Kings in strong tournaments *

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Abstract: A k-king in a digraph D is a vertex which can reach every other vertex by a directed path of length at most k. Every tournament with no vertex of in-degree zero has at least three 2-kings. In this paper, we present the structure of tournaments which have exactly three 2-kings and prove that every strong tournament, containing at least $k+2$ vertices with $k \geq 3$, has at least $k+1$ k-kings.
Keywords: Digraphs; Tournaments; Kings

1 Terminology and introduction

We only consider finite digraphs without loops and multiple arcs. Let D be a digraph with vertex set $V(D)$ and arc set $A(D)$. For any $x, y \in V(D)$, we will also write $x \rightarrow y$ if $xy \in A(D)$. For a vertex x in D, its *out-neighborhood* $N^+(x) = \{y \in V(D) : xy \in A(D)\}$ and its *in-neighborhood* $N^-(x) = \{y \in V(D) : yx \in A(D)\}$. For disjoint subsets X and Y of $V(D)$, $X \rightarrow Y$ means that every vertex of X dominates every vertex of Y. We say that X *strictly dominates* Y, if $X \rightarrow Y$ and there is no arc from Y to X. For distinct vertices x and y, the *distance* $d(x, y)$ is the length of a shortest directed path from x to y. For any $x \in V(D)$ and $S \subseteq V(D)$, define $d(x, S) = \min\{d(x, s) : s \in S\}$. For $S \subseteq V(D)$, we denote by $D[S]$ the subdigraph of D induced by the vertex set S. A digraph D is *semicomplete* if there is at least one arc between any pair of distinct vertices of D. A *tournament* is a semicomplete digraph with no cycle of length 2.

A *k-king* in a digraph D is a vertex x which can reach every other vertex by a directed path of length at most k, that is, $d(x, y) \leq k$, for any $y \in V(D) - x$. In a number of papers (see, [1-9]), kings were investigated. Observe that every tournament has a 2-king. In fact, the vertex of maximum out-degree is a 2-king. In [5], Moon proved that every tournament

*This work is supported by the National Natural Science Foundation of China (61070229)(11026163) and the Natural Science Foundation for Young Scientists of Shanxi Province, China (2011021004).
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with no vertex of in-degree zero has at least three 2-kings. It seems quite natural to ask how many k-kings there can be in the tournament with no vertex of in-degree zero. To the knowledge of the authors this problem has not previously been addressed in the literature. If a tournament D is not strong, then every vertex of the unique initial strong component D' strictly dominates every vertex outside of D'. Hence the number of k-kings in D' is the number of k-kings in D. Thus we only need to study the problem in a strong tournament rather than a tournament with no vertex of in-degree zero. In Section 2, we prove that every strong tournament, containing at least $k+2$ vertices with $k \geq 3$, has at least $k+1$ k-kings and present the structure of tournaments which have exactly three 2-kings. For concepts not defined here we refer the reader to [2].

2 Main results

We begin with the following lemma.

Lemma 2.1 [4]. *Let $\{x\}, U_0, U_1, \ldots, U_s$ be disjoint sets of vertices in a digraph D. Let also $d(x, U_0) = t$ and $U_{i+1} \subseteq N^+(U_i)$ for every $i = 0, 1, \ldots, s-1$. Then $d(x, U_s) \leq t + s$.*

Observe that every tournament has a 2-king. In fact, the vertex of maximum out-degree is a 2-king. Furthermore, if a tournament D has a vertex of in-degree zero, then this vertex of in-degree zero is the only 2-king in D. In [5], Moon proved the following.

Theorem 2.2 [5]. *Every tournament with no vertex of in-degree zero has at least three 2-kings.*

Next, we consider the k-kings of strong tournaments with $k \geq 3$.

Theorem 2.3. *If $D = (V(D), A(D))$ is a strong tournament, containing at least $k+2$ vertices with $k \geq 3$, then there are at least $k+1$ k-kings in D. Furthermore, if there are exactly k+1 k-kings in D, then there is a path $P = p_0p_1 \ldots p_k$, such that $d(p_0, p_k) = k$, $p_0, p_1, \ldots, p_k$ are exactly $k+1$ k-kings and $\{p_1, p_2, \ldots, p_k\} \to V(D) - V(P)$.*

Proof. Let X denote the set of all k-kings in D and let $Y = V(D) - X$. Clearly our theorem is true if $Y = \emptyset$, so we may assume that Y is not empty; let $w \in Y$ be arbitrary. Now define W_i as follows: $W_i = \{v \in V(D) : d(w, v) = i\}$ for all $i = 0, 1, \ldots, m$, where $m = \max\{d(w, v) : v \in V(D)\}$. As $w \in Y$, $m \geq k+1 \geq 4$. By the definition of the set W_i, we observe that $W_i \to W_0 \cup W_1 \cup \cdots \cup W_{i-2}$, for all $2 \leq i \leq m$ and $W_{i+1} \subseteq N^+(W_i)$ for all $0 \leq i \leq m-1$. We first prove two claims.

Claim A. For any $x \in W_i$, if $i \geq 2$, then $d(x, W_0 \cup W_1 \cup \cdots \cup W_{i-1}) \leq 2$ and $d(x, W_i) \leq 3$; if $i = 1$ and $N^+(x) \cap W_2 \neq \emptyset$, then $d(x, W_0 \cup W_1) \leq 3$.

If $i \geq 2$, then, by $W_i \to W_0 \cup W_1 \cup \cdots \cup W_{i-2}$ and $W_{j+1} \subseteq N^+(W_j)$ with $0 \leq j \leq i-1$, we have $d(x, W_0 \cup W_1 \cup \cdots \cup W_{i-1}) \leq 2$ and $d(x, W_i) \leq$

3. Suppose $i = 1$ and let $v \in N^+(x) \cap W_2$. By the above argument, $d(v, W_0 \cup W_1) \le 2$ and so $d(x, W_0 \cup W_1) \le d(x, v) + d(v, W_0 \cup W_1) \le 3$.

Claim B. For any $y \in W_i$, $1 \le i \le m-1$, then either (i) or (ii) below holds:

(i) For every $z \in W_{i+1}$, there is a (w, z)-path in $D - y$.

(ii) $d(y, W_{i+1}) \le 3$.

Assume that neither (i) nor (ii) holds. This implies that there exist vertices z_1 and z_2 in W_{i+1} such that there is no (w, z_1)-path in $D - y$ and there is no (y, z_2)-path of length at most 3 in D. Let $P = p_0p_1 \dots p_{i+1}$ be a shortest path from w to z_1 in D and let $R = r_0r_1 \dots r_{i+1}$ be a shortest path from w to z_2 in D. Clearly $y = p_i \to z_1$ and y does not dominate z_2, which implies that $z_1 \ne z_2$ and $r_i \ne y$. If $z_1 \to r_i$, then $yz_1r_iz_2$ is a (y, z_2)-path of length 3 and if $r_i \to z_1$, then $r_0r_1 \dots r_iz_1$ is a (w, z_1)-path in $D - y$, a contradiction. The proof of Claim B is complete.

We now prove the theorem by induction on $|V(D)|$. If $|V(D)| = k + 2$, then, by $Y \ne \emptyset$, there is a path $p_0p_1 \dots p_{k+1}$ in D such that $d(p_0, p_{k+1}) = k + 1$. Observe that $p_1p_2 \dots p_{k+1}$ is the desired path.

Now assume that $|V(D)| \ge k + 3$ and that the theorem holds for all smaller strong tournaments with at least $k + 2$ vertices. We consider the following two cases.

Case 1. There exists a vertex $y \in Y \cap W_i$, $1 \le i \le m - 1$, such that (i) of Claim B holds.

We first prove that, for every $q \in V(D) - y$, there is a (w, q)-path in $D - y$, that is, w can reach every vertex of $V(D) - y$ in $D - y$. Let $q \in W_j$ be arbitrary and let $P = p_0p_1 \dots p_j$ be a shortest path from w to q in D. If $j \le i$, then clearly $y \notin V(P)$ and so we are done. If $j \ge i + 1$, then by (i) of Claim B, there is a (w, p_{i+1})-path in $D - y$, which together with $p_{i+2} \dots p_j$ forms a (w, p_j)-path in $D - y$.

Let $u \in W_m$ be arbitrary and let $R = r_0r_1 \dots r_l$ be a shortest path from w to u in $D - y$. As $l \ge m \ge k + 1 \ge 4$, by the minimality of R, $r_l \to r_0$ and so $(D - y)[V(R)]$ is strong. Let $Q_1, Q_2, \dots, Q_s (s \ge 1)$ be an acyclic ordering of the strong components of $D - y$. As w can reach every vertex of $V(D) - y$ in $D - y$, the vertex w belongs to an initial strong component of $D - y$, say, Q_1. Again as $D - y$ is also a tournament, Q_1 is the unique initial strong component of $D - y$. Since $(D - y)[V(R)]$ is strong and $w \in V(R)$, we have $V(R) \subseteq V(Q_1)$. Hence Q_1 has at least $k + 2$ vertices.

Now we claim that every k-king in Q_1 is also a k-king in D. Let x be a k-king in Q_1 and let $z \in V(D) - x$ be arbitrary. If $z \in V(Q_1)$, then, by the choice of x, $d(x, z) \le k$. If $z \in V(Q_t)$ for $t \ge 2$, then $x \to z$ and so $d(x, z) \le k$. Suppose $z = y$ and $d(x, y) > k$. It follows from the above argument that $d(x, V(D) - y) \le k$. Therefore, $d(x, y) = k + 1$. Let $W_i^{(x)} = \{v \in V(D) : d(x, v) = i\}$, $i = 0, 1, \dots, k + 1$. As $W_{k+1}^{(x)} = y$ and

$k+1 \geq 4$, by Claim A, y is a 3-king and so y is a k-king, a contradiction to the fact $y \in Y$. Therefore, the claim is true.

Using the induction hypothesis for Q_1, we obtain that there are at least $k+1$ k-kings in Q_1 and so there are at least $k+1$ k-kings in D. If there are precisely $k+1$ k-kings in D, then there are precisely $k+1$ k-kings in Q_1, and, thus, there is a path $P = p_0 p_1 \dots p_k$ in Q_1, which is a shortest possible (p_0, p_k)-path in Q_1 such that $\{p_0, p_1, \dots, p_k\} = X$ and $\{p_1, p_2, \dots, p_k\} \to V(Q_1) - V(P)$. If there is no arc from y to a vertex in $\{p_1, p_2, \dots, p_k\}$, then clearly $\{p_1, p_2, \dots, p_k\} \to V(D) - V(P)$ and we are done. So assume that there is an arc yp_s, where $1 \leq s \leq k$. We will show that y is another k-king in D and, thus, obtain a contradiction to our assumption on the existence of the arc yp_s.

By $d(p_s, V(D) - (V(Q_1) \cup \{y\})) = 1$ and $y \to p_s$, we have $d(y, V(D) - V(Q_1)) \leq 2$. Let $z \in V(Q_1) - V(P)$. As $\{p_1, p_2, \dots, p_k\} \to V(Q_1) - V(P)$, we have $p_s \to z$. So $d(y, z) \leq 2$. To demonstrate that y is a k-king, it is now sufficient to prove that $d(y, p_j) \leq k$ for every $j \in \{0, 1, \dots, k\}$. Suppose $s < k$. For $j > s$, $d(y, p_j) \leq d(y, p_s) + d(p_s, p_j) \leq 1 + j - s \leq 1 + k - 1 \leq k$. For $0 \leq j < s$, p_j is dominated by either p_s or p_{s+1} as P is a shortest (p_0, p_k)-path, thus, $d(y, p_j) \leq 3$. Now suppose $s = k$. For $j \in \{0, 1, \dots, k-2\}$, since $p_k \to p_j$, $d(y, p_j) \leq 2$. As $p_k p_{k-2} p_{k-1}$ is a path of length 2, $d(y, p_{k-1}) \leq 3 \leq k$. Hence, $d(y, \{p_0, p_1, \dots, p_k\} \leq k$.

Case 2. For every $i = 1, 2, \dots, m-1$ and every $y \in Y \cap W_i$, (ii) of Claim B holds.

If $|X| \geq k+2$, then we are done. Hence, assume that $|X| \leq k+1$. By Claim A, $m \geq 4$ and (ii) of Claim B, we have $W_m, W_{m-1} \subseteq X$. Let $P = p_0 p_1 \dots p_m$ be a shortest path from w to a vertex $p_m \in W_m$. It can be observed that $p_i \in W_i$, for $0 \leq i \leq m$. Now we show that $p_{m-i} \in X$ for $i \in \{0, 1, \dots, k-2\}$. If $i = 0, 1$, then, by $W_m, W_{m-1} \subseteq X$, $p_m, p_{m-1} \in X$. Suppose that $i \in \{2, 3, \dots, k-2\}$. For any $i \in \{2, 3, \dots, k-2\}$ and $z \in V(D) - p_{m-i}$, if $z \in W_{m-i+1}$, then, by (ii) of Claim B, $d(p_{m-i}, z) \leq 3$; if $z \in W_{m-i+2} \cup \dots \cup W_m$, then by Lemma 2.1 and (ii) of Claim B , $d(p_{m-i}, z) \leq 3 + m - (m-i+1) = i+2 \leq k$; if $z \in W_0 \cup \dots \cup W_{m-i}$, then, by Claim A, $d(p_{m-i}, z) \leq 3$. This implies $p_{m-i} \in X$.

Claim C. $d(p_{m-k+1}, W_{m-k+2} \cup \dots \cup W_{m-1}) \leq k$ and $d(p_{m-k}, W_{m-k+1} \cup \dots \cup W_{m-2}) \leq k$.

For any $z \in W_{m-k+2} \cup \dots \cup W_{m-1}$, by (ii) of Claim B and Lemma 2.1, $d(p_{m-k+1}, z) \leq 3 + (m-1) - (m-k+2) = k$. Similarly, we can obtain the latter inequality. The proof of Claim C is complete.

To complete the proof of this theorem, first we show that $|W_m| = |W_{m-1}| = 1$. Suppose that $|W_m| \geq 2$ and $|W_{m-1}| \geq 2$. By $|X| \leq k+1$, $p_{m-k+2}, \dots, p_m \subseteq X$, $W_{m-1} \subseteq X$ and $W_m \subseteq X$, we have $|W_{m-1}| = |W_m| = 2$, say $W_{m-1} = \{p_{m-1}, x\}$ and $W_m = \{p_m, y\}$, respectively. Therefore $X = \{p_{m-k+2}, \dots, p_{m-1}, x, p_m, y\}$ and $p_{m-k+1} \in Y$. By Claims A and

C and $d(p_{m-k+1}, p_m) = k-1$, we have $d(p_{m-k+1}, V(D)-y) \le k$. Hence $d(p_{m-k+1}, y) \ge k+1$. Combining this with $d(p_{m-k+1}, p_{m-1}) = k-2$ and $d(p_{m-k+1}, p_m) = k-1$, we have $y \to p_{m-1}$ and $y \to p_m$. As D is strong, $x \to y$. Again, by $d(p_{m-k+1}, p_m) = k-1$, we have $x \to p_{m-1}$. If $p_{m-2} \to x$, then $p_{m-k+1} \dots p_{m-2}xy$ is a path of length $k-1$, a contradiction. Assume $x \to p_{m-2}$. Since $x \in W_{m-1}$ and $W_{m-1} \subseteq N^+(W_{m-2})$, there exists a vertex $z \neq p_{m-2}$ in W_{m-2} such that $z \to x$. Clearly, $d(z, \{x, y, p_{m-1}, p_m\}) \le 3$. Combining this with Claim A, we can obtain that $z \in X$, a contradiction to $|X| \le k+1$.

Suppose $|W_m| = 1$ and $|W_{m-1}| \ge 2$. By Claims A and C, and $d(p_{m-k+1}, p_m) = k-1$, we have $p_{m-k+1} \in X$. Combining this with $W_{m-1} \subset X$ and $|X| \le k+1$, we have that $|W_{m-1}| = 2$, say $W_{m-1} = \{p_{m-1}, x\}$. Hence, $X = \{p_{m-k+1}, \dots, p_{m-1}, x, p_m\}$ and $p_{m-k} \in Y$. By Claims A and C, and $d(p_{m-k}, p_m) = k$, we obtain that $d(p_{m-k}, V(D)-x) \le k$. Hence $d(p_{m-k}, x) \ge k+1$. It is not difficult to obtain that $x \to p_{m-2}, x \to p_{m-1}$. Since $W_{m-1} \subseteq N^+(W_{m-2})$ and $x \in W_{m-1}$, there exists a vertex $z \in W_{m-2}$ such that $z \to x$. By Claim A and $z \to x \to p_{m-1} \to p_m$, we can obtain $z \in X$, a contradiction to $|X| \le k+1$.

Suppose $|W_{m-1}| = 1$ and $|W_m| \ge 2$. By $W_m \subseteq N^+(W_{m-1})$, we have $W_{m-1} \to W_m$. This together with Claims A and C, we can obtain $p_{m-k}, p_{m-k+1} \in X$. Thus $\{p_{m-k}, \dots, p_{m-1}\} \cup W_m \subseteq X$ and so $|X| \ge k+2$, a contradiction.

Hence $|W_m| = |W_{m-1}| = 1$. By Claims A and C and $d(p_{m-k}, p_m) = k$, we have $X = \{p_{m-k}, p_{m-k+1}, \dots, p_{m-1}, p_m\}$. Now it suffices to prove that $\{p_{m-k+1}, \dots, p_{m-1}, p_m\} \to V(D)-X$. Assume that this is not true. Thus, there is a vertex $q \in V(D)-X$ which dominates a vertex $p_i \in \{p_{m-k+1}, \dots, p_m\}$. Observe $q \in \{W_{m-k}, \dots, W_{m-2}\}$. Let R be a shortest path from w to q. Then $Rp_ip_{i+1} \dots p_m$ is a path from w to p_m. A similar argument to the proof of Claim C, we can obtain that $d(q, W_i \cup \dots \cup W_{m-1}) \le k$. This together with Claim A and $d(q, p_m) \le k$, we have $q \in X$, a contradiction. The proof of Theorem 2.3 is complete. □

Remark. Applying Theorem 2.2, if $D = (V(D), A(D))$ is a strong tournament, then there exist at least three 2-kings in D. But if there are exactly three 2-kings in D, then the analogous result in Theorem 2.3 does't hold. See Figure 1. It is easy to check that x_3, x_4, x_5 are 2-kings and x_1, x_2 are not 2-kings. If there exists a path $P = p_0p_1p_2$ such that $d(p_0, p_2) = 2$, and $\{p_1, p_2\} \to V(D) - V(P)$, then $d^-(p_1) = d^-(p_2) = 1$. But in Figure 1, the vertex of in-degree one is unique, which is x_4.

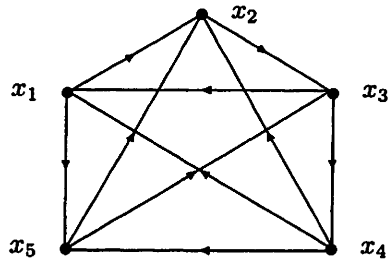

Figure 1.

We present below the structure of tournaments which have exactly three 2-kings.

Theorem 2.4. *Let D be a tournament with no vertex of in-degree zero. Then D has exactly three 2-kings x, y, z if and only if $V(D) - \{x, y, z\}$ can be partitioned into four sets B_1, B_2, B_3, B_4 (possibly empty) such that $x \to z \to y \to x$, $B_1 \to x \to B_2 \cup B_3 \cup B_4$, $B_2 \to y \to B_1 \cup B_3 \cup B_4$, $B_3 \to z \to B_1 \cup B_2 \cup B_4$, $B_1 \to B_3 \to B_2 \to B_1$ and the directions of arcs between B_4 and B_1, B_2, B_3 are arbitrary such that there is no vertex $v \in B_4$ such that $N^+(v) \cap B_1 \neq \emptyset$, $N^+(v) \cap B_2 \neq \emptyset$, $N^+(v) \cap B_3 \neq \emptyset$.*

Proof. It is not difficult to check that the vertex of maximum out-degree x_1 is a 2-king of D, a 2-king x_2 of $D[N^-(x_1)]$ is a 2-king of D and a 2-king of $D[N^-(x_2)]$ is a 2-king of D as well. We can also see that a tournament has unique a 2-king if and only if it contains a vertex of in-degree zero.

We may, without loss of generality, assume that x is the vertex of maximum out-degree in D, y is a 2-king of $D[N^-(x)]$ and z is a 2-king of $D[N^-(y)]$. Clearly, x, y, z are three 2-kings of D. Set $B_1 = N^-(x) - y$. Since D has exactly three 2-kings, there is only a 2-king of $D[N^-(x)]$ and so $y \to B_1$ and $x \to z$. Set $B_2 = N^-(y) - z$ and so $x \to B_2$. Similarly, there is only a 2-king in $D[N^-(y)]$ and so $z \to B_2$. Set $B_3 = N^-(z) \cap N^+(x)$ and $B_4 = N^+(z) \cap N^+(x)$. Note that $y \to B_3 \cup B_4$.

Claim A. Let $x_1x_2x_3x_1$ be a 3-cycle of D. If $N^-(x_1) \cap N^-(x_2) \neq \emptyset$, then there exists a 2-king of D in $N^-(x_1) \cap N^-(x_2)$.

Let w be a 2-king of $D[N^-(x_1) \cap N^-(x_2)]$. Note that for any $w' \in \{x_1, x_2, x_3\} \cup (N^-(x_1) \cap N^-(x_2))$, $d(w, w') \leq 2$. For any $w'' \in V(D) - \{x_1, x_2, x_3\} \cup (N^-(x_1) \cap N^-(x_2))$, $x_1 \to w''$ or $x_2 \to w''$ and so $d(w, w'') \leq 2$. Hence we have shown that w is a 2-king of D. The proof of Claim A is complete.

Note that none of $B_1 \cup B_2 \cup B_3 \cup B_4$ contains a 2-king as x, y, z are three 2-kings. Since $N^-(x) \cap N^-(z) \subset B_1$ and none of B_1 is a 2-king of D, by Claim A, $N^-(x) \cap N^-(z) = \emptyset$ and so $z \to B_1$. Again, since xz_1yx, where $z_1 \in B_2$, is a 3-cycle, $N^-(x) \cap N^-(z_1) \subset B_1$ and none of B_1 is a 2-kings of D, by Claim A, $B_2 \to B_1$. Since xzy_1x, where $y_1 \in B_1$, is a 3-cycle, $N^-(z) \cap N^-(y_1) \subset B_3$ and none of B_3 is a 2-king of D, by Claim A, $B_1 \to B_3$. Since $zyuz$, where $u \in B_3$, is a 3-cycle, $N^-(y) \cap N^-(u) \subset B_2$ and none of B_2 is a 2-king, by Claim A, $B_3 \to B_2$.

Suppose, on the contrary, that there exists $v \in B_4$ such that $N^+(v) \cap B_1 \neq \emptyset$, $N^+(v) \cap B_2 \neq \emptyset$ and $N^+(v) \cap B_3 \neq \emptyset$. Let $F = \{v \in B_4 : N^+(v) \cap B_1 \neq \emptyset, N^+(v) \cap B_2 \neq \emptyset \text{ and } N^+(v) \cap B_3 \neq \emptyset\}$ and let v' be a 2-king of $D[F]$ and let $v'b_1, v'b_2, v'b_3 \in A(D)$, where $b_1 \in B_1, b_2 \in B_2$ and $b_3 \in B_3$. By $B_1 \to B_3 \to B_2 \to B_1$ and $B_1 \to x$, $B_3 \to z$, $B_2 \to y$, we have $d(v', \{x, y, z\} \cup B_1 \cup B_2 \cup B_3) \leq 2$. By the definition of F, v' can reach every vertex of F by a directed path of length at most 2. For any $v'' \in B_4 - F$, there exists one of $\{b_1, b_2, b_3\}$ dominates v'', say b_1. Since $v' \to b_1 \to v''$, we have $d(v', v'') \leq 2$. Hence v' is a 2-king of D, a contradiction to the fact that D has exactly three 2-kings.

We now show the sufficiency. By the definition of D, we can check that x, y, z are 2-kings; for any $y' \in B_1, d(y', y) = 3$; for any $z' \in B_2, d(z', z) = 3$; for any $u' \in B_3, d(u', x) = 3$; for any $v' \in B_4$, either $d(v', x) \geq 3$ or $d(v', y) \geq 3$ or $d(v', z) \geq 3$. The proof of Theorem 2.4 is complete. □

Acknowledgement

The authors would like to thank the anonymous referee for some suggestions and useful comments on the original manuscript.

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