The goodness of path or cycle with respect to multiple copies of complete graphs of order three

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Abstract

The notation tK_3 represents a graph with t copies of complete graph K_3. In this note we discuss the goodness of path P_n or cycle C_n with respect to tK_3. Furthermore, this result provides the computation of Ramsey number $R(G, tK_3)$ when G is a set of disjoint paths or cycles.

Keywords: *cycle, (G, H)-free, H-good, path, Ramsey number.*

1 Introduction

All graphs in this paper are finite, undirected and simple. Let G and H be two graphs, where H is a subgraph of G, we define $G - H$ as a graph obtained from G by deleting the vertices of H and all edges incident to them. Let t be a natural number and G_i be a connected graph with the vertex set V_i and the edge set E_i for every $i = 1, 2, ..., t$. *The disjoint union of graphs*, $\bigcup_{i=1}^{t} G_i$, has the vertex set $\bigcup_{i=1}^{t} V_i$ and the edge set $\bigcup_{i=1}^{t} E_i$. Furthermore, if each G_i is isomorphic to a connected graph G then we denote by tG the disjoint union of t copies of G.

For graphs G and H, *the Ramsey number* $R(G, H)$ is the minimum n such that in every coloring of the edges of the complete graph K_n with two colors, say red and blue, there is a red copy of G or a blue copy of H. A graph F is called *(G, H)-free* if F contains no subgraph isomorphic to G and its complement $\overline{F}$ contains no subgraph isomorphic to H. The Ramsey number $R(G, H)$ can be equivalently defined as the smallest natural number n such that no (G, H)-free graph on n vertices exists.

Determining $R(G, H)$ is a notoriously hard problem. Burr [7] showed that the problem of determining whether $R(G, H) \leq n$ for a given n is

NP-hard. Furthermore in Shaeffer [19] one can find a rare natural example of a problem higher than NP-hard in the polynomial hierarchy of computational complexity theory, that is, Ramsey arrowing is $\prod_2^p$-complete. The few known values of $R(G, H)$ are collected in the dynamic survey of Radziszowski [16].

Burr [6] proved the general lower bound

$$R(G,H) \geq (|V(G)| - 1)(\chi(H) - 1) + s(H), \tag{1}$$

where G is a connected graph, $\chi(H)$ denotes the chromatic number of H and $s(H)$ is its *chromatic surplus*, namely, the minimum cardinality of a color class taken over all proper colorings of H with $\chi(H)$ colors. Motivated by this inequality, the graph G is said to be H*–good* if equality holds in (1). Chvátal [11] proved that trees are K_m–good graphs.

Faudree and Schelp [12] conjectured that C_n is K_m-good for $n \geq m \geq 3$, except for $n = m = 3$. The conjecture has been verified for $n \geq m^2 - 2$ (Bondy and Erdős [4]), for $m = 3$ (Chartrand and Schuster [8]), $m = 4$ (Yang, Huang and Zhang [18]), $m = 5$ (Bollobás, Jayawardene, Yang, Huang, Rousseau and Zhang [2]), $m = 6$ (Schiermeyer [17]) and $m = 7$ (Chen, Cheng and Zhang [9]). More recently, Nikiforov [15] proved the conjecture for all $m \geq 3$ and $n \geq 4m + 2$. Other result concerning the goodness of graphs with the chromatic surplus one can be found in Lin et al. [14]. However, the goodness of path P_n or cycle C_n with respect to tK_m for $t \geq 2$ is still open.

In this paper we establish the goodness of P_n or C_n with respect to tK_3 for $t \geq 2$ and sufficiently large n.

Theorem 1 *Let $t \geq 2$ be an integer and $f(t) = 6t^2 - 15t + 9$. If $n \geq f(t)$ then $R(P_n, tK_3) = 2n + t - 2$.*

Theorem 2 *Let $t \geq 2$ be an integer and $g(t) = 6t^2 - 3t + 1$. If $n \geq g(t)$ then $R(C_n, tK_3) = 2n + t - 2$.*

For the proof of Theorem 2 we use the following result of Bondy and the above mentioned result of Nikiforov.

Lemma 1 (Bondy, 1975 [3]) *Let G be a graph of order n. If the minimum degree of G satisfies $\delta(G) \geq \frac{n}{2}$ then either G is pancyclic or n is even and $G \simeq K_{\frac{n}{2},\frac{n}{2}}$.*

Theorem 3 (Nikiforov, 2005 [15]) *Let $m \geq 3$ be an integer. If $n \geq 4m + 2$ then $R(C_n, K_m) = (n - 1)(m - 1) + 1$.*

By extending previous results of Baskoro [1] and Stahl [20], Bielak [5] and Sudarsana et al. [21] recently proved a formula for $R(G, H)$ when every connected component of G is a H-good graph. This result motivates the study of general families of H-good graphs. In particular, Theorems 1 and 2 provide the following computation of $R(G, tK_3)$ when G is a set of disjoint paths or cycles.

Corollary 1 *Let $t \geq 2$ be an integer and $g(t) = 6t^2 - 3t + 1$. Let $G \simeq \bigcup_{i=1}^{k} l_i G_i$, where $l_i \geq 1$ and each G_i is a path or cycle of order n_i.*

If $n_1 \geq n_2 \geq ... \geq n_k \geq g(t)$ then

$$R(G, tK_3) = \max_{1 \leq i \leq k} \left\{ n_i + \sum_{j=1}^{i} l_j n_j \right\} + t - 2. \tag{2}$$

2 Proof of Theorems

We first show Theorem 1 for the case $t = 2$.

Lemma 2 *Let $n \geq 3$ be an integer. Then, $R(P_n, 2K_3) = 2n$.*

Proof. The lower bound $R(P_n, 2K_3) \geq 2n$ follows from the fact that $2K_{n-1} \cup K_1$ is a $(P_n, 2K_3)$-free graph on $2n - 1$ vertices.

Now we will prove that $R(P_n, 2K_3) \leq 2n$. Let F be an arbitrary graph of order $2n$ that contains no P_n. Select a path $P = x_1 x_2 ... x_m$ of maximal length in F, delete the vertices of P and select a second maximal length path $Q = y_1 y_2 ... y_k$ in $F - P$. Paths P and Q have $m < n$ and $k < n$ vertices, respectively, so deleting P and Q leaves at least two vertices z_1, z_2. Maximality of path length then shows that $\{x_1, y_1, z_1\}$ is an independent set and so is $\{x_m, y_k, z_2\}$. Therefore, we have a copy of $2K_3$ in $\overline{F}$. □

We are now ready to prove the first theorem.

Proof of Theorem 1. The graph $2K_{n-1} \cup K_{t-1}$ shows the lower bound $R(P_n, tK_3) \geq 2n + t - 2$.

In order to prove the upper bound $R(P_n, tK_3) \leq 2n + t - 2$ we use induction on t. For $t = 2$, Lemma 2 gives $R(P_n, 2K_3) = 2n$ and hence the assertion holds for $n \geq f(2) = 3$. Assume that the assertion is true for $n \geq f(t-1)$, that is $R(P_n, (t-1)K_3) \leq 2n + t - 3$. We shall show that the theorem is also valid for $n \geq f(t)$. Let F be an arbitrary graph on $2n+t-2$ vertices. We will show that F contains P_n or $\overline{F}$ contains tK_3. Since $2n+t-2 > 2n-1 = R(P_n, K_3)$ (Chvátal [11]), it follows that F contains P_n or $\overline{F}$ contains K_3. If F contains P_n then we are done. If $\overline{F}$ contains K_3 then the subgraph $F - \overline{K}_3$ of F has $2(n-1)+t-3$ vertices. Note that, since $t \geq 3$, we have $n \geq f(t) > f(t-1)+1$. By the induction on t, $F - \overline{K}_3$ contains P_{n-1} or the complement of $F - \overline{K}_3$ contains $(t-1)K_3$. If the complement of $F - \overline{K}_3$ contains $(t-1)K_3$ then we have a tK_3 in $\overline{F}$ and the proof is done. Thus we may assume that $F - \overline{K}_3$ contains P_{n-1}. Note that, since $R(P_n, P_m) = n + \lfloor \frac{m}{2} \rfloor - 1$ for $n \geq m \geq 2$ (Gerencsér and Gyárfás [13]), we have $R(P_n, tK_2) = n+t-1$ for $n \geq 2t$. Therefore, since $n \geq f(t) > 2t$ for $t \geq 3$, it follows that the subgraph $F - P_{n-1}$ of order $n+t-1$ contains P_n or the complement of $F - P_{n-1}$ contains tK_2. If $F - P_{n-1}$ contains P_n then we are done. Hence F contains P_{n-1}, — say $P_{n-1} = p_1p_2...p_{n-2}p_{n-1}$ —, and that $\overline{F}$ contains tK_2, — say $a_1b_1, a_2b_2, ..., a_tb_t$. It is clear that the graphs P_{n-1} and tK_2 have no vertices in common.

Assume that F contains no P_n. We will show that $\overline{F}$ contains tK_3. Thus the end vertices p_1 and p_{n-1} of path P_{n-1} must not be adjacent to any vertices in tK_2. Therefore the set $D = \{p_1, a_1, b_1\} \cup \{p_{n-1}, a_2, b_2\}$ forms a $2K_3$ in $\overline{F}$. Let us now consider the relation between the vertices in $A = \{p_2, p_3, ..., p_{n-2}\}$ and in $B = \{a_3, b_3, a_4, b_4, ..., a_t, b_t\}$.

Since there is no P_n in F, it follows that every two consecutive vertices p_i, p_{i+1} in A can not be adjacent to any vertices in B for every $i \in \{2, 3, ..., n-2\}$. Suppose that the neighborhood $N_A(u)$ in A of a vertex $u \in B$ satisfies $|N_A(u)| \geq 3t-1$. Let $p_i, p_j \in N_A(u)$ with $i < j$. Note that $j - i > 1$ since otherwise we can extend P_{n-1} to a path of length $n-1$ containing u. If $p_{i+1}p_{j+1}$ is an edge for a pair p_i, p_j in A then $P' = p_1p_2....p_iup_jp_{j-1}p_{j-2}...p_{i+1}p_{j+1}p_{j+2}....p_{n-1}$ is a new path of order n in F. If $p_{i+1}p_{j+1}$ is not an edge for every pair p_i, p_j in A then $\{p_{i+1} : p_i \in N_A(u)\} \cup \{u\}$ is a set of $3t$ independent vertices in F and we obtain a tK_3 in $\overline{F}$. Hence, for each $u \in B$ we have $|N_A(u)| \leq 3t-2$. Therefore,

$$\left| A \setminus \bigcup_{u \in B} N_A(u) \right| \geq n - 3 - (3t-2)2(t-2).$$

Since $n \geq f(t)$, it follows that there are at least $t-2$ vertices in A which are adjacent to no vertex in B and hence together with D we have a tK_3 in $\overline{F}$. This concludes the proof of Theorem 1. □

We next prove that the following lemma deals with the goodness of cycle C_n with respect to tK_2.

Lemma 3 *Let $n \geq 3$ and $t \geq 1$ be integers. Then,*

$$R(C_n, tK_2) = \begin{cases} n+t-1, & t \leq \lfloor \frac{n}{2} \rfloor; \\ 2t + \lceil \frac{n}{2} \rceil - 1, & t > \lfloor \frac{n}{2} \rfloor. \end{cases}$$

Proof. We consider two cases.

Case 1: $t \leq \lfloor \frac{n}{2} \rfloor$.

Observe that $K_{n-1} \cup K_{t-1}$ is a (C_n, tK_2)–free graph on $n+t-2$ vertices and hence $R(C_n, tK_2) \geq n+t-1$.

To prove the upper bound $R(C_n, tK_2) \leq n+t-1$ we use induction on t. For $t = 1$, the assertion holds from the fact that $R(C_n, K_2) = n$. Assume that the lemma is true for $t-1$. Let F be an arbitrary graph on $n+t-1$ vertices containing no C_n. We will show that its complement $\overline{F}$ contains tK_2. By the induction hypothesis, $\overline{F}$ contains $(t-1)K_2$. Let $\{a_1b_1, \cdots, a_{t-1}b_{t-1}\}$ be a set of independent edges in $\overline{F}$ and denote by $B = \{a_1, b_1, ..., a_{t-1}, b_{t-1}\}$.

Suppose on the contrary that $\overline{F}$ contains no tK_2. Let us consider the subgraph $F[A]$ of F induced by $A = V(F) \backslash B$, which has $n-t+1$ vertices. If there are two non adjacent vertices in $F[A]$, say x and y, then the subgraph of $\overline{F}$ induced by $\{x, y\} \cup B$ contains tK_2. Therefore, $F[A]$ is a complete graph of order $n-t+1$.

We now consider the relation between the vertices in $F[A]$ and in B. For every i, the neighborhood in F of $\{a_i, b_i\}$ has at most one vertex in $F[A]$, since otherwise we can replace the edge a_ib_i in $\overline{F}$ by two independent edges which, together with $\{a_jb_j, 1 \leq j \leq t-1, j \neq i\}$ produce a copy of tK_2 in $\overline{F}$. Thus we may assume that each b_i is adjacent in F to all but at most one vertex in $F[A]$. Now, let us consider the subgraph $F[D]$ of F induced by $D = A \cup \{b_1, b_2, ..., b_{t-1}\}$. The graph $F[D]$ has order n and minimum degree $\delta(F[D]) \geq n-t$. Since $t \leq \lfloor \frac{n}{2} \rfloor$, it follows that $\delta(F[D]) \geq \lceil \frac{n}{2} \rceil \geq \frac{n}{2}$. Lemma 1 now implies that $F[D]$ contains a cycle of order n, contradicting our assumption on F. Hence $\overline{F}$ contains a copy of tK_2 as claimed.

Case 2. $t > \lfloor \frac{n}{2} \rfloor$.

The lower bound $R(C_n, tK_2) \geq 2t + \lceil \frac{n}{2} \rceil - 1$ is obtained from the fact that $K_{\lceil \frac{n}{2} \rceil - 1} + \overline{K}_{2t-1}$ is a (C_n, tK_2)–free graph on $2t + \lceil \frac{n}{2} \rceil - 2$ vertices.

To show the upper bound $R(C_n, tK_2) \leq 2t + \lceil \frac{n}{2} \rceil - 1$ we argue as follows. Let F be a graph of order $2t + \lceil \frac{n}{2} \rceil - 1$ containing no C_n. By induction on t, we will show that $\overline{F}$ contains t independent edges. For $t = \lfloor \frac{n}{2} \rfloor$, we obtain Case 1. Therefore, $t > \lfloor \frac{n}{2} \rfloor$. By deleting a pair of non adjacent vertices u and v from F, the subgraph $F-\{u,v\}$ of F has $2(t-1)+\lceil\frac{n}{2}\rceil-1$ vertices, contains no C_n and $t-1\geq\lfloor\frac{n}{2}\rfloor$. By the induction hypothesis, the complement of $F-\{u,v\}$ contains $(t-1)K_2$ and, together with the edge uv, we have a tK_2 in $\overline{F}$. □

We next prove the following weaker form of Theorem 2.

Lemma 4 *Let $t\geq 2$ be an integer and $g(t)=6t^2-3t+1$. If F is a graph of order $2n+t-2$ containing C_{n-1} and $n\geq g(t)$ then F contains C_n or $\overline{F}$ contains tK_3.*

Proof. Let F be a graph on $2n+t-2$ vertices containing C_{n-1}. We will prove that F contains C_n or $\overline{F}$ contains tK_3.

Since F contains C_{n-1}, it follows that the subgraph $F-C_{n-1}$ of F has $n+t-1$ vertices. Note that if $t\geq 2$ then $n\geq g(t)>2t$, and hence Lemma 3 implies that the subgraph $F-C_{n-1}$ contains C_n or the complement of $F-C_{n-1}$ contains tK_2. If $F-C_{n-1}$ contains C_n then we are done.

Thus let F be a graph of order $n\geq g(t)$ containing C_{n-1} with vertex set, say $c_1,c_2,\ldots,c_{n-1}$ and edges c_ic_{i+1} (subscripts modulo $(n-1)$), and that $\overline{F}$ contains t disjoint copies $K_2^1,K_2^2,\ldots,K_2^t$ of the complete graph with two vertices. It is clear that the subgraphs C_{n-1} and tK_2 have no vertices in common.

Assume that F contains no C_n. We will show that $\overline{F}$ contains tK_3. Let us consider the relation between the vertices in $A=\{c_1,c_2,\ldots,c_{n-1}\}$ and in $B=V(K_2^1)\cup V(K_2^2)\cup\ldots\cup V(K_2^t)$. Suppose that the neighborhood $N_A(u)$ in A of a vertex $u\in B$ satisfies $|N_A(u)\cap V(C_{n-1})|\geq 3t-1$. Let $c_i,c_j\in N_A(u)\cap V(C_{n-1})$ with $i<j$. Note that $j-i>1$ since otherwise we can extend C_{n-1} to a cycle of length n containing u. If c_{i+1} and c_{j+1} are adjacent in F then we also have the cycle $\{c_iuc_jc_{j-1}\ldots c_{i+1}c_{j+1}c_{j+2}\ldots c_{n-1}c_1c_2\ldots c_i\}$ of length n in F. If $c_{i+1}c_{j+1}$ is not an edge for every pair $c_i,c_j\in N_A(u)\cap V(C_{n-1})$ then $\{c_{i+1}:c_i\in N_A(u)\cap V(C_{n-1})\}\cup\{u\}$ is a set of $3t$ independent vertices in F so that $\overline{F}$ contains tK_3. Hence, for each $u\in B$ we have $|N_F(u)\cap V(C_{n-1})|\leq 3t-2$. Therefore,

$$\left|A\setminus\bigcup_{u\in B}N_A(u)\right|\geq n-1-2t(3t-2).$$

Since $n\geq g(t)$, it follows that there are at least t vertices in A which are adjacent to no vertex in B and hence $\overline{F}$ contains tK_3. This concludes the proof of lemma. □

The following lemma provides the cases $n=19$ and $t=2$ of Theorem 2.

Lemma 5 $R(C_{19}, 2K_3) = 38$.

Proof. The graph $2K_{18} \cup K_1$ provides $R(C_{19}, 2K_3) \geq 38$.

We will prove that $R(C_{19}, 2K_3) \leq 38$. Let F be a graph on 38 vertices. We shall show that F contains C_{19} or $\overline{F}$ contains $2K_3$. Theorem 3 guarantees that F contains C_{19} or $\overline{F}$ contains K_3. If F contains C_{19} then we are done. Thus we may assume that $\overline{F}$ contains K_3. Then the subgraph $F - \overline{K}_3$ of F has 35 vertices. Again, Theorem 3 implies that the subgraph $F - \overline{K}_3$ contains C_{18} or the complement of $F - \overline{K}_3$ contains K_3. If the complement of $F - \overline{K}_3$ contains K_3 then we obtain $2K_3$ in $\overline{F}$ and the proof is done. Therefore F contains C_{18}. Now by taking $n = 19$ and $t = 2$, Lemma 4 gives that F contains C_{19} or $\overline{F}$ contains $2K_3$. □

Our last Lemma handles the case $t = 2$ of Theorem 2.

Lemma 6 *Let $n \geq 19$ be an integer. Then, $R(C_n, 2K_3) = 2n$.*

Proof. The graph $2K_{n-1} \cup K_1$ gives $R(C_n, 2K_3) \geq 2n$.

We will prove the upper bound $R(C_n, 2C_3) \leq 2n$ by induction on n. For $n = 19$, the assertion holds by Lemma 5. Assume that the assertion is true for $n - 1$, that is $R(C_{n-1}, 2K_3) \leq 2(n-1)$. We shall show that the lemma is also valid for n. Let F be an arbitrary graph of order $2n$. We will show that F contains C_n or $\overline{F}$ contains $2K_3$. By induction on n, we have that F contains C_{n-1} or $\overline{F}$ contains $2K_3$. If $\overline{F}$ contains $2K_3$ then the proof is done. This concludes that F contains C_{n-1}. For $t = 2$, Lemma 4 now guarantees that we have a cycle C_n in F or a copy of $2K_3$ in $\overline{F}$. □

We are now ready to prove the second theorem.

Proof of Theorem 2. The graph $2K_{n-1} \cup K_{t-1}$ provides the lower bound $R(C_n, tK_3) \geq 2n + t - 2$.

In order to show the upper bound $R(C_n, tK_3) \leq 2n + t - 2$ we use induction on t. For $t = 2$, Lemma 6 gives $R(C_n, 2K_3) = 2n$ and hence the assertion holds for $n \geq g(2) = 19$. Let us assume that the assertion is true for $n \geq g(t-1)$, that is $R(C_n, (t-1)K_3) \leq 2n + t - 3$. We shall show that the theorem is also valid for $n \geq g(t)$. Let F be a graph of order $2n + t - 2$. We will show that F contains C_n or $\overline{F}$ contains tK_3. Since $2n + t - 2 > 2n - 1$, it follows that F contains C_n or $\overline{F}$ contains K_3. If F contains C_n then we are done. If $\overline{F}$ contains K_3 then the subgraph $F - \overline{K}_3$ of F has $2(n-1) + t - 3$ vertices. Note that, since $t \geq 2$, we have $n \geq g(t) > g(t-1) + 1$. By induction on t, the subgraph $F - \overline{K}_3$ contains C_{n-1} or the complement of $F - \overline{K}_3$ contains $(t-1)K_3$. If the complement of $F - \overline{K}_3$ contains $(t-1)K_3$ then we have a tK_3 in $\overline{F}$ and hence the proof is done. Thus we conclude that F contains C_{n-1}. Lemma 4 now implies that F contains C_n or $\overline{F}$ contains tK_3. The proof of Theorem 2 is now complete. □

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