On the Q-index and index of triangle-free quasi-tree graphs*

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Abstract

A connected graph G is called a quasi-tree graph, if there exists $v_0 \in V(G)$ such that $G - v_0$ is a tree. In this paper, among all triangle-free quasi-tree graphs of order n with $G-v_0$ being a tree and $d(v_0) = d_0$, we determine the maximal and the second maximal signless Laplacian spectral radii together with the corresponding extremal graphs. By an analogous manner, we obtained similar results on the spectral radius of triangle-free quasi-tree graphs.

AMS subject classifications: 05C50

Key words: quasi-tree graph; triangle-free; signless Laplacian; spectral radius

*Supported by the National Natural Science Foundation of China (Nos. 11171290, 11271315, 11201417) and the Natural Science Foundation of Jiangsu Province (BK20151295).

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1 Introduction

All graphs considered here are undirected and simple. Let $G = G[V(G), E(G)]$ be a graph of order n, $A(G)$ be the adjacency matrix of G and $D(G)$ be the diagonal matrix of degrees of G. The matrix $Q(G) = D(G) + A(G)$ is called the signless Laplacian matrix (or Q-matrix) of G. Since $A(G)$ and $Q(G)$ are symmetric, it follows that their eigenvalues are all real. As usual, we call the largest eigenvalue of $A(G)$ the spectral radius (or index) of G, denoted by $\rho(G)$, and the largest eigenvalue of $Q(G)$ the signless Laplacian spectral radius (or Q-index) of G, denoted by $q(G)$. For a connected graph G, $Q(G)$ is nonnegative and irreducible. By the Perron-Frobenius Theorem, $q(G)$ is simple and there is a unique positive unit eigenvector corresponding to it. We shall refer to such an eigenvector as the Perron vector of $Q(G)$.

The spectral Turán extremal problem is to determine the largest (or smallest) eigenvalue of a graph not containing a subgraph H. Nikiforov [9] proved a spectral extremal Turán theorem: let $\rho(G)$ be the largest eigenvalue of the adjacency matrix of G not containing a complete graph K_t of order t as a subgraph, then $\rho(G) \leq \rho(T_{n,t-1})$ with equality if and only if $G = T_{n,t-1}$. Further, Nikiforov explicitly advocated the study of spectral Turán problems in many publications and presented [12] a comprehensive survey on these topics.

The signless Laplacian eigenvalues of a graph have recently attracted more and more researchers' attention, see [1] and its references. Recently, Freitas, Nikiforov and Patuzzi [3] presented an even newer trend in spectral extremal graph theory as follows:

Problem A *Given a graph F, what is the maximum Q-index of a graph G of order n, with no subgraph isomorphic to F?*

This problem has been solved for several classes of forbidden subgraphs, see [2], [3], [5], [11], [13] and their references. In particular, in [3] it has been solved for forbidden cycles C_4 and C_5. For longer cycles, a general conjecture has been stated in [3] and was investigated in [10] and [13].

A connected graph G is called a quasi-tree graph, if there exists $v_0 \in V(G)$ such that $G - v_0$ is a tree. Liu and Lu [8] first gave the concept of a quasi-tree graph and determined the maximal and the second maximal spectral radii among all quasi-tree graphs of order n with $G - v_0$ being a tree and $d(v_0) = d_0$. Since then, quasi-tree graphs have been investigated by many authors. For example, Geng and Li [4] determined the quasi-tree graph and the quasi-unicyclic graph which have the maximal spectral radii among all the quasi-tree graphs and quasi-unicyclic graphs of order n with k pendant vertices, respectively. Li, Shiu and Chan [7] determined the bipartite quasi-tree graphs which have the maximum and the second large Laplacian spectral radii among all bipartite quasi-tree graphs of order n with $G-v_0$ being a tree and $d(v_0) = d_0$.

Denote by $\mathcal{Q}_t(n, d_0)$ the set of all triangle-free quasi-tree graphs of order n with $G - v_0$ being a tree and $d(v_0) = d_0$. Motivated by Problem A and the results on the quasi-tree graphs and the quasi-unicyclic graphs, in this paper we determine the maximal and the second maximal signless Laplacian spectral radii together with the corresponding extremal graphs among all quasi-tree graphs in $\mathcal{Q}_t(n, d_0)$. By an analogous manner, we obtained similar results on the spectral radius of triangle-free quasi-tree graphs.

2 Preliminaries

Denote by C_n and P_n the cycle and the path of order n, respectively. Let $G - xy$ denote the graph that arises from G by deleting the edge $xy \in E(G)$. Similarly, $G + xy$ is a graph that arises from G by adding an edge $xy \notin E(G)$, where $x, y \in V(G)$. For $v \in V(G)$, $N_G(v)$ denotes the set of all neighbors of vertex v in G, and $d(v) = |N_G(v)|$ denotes the degree of vertex v in G. Let $\Delta(G) = \Delta$ be the maximum degree of G. A pendant vertex of G is a vertex of degree 1.

Let $d_0 \geq 2$ and Q_{n,d_0}, depicted in Fig. 1, be a graph obtained from the star $K_{1,n-2}$ and an isolated vertex v_0 by adding d_0 edges joining v_0 to the pendent vertices of $K_{1,n-2}$, respectively. Clearly, $Q_{n,d_0} \in \mathcal{Q}_t(n, d_0)$. $Q^s_{n,d_0} \in \mathcal{Q}_t(n, d_0)$ is depicted in Fig. 2, where $0 \leq s \leq n - d_0 - 2$ and $Q^0_{n,d_0} = Q_{n,d_0}$. $Q^{s,t}_{k,l} \in \mathcal{Q}_t(n, d_0)$ is depicted in Fig. 3, where $0 \leq s \leq t \leq d_0$, $s + t = d_0$, $k \geq 0, l \geq 0$, $k + l = n - d_0 - 3$, and $Q^{0,d_0}_{0,n-d_0-3} = Q_{n,d_0}$.

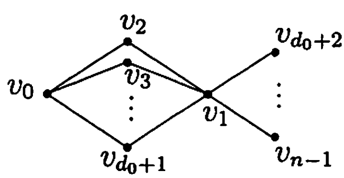

Fig. 1. Q_{n,d_0}

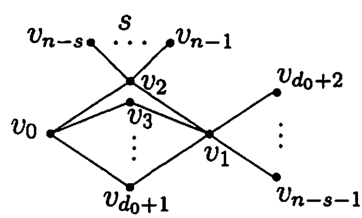

Fig. 2. Q^s_{n,d_0}

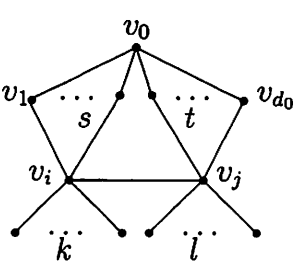

Fig. 3. $Q^{s,t}_{k,l}$

Lemma 2.1. *$q(Q_{n,d_0})$, $q(Q^{n-d_0-2}_{n,d_0})$, $q(Q^1_{n,d_0})$ and $q(Q^{0,d_0}_{1,n-d_0-4})$*

are the largest roots of the following polynomials respectively:

$$\begin{aligned}
f_{d_0}(x) &\triangleq x^3-(n+d_0+1)x^2+(d_0n+2n-2)x-d_0n,\\
g_{d_0}(x) &\triangleq x^3-(n+3)x^2-(d_0^2-nd_0+d_0-2n-4)x-2n,\\
h_{d_0}(x) &\triangleq x^4-(n+d_0+3)x^3+(2d_0+d_0n+5n-5)x^2\\
&\quad -(5n+4d_0n-5d_0-9)x+2d_0n,\\
r_{d_0}(x) &\triangleq x^5-(d_0+n+3)x^4+(2d_0+5n+d_0n-3)x^3\\
&\quad +(3d_0-7n-4d_0n+11)x^2\\
&\quad +(2n-3d_0+4d_0n-2)x-d_0n.
\end{aligned}$$

Proof. Let $V(Q_{n,d_0})=\{v_0,v_1,\dots,v_{n-1}\}$, and $X=(x_0,x_1,\dots,x_{n-1})^T$ be the Perron vector of $Q(Q_{n,d_0})$, where x_i corresponds to the vertex v_i $(0\le i\le n-1)$. By the symmetry of Q_{n,d_0}, we have $x_2=x_3=\cdots=x_{d_0+1}$, $x_{d_0+2}=\cdots=x_{n-1}$. From the eigenvalue equation $Q(Q_{n,d_0})X=q(Q_{n,d_0})X$, we have

$$\begin{aligned}
(q(Q_{n,d_0})-d_0)x_0 &= d_0x_2,\\
(q(Q_{n,d_0})-n+2)x_1 &= d_0x_2+(n-d_0-2)x_{n-1},\\
(q(Q_{n,d_0})-2)x_2 &= x_0+x_1,\\
(q(Q_{n,d_0})-1)x_{n-1} &= x_1.
\end{aligned}$$

Since $X=(x_0,x_1,\dots,x_{n-1})^T$ be an eigenvector of $q(Q_{n,d_0})$, then $X\neq 0$. This implies that

$$\begin{vmatrix}
q(Q_{n,d_0})-d_0 & 0 & -d_0 & 0\\
0 & q(Q_{n,d_0})-n+2 & -d_0 & -(n-d_0-2)\\
-1 & -1 & q(Q_{n,d_0})-2 & 0\\
0 & -1 & 0 & q(Q_{n,d_0})-1
\end{vmatrix}=0.$$

Therefore, $q(Q_{n,d_0})$ is the largest root of the equation

$$\begin{vmatrix}
x-d_0 & 0 & -d_0 & 0\\
0 & x-n+2 & -d_0 & -(n-d_0-2)\\
-1 & -1 & x-2 & 0\\
0 & -1 & 0 & x-1
\end{vmatrix}=0.$$

By an easy computation, we can obtain $q(Q_{n,d_0})$ is the largest root of the polynomial

$$f_{d_0}(x) = x^3 - (n + d_0 + 1)x^2 + (d_0 n + 2n - 2)x - d_0 n.$$

Similarly, we can obtain $q(Q_{n,d_0}^{n-d_0-2})$, $q(Q_{n,d_0}^{1})$ and $q(Q_{1,n-d_0-4}^{0,d_0})$ are the largest roots of the following polynomials respectively:

$$\begin{aligned} g_{d_0}(x) &\triangleq x^3 - (n+3)x^2 - (d_0^2 - nd_0 + d_0 - 2n - 4)x - 2n, \\ h_{d_0}(x) &\triangleq x^4 - (n + d_0 + 3)x^3 + (2d_0 + d_0 n + 5n - 5)x^2 \\ &\quad -(5n + 4d_0 n - 5d_0 - 9)x + 2d_0 n, \\ r_{d_0}(x) &\triangleq x^5 - (d_0 + n + 3)x^4 + (2d_0 + 5n + d_0 n - 3)x^3 \\ &\quad +(3d_0 - 7n - 4d_0 n + 11)x^2 \\ &\quad +(2n - 3d_0 + 4d_0 n - 2)x - d_0 n. \end{aligned}$$

This completes the proof. □

Lemma 2.2. *([6]) Let G be a connected graph and $q(G)$ be the spectral radius of $D(G)+A(G)$. Let u, v be two vertices of G and d_v be the degree of vertex v. Suppose $v_1, v_2, \ldots, v_s$ $(1 \leq s \leq d_v)$ are some vertices of $N_G(v) \setminus N_G(u)$ and $x = (x_1, x_2, \ldots, x_n)^T$ is the Perron vector of $D(G) + A(G)$, where x_i corresponds to the vertex v_i $(1 \leq i \leq n)$. Let G^* be the graph obtained from G by deleting the edges (v, v_i) and adding the edges (u, v_i) $(1 \leq i \leq s)$. If $x_u \geq x_v$ then $q(G) < q(G^*)$.*

As immediate consequences of Lemma 2.2, we have the following.

Lemma 2.3. *Let G be a connected graph and let $e = uv$ be a non-pendant edge of G with $N(u) \cap N(v) = \emptyset$. Let G^* be the graph obtained from G by deleting the edge uv, identifying u with v, and adding a pendant edge denoted still by e to $u(= v)$. Then $q(G) < q(G^*)$.*

Lemma 2.4. *Let G, G', G'' be connected, mutually disjoint graphs. Suppose that u, v are two vertices of G, u' is a vertex of G' and u'' is a vertex of G''. Let G_1 be the graph obtained from G, G', G'' by identifying, respectively, u with u' and v with u''. Let G_2 be the graph obtained from G, G', G'' by identifying vertices u, u', u''. Let G_3 be the graph obtained from G, G', G'' by identifying vertices v, u', u''. Then either $q(G_1) < q(G_2)$ or $q(G_1) < q(G_3)$.*

Lemma 2.5. *([1]) Let G be a connected graph, containing at least one edge. Then $q(G) \geq \Delta + 1$, with equality if and only if G is the star $K_{1,n-1}$.*

3 The Q-index of a triangle-free quasi-tree graph

When $d_0 = 1$, $\mathcal{Q}_t(n, 1)$ is the set of all trees of order n. Let S_n^2 be a tree obtained by attaching a pendant edge to a pendant vertex of the star $K_{1,n-2}$. It is well known that the star $K_{1,n-1}$ alone has the maximal and S_n^2 alone has the second maximal signless Laplacian spectral radius among the trees of order n. Next we assume that $d_0 \geq 2$.

Theorem 3.1. *Let $n \geq 4$, $d_0 \geq 2$, $G \in \mathcal{Q}_t(n, d_0)$. Then $q(G) \leq q(Q_{n,d_0})$ with equality if and only if $G = Q_{n,d_0}$, where $q(Q_{n,d_0})$ is the largest root of the equation*

$$x^3 - (n + d_0 + 1)x^2 + (d_0 n + 2n - 2)x - d_0 n = 0.$$

Proof. For $n = 4, 5$, by Matlab, it is easy to see that Theorem 3.1 holds. Next, we assume that $n \geq 6$, $d_0 \geq 2$. Choose $G \in \mathcal{Q}_t(n, d_0)$ such that $q(G)$ is as large as possible. Let $V(G) = \{ v_0, v_1, \ldots, v_{n-1} \}$, and $X = (x_0, x_1, \ldots, x_{n-1})^T$ be the Perron vector of $Q(G)$, where x_i corresponds to the vertex v_i $(0 \leq i \leq$

$n-1$). Assume that $G - v_0$ is a tree and $d(v_0) = d_0$. The vertices of G may be coloured such that v_0 is black, the vertices in $N_G(v_0)$ are red, the pendant vertices are green and the others are white. Since G is triangle-free, it follows that arbitrary two red vertices of G are nonadjacent.

Firstly, we show that no pair of white vertices are adjacent. Otherwise, let v_iv_j be an edge of G such that both v_i and v_j are white. Applying Lemma 2.3 to the edge v_iv_j, we get $G^* \in \mathcal{Q}_t(n, d_0)$ and $q(G) < q(G^*)$, a contradiction. Therefore no pair of white vertices are adjacent.

Secondly, we show that there is unique white vertex in G. Otherwise, assume that v_i, v_j are two white vertices of G. Then v_i, v_j are nonadjacent. Let

$$N(v_i) = \{ v_{i_1}, v_{i_2}, \ldots, v_{i_s} \}, \qquad N(v_j) = \{ v_{j_1}, v_{j_2}, \ldots, v_{j_t} \},$$

where $s \geq 2$ and $t \geq 2$. Assume that v_{i_1}, v_{j_1} are in the unique path $v_i - v_j$ of $G - v_0$. If $x_i \geq x_j$, let

$$G^* = G - \{ v_jv_{j_2}, \ldots, v_jv_{j_t} \} + \{ v_iv_{j_2}, \ldots, v_iv_{j_t} \}.$$

If $x_i < x_j$, let

$$G^* = G - \{ v_iv_{i_2}, \ldots, v_iv_{i_s} \} + \{ v_jv_{i_2}, \ldots, v_jv_{i_s} \}.$$

Then $G^* \in \mathcal{Q}_t(n, d_0)$. By Lemma 2.2, we have $q(G) < q(G^*)$, a contradiction. Therefore there is unique white vertex in G.

Thirdly, by Lemma 2.4, we claim that all green vertices are attached at the same red vertex or white vertex of G.

Combining the above arguments, we have $G = Q_{n,d_0}$ or $G = Q_{n,d_0}^{n-d_0-2}$. When $d_0 = 2$ or $d_0 = n-2$, $Q_{n,d_0} = Q_{n,d_0}^{n-d_0-2}$. Next we show that $q(Q_{n,d_0}^{n-d_0-2}) < q(Q_{n,d_0})$ for $3 \leq d_0 \leq n-3$.

Since $Q_{n,d_0}^{n-d_0-2}$ contains K_{2,d_0} as a proper subgraph, it follows that

$$q(Q_{n,d_0}^{n-d_0-2}) > q(K_{2,d_0}) = d_0 + 2.$$

By Lemma 2.1, we have $q(Q_{n,d_0})$ and $q(Q_{n,d_0}^{n-d_0-2})$ are the largest roots of the polynomials $f_{d_0}(x)$ and $g_{d_0}(x)$ respectively. Since

$$g_{d_0}(x) - f_{d_0}(x) = (d_0 - 2)(x^2 - (d_0 + 3)x + n) > 0$$

for $x \geq q(Q_{n,d_0}^{n-d_0-2}) > d_0 + 2$, it follows that $q(Q_{n,d_0}^{n-d_0-2}) < q(Q_{n,d_0})$ for $3 \leq d_0 \leq n - 3$.

Combining the above arguments, we have $G = Q_{n,d_0}$ and $q(Q_{n,d_0})$ is the largest root of the equation $x^3 - (n + d_0 + 1)x^2 + (d_0 n + 2n - 2)x - d_0 n = 0$. The proof is completed. □

Theorem 3.2. *Let $n \geq 6$, $d_0 \geq 2$, $G \in \mathcal{Q}_t(n, d_0) \setminus \{ Q_{n,d_0} \}$. Then $q(G) \leq q(Q_{n,d_0}^1)$ with equality if and only if $G = Q_{n,d_0}^1$ or $Q_{n,3}^{n-5}$. Moreover $q(Q_{n,3}^{n-5}) = q(Q_{n,3}^1)$, and $q(Q_{n,d_0}^1)$ is the largest root of the equation*

$$\begin{aligned} &x^4 - (n + d_0 + 3)x^3 + (2d_0 + d_0 n + 5n - 5)x^2 \\ &-(5n + 4d_0 n - 5d_0 - 9)x + 2d_0 n = 0. \end{aligned}$$

Proof. Choose $G \in \mathcal{Q}_t(n, d_0) \setminus \{ Q_{n,d_0} \}$ such that $q(G)$ is as large as possible. Let $V(G) = \{ v_0, v_1, \ldots, v_{n-1} \}$, and $X = (x_0, x_1, \ldots, x_{n-1})^T$ be the Perron vector of $Q(G)$, where x_i corresponds to the vertex v_i $(0 \leq i \leq n-1)$. Assume that $G - v_0$ is a tree and $d(v_0) = d_0$. The vertices of G may be coloured so that v_0 is black, the vertices in $N_G(v_0)$ are red, the pendant vertices are green and the others are white. Since G is triangle-free, it follows that arbitrary two red vertices of G are nonadjacent.

Firstly, we show that no pair of white vertices are adjacent. Otherwise, let $v_i v_j$ be an edge such that both v_i and v_j are white. Applying Lemma 2.3 to the edge $v_i v_j$, we get a graph $G^* \in \mathcal{Q}_t(n, d_0)$ with $q(G) < q(G^*)$. If $G^* \neq Q_{n,d_0}$, then $G^* \in \mathcal{Q}_t(n, d_0) \setminus \{ Q_{n,d_0} \}$, a contradiction. If $G^* = Q_{n,d_0}$, then $G = Q_{k,l}^{s,t}$, depicted in Fig. 3, where $0 \leq s \leq t \leq d_0$, $s + t = d_0$, $k \geq 0, l \geq 0$, $k + l = n - d_0 - 3$, $s + k \geq 1$.

If $s+k \geq 2$ and $t+l \geq 2$, applying Lemma 2.2 to the vertices v_i and v_j, we can get $G^{**} = Q_{n,d_0}^{s',t'} \in \mathcal{Q}_t(n, d_0) \setminus \{ Q_{n,d_0} \}$ such that $q(G) < q(G^{**})$, a contradiction.

If $s+k \geq 2$, $t = 1$ and $l = 0$, then $s = 1$ and $d_0 = 2$. Applying Lemma 2.3 to the vertices v_j and v_2, we can get $G^{**} = Q_{n,d_0}^{1} \in \mathcal{Q}_t(n, d_0) \setminus \{ Q_{n,d_0} \}$ such that $q(G) < q(G^{**})$, a contradiction.

If $s = 1$ and $k = 0$, applying Lemma 2.3 to the edge $v_1 v_i$, we get $q(G) < q(Q_{n,d_0}^{1})$, a contradiction.

If $s = 0$ and $k = 1$, by Lemma 2.1, we have $q(Q_{n,d_0}^{1})$ and $q(Q_{1,n-d_0-4}^{0,d_0})$ are the largest roots of the polynomials $h_{d_0}(x)$ and $r_{d_0}(x)$ respectively. It follows that

$$r_{d_0}(x) - xh_{d_0}(x) = 2x^3 + (2-2n-2d_0)x^2 + (2n-3d_0+2d_0n-2)x - d_0n$$

$$= 2x(x - d_0 - 1)(x - n + 2) + (d_0 + 2)x - d_0n > 0$$

for $x \geq q(Q_{1,n-d_0-4}^{0,d_0}) > n - 2 \geq d_0 + 2$. Therefore $q(G) = q(Q_{1,n-d_0-4}^{0,d_0}) < q(Q_{n,d_0}^{1})$, a contradiction. Therefore no pair of white vertices are adjacent.

Secondly, similarly to the proof of Theorem 3.1, we can prove that in G there is unique white vertex and that there is unique red vertex with green vertices attached.

Combining the above arguments, we have $G = Q_{n,d_0}^{s}$ with $1 \leq s \leq n - d_0 - 2$. If $d_0 = n - 3$, then $s = 1$. Namely $G = Q_{n,n-3}^{1}$. Next we assume that $d_0 \leq n - 4$ and show that $G = Q_{n,d_0}^{1}$ or $Q_{n,3}^{n-5}$.

For $d_0 = 2$, since $G \neq Q_{n,2}$, it follows that $d(v_1) \geq 3$ and $d(v_2) \geq 3$. If $G \neq Q_{n,2}^{1}$, then $d(v_1) \geq 4$ and $d(v_2) \geq 4$. Applying Lemma 2.4 to the vertices v_1 and v_2, we can get $G^* = Q_{n,2}^{1}$ such that $q(G) < q(G^*)$, a contradiction. Therefore $G = Q_{n,2}^{1}$.

For $d_0 \geq 3$ and $d(v_1) \geq d_0 + 1$, we show that $G = Q_{n,d_0}^{1}$. Otherwise, assume that $G = Q_{n,d_0}^{s}$ with $2 \leq s \leq n - d_0 - 3$. If

$x_1 \geq x_2$, let

$$G^* = G - v_2 v_{n-s} + v_1 v_{n-s}.$$

If $x_1 < x_2$, let

$$G^* = G - v_1 v_{n-s-1} + v_2 v_{n-s-1}.$$

Then $G^* \in \mathcal{Q}_t(n, d_0) \backslash \{ Q_{n,d_0} \}$. By Lemma 2.2, we have $q(G) < q(G^*)$, a contradiction. Therefore $G = Q^1_{n,d_0}$.

For $d_0 \geq 3$ and $d(v_1) = d_0$, we have $G = Q^{n-d_0-2}_{n,d_0}$, By Lemma 2.1, $q(Q^1_{n,d_0})$ and $q(Q^{n-d_0-2}_{n,d_0})$ are the largest roots of the polynomials $h_{d_0}(x)$ and $g_{d_0}(x)$ respectively. It follows that

$$(x - d_0)g_{d_0}(x) - h_{d_0}(x) = (d_0 - 3)(n - d_0 - 3)x(x - d_0 - 1).$$

When $d_0 = 3$, we have $(x - 3)g_3(x) = h_3(x)$. Since $q(Q^1_{n,3}) > q(K_{2,3}) = 5$ and $q(Q^{n-5}_{n,3}) > q(K_{2,3}) = 5$, it follows that $q(Q^{n-5}_{n,3}) = q(Q^1_{n,3})$. When $d_0 = n - 3$, we have $Q^{n-d_0-2}_{n,d_0} = Q^1_{n,d_0}$. When $3 < d_0 < n - 3$, for $x \geq q(Q^{n-d_0-2}_{n,d_0}) > q(K_{2,d_0}) = d_0 + 2$, we have

$$(x - d_0)g_{d_0}(x) - h_{d_0}(x) = (d_0 - 3)(n - d_0 - 3)x(x - d_0 - 1) > 0.$$

It follows that $h_{d_0}(q(Q^{n-d_0-2}_{n,d_0})) < 0$. This implies that $q(Q^{n-d_0-2}_{n,d_0}) < q(Q^1_{n,d_0})$.

Combining the above arguments, we have $q(Q^{n-5}_{n,3}) = q(Q^1_{n,3})$ and $G = Q^1_{n,d_0}$ or $Q^{n-5}_{n,3}$. By Lemma 2.1, $q(Q^1_{n,d_0})$ is the largest root of the equation

$$\begin{aligned} &x^4 - (n + d_0 + 3)x^3 + (2d_0 + d_0 n + 5n - 5)x^2 \\ &-(5n + 4d_0 n - 5d_0 - 9)x + 2d_0 n = 0. \end{aligned}$$

This completes the proof. □

Remark 3.3. *For $n = 5$ and $d_0 = 2$, $\mathcal{Q}_t(5, 2) \backslash \{ Q_{5,2} \} = \{ C_5 \}$. For $n = 5$ and $d_0 = 3$, $\mathcal{Q}_t(5, 3) \setminus \{ Q_{5,3} \} = \emptyset$.*

Note that if we add an edge e to a connected graph G, then $q(G+e) > q(G)$ as $Q(G)$ of a connected graph is irreducible. So we have the following result.

Lemma 3.4. *$q(Q_{n,d_0+1}) > q(Q_{n,d_0})$ for $2 \le d_0 \le n-3$.*

By Theorem 3.1 and Lemma 3.4, we have the following result.

Theorem 3.5. *Let $n \ge 4$ and G be a triangle-free quasi-tree graph of order n. Then $q(G) \le q(Q_{n,n-2})$ with equality if and only if $G = Q_{n,n-2}$, where $q(Q_{n,n-2})$ is the largest root of the equation $x^3 - (2n-1)x^2 + (n^2-2)x - n^2 + 2n = 0$.*

4 The index of a triangle-free quasi-tree graph

By an analogous manner as above, we can obtain the similar results on the spectral radius of triangle-free quasi-tree graphs.

Theorem 4.1. *Let $n \ge 4$, $d_0 \ge 2$, $G \in \mathcal{Q}_t(n, d_0)$. Then*

$$\rho(G) \le \frac{1}{2}\sqrt{2n + 2d_0 + 2\sqrt{5d_0^2 - 2nd_0 + 4d_0 + n^2 - 4n + 4} - 4}\,,$$

with equality if and only if $G = Q_{n,d_0}$.

Theorem 4.2. *Let $n \ge 6$, $d_0 \ge 2$, $G \in \mathcal{Q}_t(n, d_0) \setminus \{Q_{n,d_0}\}$. Then $\rho(G) \le \rho(Q^1_{n,d_0})$, with equality if and only if $G = Q^1_{n,d_0}$, where $\rho(Q^1_{n,d_0})$ is the largest root of the equation $x^6 - (n + d_0 - 2)x^4 + (n - 2d_0 + nd_0 - d_0^2 - 5)x^2 + 2d_0 + n - d_0 n + d_0^2 - 3 = 0$.*

Theorem 4.3. *Let $n \ge 4$ and G be a triangle-free quasi-tree graph of order n. Then $\rho(G) \le \sqrt{2n-4}$, with equality if and only if $G = Q_{n,n-2}$.*

Acknowledgment

We are grateful to the anonymous referees for valuable suggestions which result in an improvement of the original manuscript.

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