Forbidden subgraph conditions for Hamilton cycles in implicit claw-heavy graphs*

Junqing Cai, Yuzhong Zhang

School of Management, Qufu Normal University, Rizhao 276826, P.R. China

E-mails: caijq09@163.com, yuzhongrz@163.com

Abstract

Let $id(v)$ denote the implicit degree of a vertex v in a graph G. We define G to be implicit claw-heavy if every induced claw of G has a pair of nonadjacent vertices such that their implicit degree sum is more than or equal to $|V(G)|$. In this paper, we show that an implicit claw-heavy graph G is hamiltonian if we impose certain additional conditions on G involving numbers of common neighbors of some specific pair of nonadjacent vertices, or forbidden induced subgraphs. Our results extend two previous theorems of Chen et al. [B. Chen, S. Zhang and S. Qiao, Hamilton cycles in claw-heavy graphs, Discrete Math., 309 (2009) 2015-2019.] on the existence of Hamilton cycles in claw-heavy graphs.

Keywords: Implicit degree; Hamilton cycle; Implicit claw-heavy

1 Introduction

In this paper, we consider only undirected, finite and simple graphs. Notation and terminology not defined here can be found in [3].

For a subset S of $V(G)$, we use $\langle S\rangle$ to denote the subgraph of G induced by S. A graph T is called an *induced subgraph* of G if $T = \langle S\rangle$ for some $S \subseteq V(G)$. An induced subgraph of G with vertex set $\{u, v, w, x\}$ and edge set $\{uv, uw, ux\}$ is called a *claw* of G, with center u and end vertices v, w, x. An induced subgraph of G isomorphic to a claw with one additional edge is called a *modified claw.*

* This work is supported by Postdoctoral Science Foundation of China (No. 2015M571999), Natural Science Foundation of China (No. 11501322, 11426145, 11426143), Natural Science Foundation of Shandong Province (No. ZR2014AP002, ZR2015GZ009, 2014ZRB019TM), Scientific Research Foundation for Doctors in Qufu Normal University (No. 2012015) and Higher Educational Science and Technology Program of Shandong Province (No. J13LI09).

Let G be a graph and S be a subgraph of G. For a vertex $u \in V(G)$, define $N_S(u) = \{v \in V(S) : uv \in E(G)\}$. The degree of u in S is denoted by $d_S(u) = |N_S(u)|$. If $S = G$, we can use $N(u)$ and $d(u)$ in place of $N_G(u)$ and $d_G(u)$, respectively. For two vertices u and v in G, we use $d(u, v)$ to denote the distance between u and v in G.

In 1989, Zhu, Li and Deng [12] found that though some vertices may have small degrees, we can use some large degree vertices to replace small degree vertices in the right position considered in the proofs, so that we may construct a longer cycle. This idea leads to the definition of implicit degree. We use $N_2(v)$ to denote the vertices which are at distance 2 from v in G.

Definition 1. *([12]) Let v be a vertex of a graph G and $d(v) = l + 1$. Set $M_2 = \max\{d(u) : u \in N_2(v)\}$. If $N_2(v) \neq \emptyset$ and $d(v) \geq 2$, then let $d_1 \leq d_2 \leq \ldots \leq d_l \leq d_{l+1} \leq \ldots$ be the degree sequence of vertices of $N(v) \cup N_2(v)$. Define*

$$d^*(v) = \begin{cases} d_{l+1}, & \text{if} \quad d_{l+1} > M_2; \\ d_l, & \text{otherwise}. \end{cases}$$

Then the implicit degree of v is defined as $id(v) = \max\{d(v), d^*(v)\}$. If $N_2(v) = \emptyset$ or $d(v) \leq 1$, then define $id(v) = d(v)$.

Clearly, $id(v) \geq d(v)$ for every vertex v from the definition of implicit degree. A vertex v in a graph G of order n is called *heavy* (*implicit-heavy*) if $d(v) \geq n/2$ ($id(v) \geq n/2$). If v is not heavy (not implicit-heavy), we call it *light* (*implicit-light*). A claw of G is called *2-heavy* (*implicit 2-heavy*) if at least two of its end vertices are heavy (implicit-heavy). And G is called *2-heavy* (*implicit 2-heavy*) if all its claws are 2-heavy (implicit 2-heavy). G is called *claw-heavy* (*implicit claw-heavy*) if every induced claw of G has a pair of nonadjacent vertices u and v such that $d(u) + d(v) \geq n$ ($id(u) + id(v) \geq n$). Clearly, every 2-heavy (implicit 2-heavy) graph is claw-heavy (implicit claw-heavy), and every 2-heavy (claw-heavy) graph is implicit 2-heavy (implicit claw-heavy), but every claw-heavy (implicit claw-heavy) graph is not necessarily 2-heavy (implicit 2-heavy).

Let T be a graph, we say G is *T-free* if G does not contain an induced subgraph isomorphic to T. Note that every claw-free graph is 2-heavy. We use D, H and P_7 (a path on 7 vertices) to denote the graphs in Fig.1.

A cycle in a graph G is called a *Hamilton cycle* if it contains all vertices of G. And G is called *hamiltonian* if it contains a Hamilton cycle. Degree conditions and forbidden subgraph conditions are two important types of sufficient conditions for the existence of Hamilton cycles in graphs. The following two results are two examples of these two types of conditions, respectively.

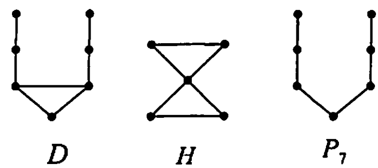

Fig.1

Theorem 1. *([7]) Let G be a graph of order $n \geq 3$. If $d(u) \geq n/2$ for each vertex $u \in V(G)$, then G is hamiltonian.*

Theorem 2. *([11]) Let G be a 2-connected graph of order $n \geq 3$. If G is claw-free and $|N(u) \cap N(v)| \geq 2$ for every pair of vertices u and v with $d(u,v) = 2$, then G is hamiltonian.*

Combining the above two types of conditions, Broersma et al. [4] gave a common generalization of Theorem 2.

Theorem 3. *([4]) Let G be a 2-connected graph of order $n \geq 3$. If G is 2-heavy and $|N(u) \cap N(v)| \geq 2$ for every pair of vertices u and v with $d(u,v) = 2$ and $\max\{d(u), d(v)\} < n/2$, then G is hamiltonian.*

In 2009, Chen et al. [6] relaxed 2-heavy in Theorem 3 to claw-heavy, and got the following result.

Theorem 4. *([6]) Let G be a 2-connected graph of order $n \geq 3$. If G is claw-heavy and $|N(u) \cap N(v)| \geq 2$ for every pair of vertices u and v with $d(u,v) = 2$ and $\max\{d(u), d(v)\} < n/2$, then G is hamiltonian.*

Our first objective in this paper is to prove that we can use implicit claw-heavy in place of claw-heavy in Theorem 4.

Theorem 5. *Let G be a 2-connected graph of order $n \geq 3$. If G is implicit claw-heavy and $|N(u) \cap N(v)| \geq 2$ for every pair of vertices u and v with $d(u,v) = 2$ and $\max\{id(u), id(v)\} < n/2$, then G is hamiltonian.*

There are many results on the existence of Hamilton cycles for claw-free graphs, the following two are known.

Theorem 6. *([5]) Let G be a 2-connected graph. If G is claw-free, P_7-free and D-free, then G is hamiltonian.*

Theorem 7. *([8]) Let G be a 2-connected graph. If G is claw-free, P_7-free and H-free, then G is hamiltonian.*

In [4], the authors extended Theorem 6 and Theorem 7 to the class of 2-heavy graphs.

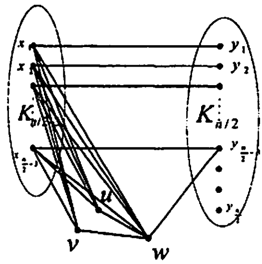

Fig.2

Theorem 8. *([4]) Let G be a 2-connected graph of order $n \geq 3$. If G is 2-heavy, and moreover P_7-free and D-free, or P_7-free and H-free, then G is hamiltonian.*

In 2009, Chen et al. [6] relaxed 2-heavy in Theorem 8 to claw-heavy, and got the following result.

Theorem 9. *([6]) Let G be a 2-connected graph. If G is claw-heavy, and moreover P_7-free and D-free, or P_7-free and H-free, then G is hamiltonian.*

Our second objective in this paper is to prove that we can use implicit claw-heavy in place of claw-heavy in Theorem 9.

Theorem 10. *Let G be a 2-connected graph of order $n \geq 3$. If G is implicit claw-heavy, and moreover P_7-free and D-free, or P_7-free and H-free, then G is hamiltonian.*

Remark 1. The graph in Fig.2 shows that our results in Theorem 5 and Theorem 10 do strengthen those in Theorem 4 and Theorem 9, respectively. Let $n \geq 12$ be an even integer and $K_{n/2-3} \cup K_{n/2}$ denote the union of two complete graphs $K_{n/2-3}$ and $K_{n/2}$. And let $V(K_{n/2-3}) = \{x_1, x_2, \ldots, x_{n/2-3}\}$ and $V(K_{n/2}) = \{y_1, y_2, \ldots, y_{n/2}\}$. We construct a graph G with $V(G) = V(K_{n/2-3} \cup K_{n/2}) \cup \{u, v, w\}$ and $E(G) = E(K_{n/2-3} \cup K_{n/2}) \cup \{x_i y_i : i = 1, 2, \ldots, n/2-3\} \cup \{uw, vw, wy_{n/2-3}\} \cup \{ux_i, vx_i, wx_i : i = 1, 2, \ldots, n/2-3\}$. It is easy to see that G is a hamiltonian graph not satisfying the conditions of Theorem 4 or Theorem 9. But since $id(u) = id(v) = n/2$ and $id(y_{n/2-3}) \geq d(y_{n/2-3}) = n/2+1$, G satisfies the conditions of Theorem 5 and Theorem 10.

Remark 2. It is clear that every P_6-free graph is also $\{P_7, D\}$-free. Thus the result in [9] (Let G be a 2-connected graph of order $n \geq 3$. If G is implicit claw-heavy and P_6-free, then G is hamiltonian.) is the corollary of our result in Theorem 10.

2 Lemmas

For a cycle C in G with a given orientation and a vertex x in C, x^+ and x^- denote the successor and the predecessor of x in C, respectively. We define $x^{+2} = (x^+)^+$ and $x^{-2} = (x^-)^-$. And for any $I \subseteq V(C)$, let $I^- = \{x : x^+ \in I\}$ and $I^+ = \{x : x^- \in I\}$. For two vertices $x, y \in C$, xCy denotes the subpath of C from x to y. We use $y\bar{C}x$ for the path from y to x in the reversed direction of C. A similar notation is used for paths.

A cycle C is called *implicit-heavy* if it contains all implicit-heavy vertices of G; it is called *extendable* if there exists a longer cycle in G containing all vertices of C. Our proofs of Theorem 5 and Theorem 10 are based on the following lemmas. The first lemma is implicit in the work of Li, Ning and Cai [10].

Lemma 1. *([10]) Every 2-connected graph contains an implicit-heavy cycle.*

Lemma 2. *([1]) Let G be a 2-connected nonhamiltonian graph of order $n \geq 3$ and C be a nonextendable cycle of G. If P is a path of G connecting x and y such that $V(C) \subset V(P)$, then $xy \notin E(G)$ and $d(x) + d(y) < n$.*

Lemma 3. *([10]) Let G be a 2-connected graph, $P = x_1x_2 \ldots x_p$ with $x_1 = x$ and $x_p = y$ be a path connecting x and y in G and $xy \notin E(G)$.*
(i) If $d(u) < id(x)$ for each vertex $u \in N_{G-V(P)}(x) \cup \{x\}$, then there exists a vertex $x_s \in N_P(x)^-$ such that $d(x_s) \geq id(x)$.
(ii) If $d(v) < id(y)$ for each vertex $v \in N_{G-V(P)}(y) \cup \{y\}$, then there exists a vertex $x_t \in N_P(y)^+$ such that $d(x_t) \geq id(y)$.

Lemma 4. *Let G be a 2-connected nonhamiltonian graph of order $n \geq 3$ and C be a nonextendable cycle of G. If P is a path connecting x and y in G such that $V(C) \subset V(P)$, then $xy \notin E(G)$ and $id(x) + id(y) < n$.*

Proof. Clearly, $xy \notin E(G)$. Suppose to the contrary that $id(x)+id(y) \geq n$. For convenience, let $P = x_1x_2 \ldots x_p$ with $x_1 = x$ and $x_p = y$. Set $R = G - V(P)$. By the choice of C, G has no cycle containing all vertices of P. In particular, x and y have no common neighbors in R. By Lemma 2, we can assume that $d(x) < id(x)$. If there exists a vertex $u \in N_R(x)$ such that $d(u) \geq id(x)$, then $P_1 = ux_1x_2 \ldots x_p$ is a path of G with $V(P) \subseteq V(P_1)$. If there exists no vertex $u \in N_R(x)$ such that $d(u) \geq id(x)$, then by Lemma 3 (i), there exists a vertex $x_s \in N_P(x)^-$ such that $d(x_s) \geq id(x)$. Thus $P_2 = x_sx_{s-1} \ldots x_1x_{s+1}x_{s+2} \ldots x_p$ is a path of G with $V(P) \subseteq V(P_2)$. By similar argument to the path P_1 or P_2, we may obtain a path P_3 connecting v and w such that $V(P) \subseteq V(P_3)$, $d(v) \geq id(x)$ and $d(w) \geq id(y)$. So $d(v) + d(w) \geq id(x) + id(y) \geq n$. But since $V(C) \subset V(P) \subseteq V(P_3)$, by Lemma 2, $d(v) + d(w) < n$, a contradiction. □

Lemma 5. *Let G be a 2-connected graph of order n and $x, y \in V(G)$ be two nonadjacent vertices such that $id(x) + id(y) \geq n$. If $G + xy$ has a cycle C, then G has a cycle containing all vertices of C.*

Proof. Assume G does not have a cycle containing all vertices of C. Then G has a path P from x to y containing all vertices of C. Clearly, $xy \notin E(G)$, x and y have no common neighbors in $V(G) \setminus V(P)$. By similar argument as in Lemma 4, there exists a path P' from u to v such that $V(P) \subseteq V(P')$, $d(u) \geq id(x)$ and $d(v) \geq id(y)$. Thus, $d(u) + d(v) \geq id(x) + id(y) \geq n$.

For convenience, let $P' = x_1x_2 \ldots x_p$ ($x_1 = u$ and $x_p = v$) and $R = G - V(P')$. Since $V(C) \subseteq V(P')$ and G has no cycle containing all vertices of C, we can get that $uv \notin E(G)$ and $N_R(u) \cap N_R(v) = \emptyset$. Noting that $d(u) + d(v) \geq n$, then $d_{P'}(u) + d_{P'}(v) \geq |V(P')|$. There must exist a vertex $x_i \in V(P')$ such that $ux_{i+1} \in E(G)$ and $vx_i \in E(G)$. Thus, $C' = uP'x_iv\bar{P}'x_{i+1}u$ is a cycle containing all vertices of C, a contradiction. □

Lemma 6. *Let C be a nonextendable cycle in a 2-connected graph G of order n, S be a component of $G - V(C)$, and A be the set of neighbors of S on C. Then*
(a) $A \cap A^- = \emptyset, A \cap A^+ = \emptyset$, and A^- and A^+ are independent sets,
(b) Each pair of vertices from A^- or A^+ has implicit degree sum smaller than n.

Proof. The proof of (a) can be found in [2].

(b) We will prove by contradiction. Suppose without loss of generality, that there is a pair of vertices from A^+ has implicit degree sum at least n. For convenience, we give C a clockwise orientation. Let $\{x_1, x_2, \ldots, x_k\}$ be the neighbors of S on C and let $x_1, x_2, \ldots, x_k$ occur in this order along C. We may assume without loss of generality that $id(x_1^+) + id(x_2^+) \geq n$. Let P be a path connecting x_1 and x_2 such that $|V(P) \cap V(S)|$ is as large as possible and $V(P) \cap V(C) = \{x_1, x_2\}$. Then $|V(P)| \geq 3$. We orient P from x_1 to x_2. Set $P' = x_1^+Cx_2\bar{P}x_1\bar{C}x_2^+$. Clearly, $V(C) \subset V(P')$. Then by Lemma 4, we have $id(x_1^+) + id(x_2^+) < n$, a contradiction. □

3 Proofs of Theorem 5 and Theorem 10

Proof of Theorem 5. Suppose to the contrary that G is not hamiltonian. Then by Lemma 1, G contains an implicit-heavy cycle. Let C be a longest implicit-heavy cycle and give C a clockwise orientation. Then $V(G) \setminus V(C) \neq \emptyset$. Since G is 2-connected, there is a path $P = x_1u_1u_2 \ldots u_rx_2$ connecting two vertices $x_1, x_2 \in V(C)$ internally disjoint with C and such that $|V(P)| \geq 3$.

By the choice of C, all internal vertices on P are implicit-light. Since $P' = u_k\bar{P}x_1\bar{C}x_1^+$ and $P'' = u_k\bar{P}x_1Cx_1^-$ are two paths such that $V(C) \subset$

$V(P')$ and $V(C) \subset V(P'')$ for every vertex $u_k \in \{u_1, u_2, \dots, u_r\}$, we have $u_k x_1^+, u_k x_1^- \notin E(G)$, $id(u_k) + id(x_1^+) < n$ and $id(u_k) + id(x_1^-) < n$ for every vertex $u_k \in \{u_1, u_2, \dots, u_r\}$ by Lemma 4. Similarly, $u_k x_2^+, u_k x_2^- \notin E(G)$, $id(u_k) + id(x_2^+) < n$ and $id(u_k) + id(x_2^-) < n$ for every vertex $u_k \in \{u_1, u_2, \dots, u_r\}$. By Lemma 6 (b), we have $id(x_1^+) + id(x_2^+) < n$ and $id(x_1^-) + id(x_2^-) < n$. This implies that $id(x_1^-) + id(x_1^+) < n$ or $id(x_2^-) + id(x_2^+) < n$. Since G is implicit claw-heavy, $\{x_1, u_1, x_1^-, x_1^+\}$ induces a modified claw or $\{x_2, u_r, x_2^-, x_2^+\}$ induces a modified claw. Without loss of generality, we may assume that $\{x_1, u_1, x_1^-, x_1^+\}$ induces a modified claw, $x_1^- x_1^+ \in E(G)$ and $id(x_1^-)+id(x_1^+) < n$. Here we may assume $id(x_1^+) < n/2$.

Now we have $d(u_1, x_1^+) = 2$ and $\max\{id(u_1), id(x_1^+)\} < n/2$. By the condition of Theorem 5, we have $|N(u_1) \cap N(x_1^+)| \geq 2$. Thus, there is a vertex $x \in (N(u_1) \cap N(x_1^+)) \setminus \{x_1\}$. By the choice of C, it easy to see that $x \in V(C)$. From Lemma 6, we can obtain that $\{x, u_1, x_1^+, x^+\}$ induces a claw and $id(x_1^+) + id(x^+) < n$. Since $P_1 = x^+ C x u_1$ is a path with $V(C) \subset V(P_1)$, $id(x^+) + id(u_1) < n$ by Lemma 4. Noting that $id(x_1^+) + id(u_1) < n$, then $\{x, u_1, x_1^+, x^+\}$ induce a claw with no pair of nonadjacent vertices having implicit degree sum more than or equal to n, this contradicts the condition of Theorem 5. □

Proof of Theorem 10. Suppose to the contrary that G is not hamiltonian. By Lemma 1, G contains an implicit-heavy cycle. Let C be a longest implicit-heavy cycle and give C a clockwise orientation. Then $V(G) \setminus V(C) \neq \emptyset$. Since G is 2-connected, there exists a path P connecting two vertices $x_1 \in V(C)$ and $x_2 \in V(C)$ internally disjoint with C and such that $|V(P)| \geq 3$. Let $P = x_1 u_1 u_2 \dots u_r x_2$ be such a path of minimum length. Then P is an induced path unless $x_1 x_2 \in E(G)$.

By similar argument as in the proof of Theorem 5, we have the following claim.

Claim 1. $u_k x_i^+, u_k x_i^- \notin E(G)$, $id(u_k)+id(x_i^+) < n$ and $id(u_k)+id(x_i^-) < n$ for every $u_k \in \{u_1, u_2, \dots, u_r\}$ and $i = 1, 2$.

Claim 2. Either $x_1^- x_1^+ \in E(G)$ or $x_2^- x_2^+ \in E(G)$.

Proof. Suppose to the contrary that $x_1^- x_1^+ \notin E(G)$ and $x_2^- x_2^+ \notin E(G)$. Since G is implicit claw-heavy, we have $id(x_1^-) + id(x_1^+) \geq n$ and $id(x_2^-) + id(x_2^+) \geq n$ by Claim 1. This implies that $id(x_1^-) + id(x_2^-) \geq n$ or $id(x_1^+) + id(x_2^+) \geq n$. But by Lemma 6 (b), we have $id(x_1^-) + id(x_2^-) < n$ and $id(x_1^+) + id(x_2^+) < n$, a contradiction. □

Claim 3. There is some vertex in $x_i^+ C x_{3-i}^-$ not adjacent to x_i in G for

$i = 1, 2$. In particular, $x_i x_{3-i}^-, x_i x_{3-i}^+ \notin E(G)$.

Proof. Suppose without loss of generality that $x_1 x_2^- \in E(G)$. By Lemma 6, we have $x_1^- x_2^- \notin E(G)$ and $id(x_1^-) + id(x_2^-) < n$. By Claim 1, we have $u_1 x_i^- \notin E(G)$ and $id(u_1) + id(x_i^-) < n$, $i = 1, 2$. Then $\{x_1, u_1, x_1^-, x_2^-\}$ induces a claw with no pair of nonadjacent vertices such that their implicit degree sum is at least n, a contradiction. □

By Claim 3, there is a vertex in $x_i^+ C x_{3-i}^-$ not adjacent to x_i for $i = 1, 2$. Let y_i be the first vertex in $x_i^+ C x_{3-i}^-$ not adjacent to x_i for $i = 1, 2$. By Claim 2, without loss of generality, we may assume $x_1^- x_1^+ \in E(G)$. We will distinguish the following two cases.

Case 1. $x_2^- x_2^+ \in E(G)$.

Let u be a vertex in $V(P) \setminus \{x_1, x_2\}$ and let z_i be an arbitrary vertex in $x_i^+ C y_i$, $i = 1, 2$.

Claim 4. $uz_1, uz_2, z_1 x_2, z_2 x_1, z_1 z_2 \notin E(G)$.

Proof. If $uz_1 \in E(G)$, by Claim 1, $z_1 \neq x_1^+$, then $x_1 P u z_1 C x_1^- x_1^+ C z_1^- x_1$ is an implicit-heavy cycle longer than C, contradicting the choice of C. Hence $uz_1 \notin E(G)$. Similarly, $uz_2 \notin E(G)$.

If $z_1 x_2 \in E(G)$, by Claim 3, $z_1 \neq x_1^+$, then $x_1 P x_2 z_1 C x_2^- x_2^+ C x_1^- x_1^+ C z_1^- x_1$ is an implicit-heavy cycle longer than C, contradicting the choice of C. Hence $z_1 x_2 \notin E(G)$. Similarly, $z_2 x_1 \notin E(G)$.

If $z_1 z_2 \in E(G)$, by Lemma 6 (a), $z_1 \neq x_1^+$ or $z_2 \neq x_2^+$, then $x_1 P x_2 z_2^- \bar{C} x_2^+ x_2^- \bar{C} z_1 z_2 C x_1^- x_1^+ C z_1^- x_1$ (if $z_1 \neq x_1^+$ and $z_2 \neq x_2^+$) or $x_1 P x_2 z_2^- \bar{C} x_2^+ x_2^- \bar{C} z_1 z_2 C x_1$ (if $z_1 = x_1^+$ and $z_2 \neq x_2^+$) or $x_1 P x_2 \bar{C} z_1 z_2 C x_1^- x_1^+ C z_1^- x_1$ (if $z_1 \neq x_1^+$ is an implicit-heavy cycle longer than C, contradicting the choice of C. Hence $z_1 z_2 \notin E(G)$. □

Claim 5. $r \geq 2$.

Proof. Suppose $r = 1$. Then by Claim 4 and the choice of y_1 and y_2, $\{y_1, y_1^-, x_1, u_1, x_2, y_2^-, y_2\}$ induces a P_7 if $x_1 x_2 \notin E(G)$ or D if $x_1 x_2 \in E(G)$. In the latter case, it is easy to check that $\{u_1, x_1, x_2, x_1^-, x_1^+\}$ induces an H, a contradiction. □

Claim 6. $id(x_1^+) + id(x_2) < n$ and $id(x_2^+) + id(x_1) < n$.

Proof. Since $P' = x_1^+ C x_2^- x_2^+ C x_1 P x_2$ and $P'' = x_2^+ C x_1^- x_1^+ C x_2 \bar{P} x_1$ are two paths such that $V(C) \subset V(P')$ and $V(C) \subset V(P'')$, we can obtain that $id(x_1^+) + id(x_2) < n$ and $id(x_2^+) + id(x_1) < n$ by Lemma 4. □

Claim 7. $x_1x_2 \in E(G)$

Proof. Suppose $x_1x_2 \notin E(G)$. Now by the choice of P and Claim 4, we have $\{y_1, y_1^-, x_1, u_1, u_2, \ldots, u_r, x_2, y_2^-, y_2\}$ induces a path P_{r+6}. Then we can find an induced subgraph isomorphic to P_7, contradicting the hypothesis of Theorem 10. □

Claim 8. $id(x_1) > n/2$ and $id(x_2) > n/2$.

Proof. By the choice of P, we have $u_1x_2, u_rx_1 \notin E(G)$. By Claim 1 and Claim 4, we have $\{x_1, u_1, x_1^+, x_2\}$ induces a claw. Thus, by the hypothesis of Theorem 10 and Claim 6, $id(u_1) + id(x_2) \geq n$. Since $id(u_1) < n/2$, $id(x_2) > n/2$. Similarly, we can prove that $id(x_1) > n/2$. □

Claim 9. Either $d(x_1) < id(x_1)$ or $d(x_2) < id(x_2)$.

Proof. Suppose to the contrary that $d(x_1) = id(x_1)$ and $d(x_2) = id(x_2)$. Then by Claim 8, $d(x_1) + d(x_2) = id(x_1) + id(x_2) > n$. By the choice of P and by Claim 5, we have $N_{G-C}(x_1) \cap N_{G-C}(x_2) = \emptyset$. Thus, $|N_C(x_1)| + |N_C(x_2)| > |V(C)|$.

By Claim 7, $x_1x_2 \in E(G)$. Then by the choice of y_1 and y_2 and by Claim 3 and Claim 4, we have

$$|N_{x_1^- C y_1^-}(x_1)| + |N_{x_1^- C y_1^-}(x_2)| = |V(x_1^- C y_1^-)|,$$

and

$$|N_{x_2^- C y_2^-}(x_1)| + |N_{x_2^- C y_2^-}(x_2)| = |V(x_2^- C y_2^-)|.$$

Moreover, by the choice of y_1, y_2 and by Claim 4, we have $x_1y_1, x_2y_1, x_1y_2, x_2y_2 \notin E(G)$. Thus,

$$\begin{aligned} & |N_{y_1^+ C x_2^{-2}}(x_1)| + |N_{y_1^+ C x_2^{-2}}(x_2)| + |N_{y_2^+ C x_1^{-2}}(x_1)| + |N_{y_2^+ C x_1^{-2}}(x_2)| \\ > \ & |V(y_1^+ C x_2^{-2})| + |V(y_2^+ C x_1^{-2})| + 2. \end{aligned}$$

This implies that either

$$|N_{y_1^+ C x_2^{-2}}(x_1)| + |N_{y_1^+ C x_2^{-2}}(x_2)| > |V(y_1^+ C x_2^{-2})| + 1$$

or

$$|N_{y_2^+ C x_1^{-2}}(x_1)| + |N_{y_2^+ C x_1^{-2}}(x_2)| > |V(y_2^+ C x_1^{-2})| + 1.$$

Without loss of generality, we may assume that

$$|N_{y_1^+Cx_2^{-2}}(x_1)| + |N_{y_1^+Cx_2^{-2}}(x_2)| > |V(y_1^+Cx_2^{-2})| + 1.$$

Then there exists a vertex $v \in V(y_1^+Cx_2^{-2})$ such that $x_1v \in E(G)$ and $x_2v^- \in E(G)$. Now $C' = x_1^+Cv^-x_2\bar{P}x_1vCx_2^-x_2^+Cx_1^-x_1^+$ is a cycle such that $V(C) \subset V(C')$, contradicting the choice of C. □

By Claim 9, without loss of generality, we may assume $d(x_1) < id(x_1)$. Set $R = V(G) - V(C)$, $C_1 = V(x_1Cx_2)$ and $C_2 = V(x_2^+Cx_1^-)$.

Claim 10. There exists some $x \in (N_{C_1}(x_1)^- \setminus \{x_1, x_2^-\}) \cup N_{C_2}(x_1)^-$ such that $d(x) \geq id(x_1)$.

Proof. Let $d(x_1) = l+1$. By the choice of P, we have $d(u) \leq id(u) < n/2$ for every vertex $u \in R$. Then $d(w) < id(x_1)$ for every $w \in N_R(x_1) \cup (N_2(x_1) \cap R)$. Then by the definition of implicit degree and the following fact:

$$N_{C_1}(x_1)^- \cup N_{C_2}(x_1)^- = N_C(x_1)^- \subset N_C(x_1) \cup (N_2(x_1) \cap V(C)),$$

$$|N_C(x_1)^-| + |N_R(x_1)| = d(x_1) - 1 = l,$$

$$u_2 \in N_2(x_1), \quad d(u_2) < id(x_1) \quad \textit{and} \quad id(x_1) > d(x_1) = l+1,$$

there exists some $x \in (N_{C_1}(x_1)^- \setminus \{x_1, x_2^-\}) \cup N_{C_2}(x_1)^-$ such that $d(x) \geq id(x_1)$. □

By Claim 10, there exists some $x \in (N_{C_1}(x_1)^- \setminus \{x_1, x_2^-\}) \cup N_{C_2}(x_1)^-$ such that $d(x) \geq id(x_1)$. If $x \in N_{C_1}(x_1)^- \setminus \{x_1, x_2^-\}$, then $P' = x\bar{C}x_1^+x_1^-\bar{C}$ $x_2^+x_2^-\bar{C}x^+x_1Px_2$ is a path such that $V(C) \subset V(P')$. Then by Lemma 3 and the choice of C, we have $id(x) + id(x_2) < n$. But by Claim 8, $id(x) + id(x_2) \geq d(x) + id(x_2) = id(x_1) + id(x_2) > n$, a contradiction.

Suppose $x \in N_{C_2}(x_1)^-$. By Claim 4, we have $x_1x_2^+ \notin E(G)$ and $x_1x_2^{+2} \notin E(G)$. So $x \neq x_2, x_2^+$. Then $P'' = x\bar{C}x_2^+x_2^-\bar{C}x_1^+x_1^-\bar{C}x^+x_1Px_2$ is a path such that $V(C) \subset V(P'')$. Then by Lemma 3 and the choice of C, we have $id(x)+id(x_2) < n$. But by Claim 8, $id(x)+id(x_2) \geq d(x)+id(x_2) = id(x_1) + id(x_2) > n$, a contradiction.

Case 2. $x_2^-x_2^+ \notin E(G)$.

Then $\{x_2, x_2^-, x_2^+, u_r\}$ induces a claw. Since G is implicit claw-heavy, $id(x_2^-) + id(x_2^+) \geq n$ by Claim 1. Then by similar argument as in Case 1 to the graph $G + x_2^-x_2^+$, we can get that $G + x_2^-x_2^+$ has a cycle C' containing all implicit-heavy vertices of G, and such that C' is longer than C. By

Lemma 5, G has a cycle containing all vertices of C', this contradicts the choice of C. Now we complete the proof of Theorem 10. □

Acknowledgements The authors are very grateful to the anonymous referee whose helpful comments and suggestions have led to a substantially improvement of the paper.

References

[1] J. Bondy, Large cycles in graphs, Discrete Math., 1 (1971) 121-132.

[2] J. Bondy, Longest paths and cycles in graphs of high degree, Research Report CORR 80-16, Univ. of Waterloo, Waterloo, Ont., (1980).

[3] J. Bondy and U. Murty, Graph Theory with Applications, Macmillan, London, (1976).

[4] H. Broersma, Z. Ryjáček and I. Schiermeyer, Dirac's minimum degree condition restricted to claws, Discrete Math., 167-168 (1997) 155-166.

[5] H. Broersma and H. Veldman, Restrictions on induced subgaphs ensuring hamiltonicity or pancyclicity of $K_{1,3}$-free graphs, in: R. Bodendiek ed. Contemporary Methods in Graph Theory, (BI-Wiss.-Verl., Mannhein-Wien-Zürich, 1990) 181-194.

[6] B. Chen, S. Zhang and S. Qiao, Hamilton cycles in claw-heavy graphs, Discrete Math., 309 (2009) 2015-2019.

[7] G. Dirac, Some theorems on abstract graphs, Proc. London Math. Soc., 2 (1952) 69-81.

[8] R. Faudree, Z. Ryjáček and I. Schiermeyer, Forbidden subgraphs and cycle extendability, J. Combin. Math.Combin.Comput., 19 (1995) 109-128.

[9] X. Huang, Hamilton cycles in implicit claw-heavy graphs, Infor. Proce. Lett., 114 (2014) 676-679.

[10] H. Li, W. Ning and J. Cai, An implicit degree condition for cyclability in graphs, FAW-AAIM 2011, LNCS, 6681 (2011) 82-89.

[11] R. Shi, 2-Neighborhoods and hamiltonian conditions, J. Graph Theory, 16 (1992) 267-271.

[12] Y. Zhu, H. Li and X. Deng, Implicit-degrees and circumferences, Graphs Combin., 5 (1989) 283-290.