^+). So, we shall prove that, for a vertex $u \in \{x, x^-\}$, G has a rooted HIST with root u. If $u = x$, then let G'' be a near triangulation obtained from G by removing the vertex y. Since $|G| = 2n$, $|G''| = 2n - 1$, and hence G'' satisfies the rooted near 1-HIST property by the case(a). Therefore, G'' has a rooted near 1-HIST with root x. Moreover, the rooted near 1-HIST extends to a rooted HIST of G with root x by adding an edge xy. If $u = x^-$, then we may suppose that $d_G(x^-) = 3$. (Otherwise, we can find a rooted HIST of G' with root x^-, and it extends to a rooted HIST of G with root x^-). So, G has a crossing face x^-x^+z such that x^-z is contained in the boundary cycle B, where $z \neq x$. Let C be the separating cycle of x^-x^+z such that $x \notin V(C)$. By Claim 1 on the case(a), G has a face or faces in the interior of C such that we can apply an edge deletion, a (2,2)-deletion or a (2,3)-deletion. If we can apply an edge deletion, a (2,2)-deletion or a (2,3)-deletion to G so that the resulting near triangulation still contains the vertex x^- of degree three, we can finish this case. Otherwise, we must apply (2,3)-deletion to x^-x^+z since an edge deletion and a (2,2)-deletion in the interior of C do not decrease the degree of x^-. This implies that G has exactly one leaf face and x^-x^+z in the interior of C, and hence we can find a rooted HIST H of G with root x^- such that $d_H(x^-) = d_H(x^+) = 3$, and, for w different from x^- and x^+, $d_H(w) = 1$.

Thirdly, we suppose that G' is obtained from G by a (2,2)-deletion of x, y. Let x^-xx^+ and x^+yy^+ be two leaf faces of G such that $d_G(x) = d_G(y) = 2$. Since G' satisfies the rooted HIST property, for any vertex v of G' such that $d_{G'}(v) \geq 3$, we can find a rooted HIST H' with root v. Moreover, H' extends to a rooted HIST H of G with root v by adding edges xx^+, yx^+. (This operation implies that G has a rooted HIST with root x^+). So, we shall prove that, for a vertex $v \in \{x^-, y^+\}$, G has a rooted HIST with root v. By symmetry, it suffices to show that we can apply an edge deletion, a (2,2)-deletion edge or a (2,3)-deletion to G so that the resulting near triangulation still contains the vertex x^- of degree three when $d_G(x^-) = 3$. Let y^+zz^+ be a face of G such that y^+z is contained in B. If y^+zz^+ is a trivial face or a leaf face, then we can apply an edge deletion or a (2,2)-deletion to G so that the resulting near triangulation still contains the vertex x^- of degree three. So, we may suppose that y^+zz^+ is a crossing face. (Note that $x^- \neq z^+$ since $d_G(x^-) = 3$). Let C be the separating cycle of y^+zz^+ such that $x^- \notin V(C)$. By Claim 1, G has a face or faces such that we can apply an edge deletion, a (2,2)-deletion or a (2,3)-deletion in the interior of C so that the resulting near triangulation still contains x^- of degree three. (Note that even if we apply a (2.3)-deletions in the interior of C the resulting near triangulation still contains x^- of degree three since $x^- \neq z^+$). Therefore, we can find a rooted HIST of G with root x^-. So, in the case when $|G| = 2n$, the theorem is satisfied. ■

4 Conclusion

Theorem 2 asserts that if a near triangulation G has $2n$ vertices, then, for any vertex v where $d_G(v) \geq 3$, G has a rooted HIST with root v. Otherwise (i.e., $|G| = 2n - 1$), for any vertex v where $d_G(v) \geq 2$, G has a rooted near 1-HIST with root v. When we consider HISTs of a triangulation on a surface, we often consider a spanning subgraph of the triangulation which is a planar graph. For example, we obtain a HIST of a projective planar triangulation G from a near triangulation which is a spanning subgraph of G. Therefore, our result might be useful when we solve the conjecture that *every triangulation on any surface with at least four vertices has a HIST.*

References

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