On the 2-rainbow bondage number of planar graphs

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Abstract

A 2-*rainbow dominating function* (2RDF) on a graph $G = (V, E)$ is a function f from the vertex set V to the set of all subsets of the set $\{1,2\}$ such that for any vertex $v \in V$ with $f(v) = \emptyset$ the condition $\cup_{u \in N(v)} f(u) = \{1,2\}$ is fulfilled. The *weight* of a 2RDF f is the value $\omega(f) = \sum_{v \in V(G)} |f(v)|$. The 2-*rainbow domination number*, denoted by $\gamma_{r2}(G)$, is the minimum weight of a 2RDF on G. The rainbow bondage number $b_{r2}(G)$ of a graph G with maximum degree at least two, is the minimum cardinality of all sets $E' \subseteq E(G)$ for which $\gamma_{r2}(G - E') > \gamma_{r2}(G)$. Dehgardi, Sheikholeslami and Volkmann, [Discrete Appl. Math. **174** (2014), 133-139] proved that the rainbow bondage number of a planar graph does not exceed 15. In this paper we improve this result.

Keywords: rainbow domination number, rainbow bondage number.
MSC2010: 05C69.

1 Introduction

In this paper, G is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ and size $|E|$ of G are denoted by $n = n(G)$ and $m = m(G)$, respectively. For every vertex $v \in V(G)$, the *open neighborhood* $N_G(v) = N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N_G[v] = N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = \deg(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. The *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \bigcup_{v \in S} N(v)$, and the *closed neighborhood* of S is the set $N[S] = N(S) \cup S$. By $d_G(x, y) = d(x, y)$ we denote the *distance* of the vertices x and y in the graph G. The *girth* $g(G)$ of G is the length of a shortest cycle in G, and $g(G) = \infty$ when G is a forest. For the notation and terminology not defined here, we refer the reader to [12, 18].

A subset S of vertices of G is a *dominating set* if $N[S] = V$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. To measure the vulnerability or the stability of the domination in an interconnection network under edge failure, Fink et al. [10] proposed the concept of the bondage number in 1990. The *bondage number* of G, denoted by $b(G)$, is the minimum number of edges whose removal from G results in a graph with larger domination number. For more information on this topic we refer the reader to the survey article by Xu [20].

For a positive integer k, a *k-rainbow dominating function* (kRDF) of a graph G is a function f from the vertex set $V(G)$ to the set of all subsets of the set $\{1, 2, \ldots, k\}$ such that for any vertex $v \in V(G)$ with $f(v) = \emptyset$ the condition $\cup_{u \in N(v)} f(u) = \{1, 2, \ldots, k\}$ is fulfilled. The *weight* of a kRDF f is the value $\omega(f) = \sum_{v \in V(G)} |f(v)|$. The *$k$-rainbow domination number* of a graph G, denoted by $\gamma_{rk}(G)$, is the minimum weight of a kRDF of G. A *$\gamma_{rk}(G)$-function* is a k-rainbow dominating function of G with weight $\gamma_{rk}(G)$. Note that $\gamma_{r1}(G)$ is the classical domination number $\gamma(G)$. The k-rainbow domination number was introduced by Brešar, Henning, and Rall [3] and has been studied by several authors [4–8, 14–16, 19].

Let G be a graph with maximum degree at least two. The *k-rainbow bondage number* $b_{rk}(G)$ of G is the minimum cardinality of all sets $E' \subseteq E$ for which $\gamma_{rk}(G - E') > \gamma_{rk}(G)$. Since in the study of k-rainbow bondage number the assumption $\Delta(G) \geq 2$ is necessary, we always assume that when we discuss $b_{rk}(G)$, all graphs involved satisfy $\Delta(G) \geq 2$. The k-rainbow bondage number was introduced by Dehgardi, Sheikholeslami, and Volkmann in [7]. In their paper, they proved that:

Proposition A. If G is a planar graph with maximum degree at least two, then $b_{r2}(G) \leq 15$.

In this paper, we improve the stated bound in Proposition A. We make use of the following results in this paper.

Proposition B. ([4, 17]) If G is a planar graph with minimum degree 5, then G contains an edge xy with $\deg(x) = 5$ and $\deg(y) \in \{5, 6\}$.

Proposition C. ([11]) Let G be a planar graph of girth $g < \infty$ and c be the number of cut-edges in G, then

$$m(G) \leq \frac{g(n(G) - 2) - c}{g - 2}.$$

Corollary 1. Let G be a planar graph of girth $g(G) < \infty$. Then (i) $\delta(G) \leq 5$, (ii) if $g(G) \geq 4$, then $\delta(G) \leq 3$, (iii) if $g(G) \geq 6$, then $\delta(G) \leq 2$.

Proposition D. (Euler's formula) If G is a connected planar graph, then

$$|V(G)| - |E(G)| + |F(G)| = 2,$$

where $F(G)$ is the face set of G.

Proposition E. ([13]) Let v be a vertex of a planar graph G with $d(v) \geq 3$, and let $E_v = \{xy \mid x, y \in N(v) \text{ and } xy \notin E(G)\}$. Then there exists a subset $S \subseteq E_v$ such that $H = G + S$ is still a planar graph and $H[N(v)]$ is 2-connected.

Proposition F. ([7]) If G is a graph, and xyz a path of length 2 in G, then

$$b_{r2}(G) \leq d(x) + d(y) + d(z) - 3.$$

Proposition G. ([7]) If G is a connected graph of order $n \geq 3$, then

$$b_{r2}(G) \leq \delta(G) + 2\Delta(G) - 3.$$

Proposition H. ([7]) If G is a connected graph of order $n \geq 3$ and edge-connectivity $\lambda(G)$, then

$$b_{r2}(G) \leq \lambda(G) + 2\Delta(G) - 3.$$

Proposition I. ([7]) Let xyz be a path of length 2 in graph G. If $xz \notin E(G)$, then

$$b_{r2}(G) \leq d(x) + d(y) + d(z) - 2 - |N(x) \cap N(y)| - |N(x) \cap N(z)|;$$

otherwise,

$$b_{r2}(G) \leq d(x) + d(y) + d(z) - 3 - |N(x) \cap N(y)| - |N(x) \cap N(z)|.$$

Proposition J. ([7]) If T is a tree of order $n \geq 3$, then $b_{r2}(T) \leq 2$.

The next result is an immediate consequence of Propositions A, G and Corollary 1.

Corollary 2. If G is a connected planar graph with maximum degree at least two, then

$$b_{r2}(G) \leq \max\{15, 2\Delta(G) + 2\}.$$

2 Bounds on the 2-rainbow bondage number

In this section, we will improve the bound of Corollary 2 for connected planar graph.

Theorem 3. If G is a connected planar graph of order $n \geq 3$, then $b_{r2}(G) \leq \max\{15, 2\Delta\}$.

Proof. By Proposition A, we only need to prove that $b_{r2}(G) \leq 2\Delta$ for $\Delta \leq 7$. If $\lambda(G) \leq 3$, then the result follows by Proposition H. Now on, $\Delta(G) \geq \delta(G) \geq \lambda(G) \geq 4$. Assume, to the contrary, that $b_{r2}(G) \geq 2\Delta+1$. For each edge $e = xy$ in $E(G)$, we assign two variables $v_e = 1/\deg(x) + 1/\deg(y)$ and $f_e = 1/a_x + 1/a_y$, where a_x and a_y are the number of edges comprising the faces which e borders. Obviously $\sum_{e\in E} v_e = n$ and $\sum_{e \in E} f_e = |F(G)|$. By Proposition D, we have

$$\sum_{e\in E}(v_e + f_e - 1) = n - |E(G)| + |F(G)| = 2. \tag{1}$$

Now we show that for every edge $e = xy$, $v_e + f_e - 1 \leq 0$ which leads to a contradiction by (1). Assume that $e = xy \in E$ and $\deg(x) \leq \deg(y)$. First let $\deg(x) = 4$. It follows from Proposition I and our assumption $b_{r2}(G) \geq 2\Delta + 1$ that

$$2\Delta + 1 \leq b_{r2}(G) \leq 4 + \deg(y) + \Delta - 3 - |N(x) \cap N(y)| \leq 2\Delta + 1,$$

and hence $\deg(y) = \Delta \geq 4$ and x and y can have no common neighbor. So a_x and a_y are both at least 4 that implies $v_e + f_e - 1 \leq 0$.

Now let $\deg(x) = 5$. Then by Proposition I we have

$$\begin{aligned} 2\Delta + 1 &\leq b_{r2}(G) \\ &\leq 5 + \deg(y) + \Delta - 3 - |N(x) \cap N(y)| \\ &\leq \deg(y) + \Delta + 2 - |N(x) \cap N(y)| \end{aligned} \tag{2}$$

and hence $\deg(y) \geq \Delta - 1$. If $\deg(y) = \Delta - 1$, then x and y can have no common neighbor and $\deg(y) \geq 5$ by assumption, and the result follows as above. Let $\deg(y) = \Delta \geq 5$. Then (2) shows that x and y have at most one common neighbor and hence at most one of a_x and a_y is equal to 3. This leads to $v_e + f_e - 1 \leq 0$. Finally if $\deg(x) \geq 6$, then $a_x, a_y \geq 3$ implying that $v_e + f_e - 1 \leq 0$. This completes the proof. □

Theorem 4. Let G be a connected planar graph of order at least three with no vertex of degree five. Then $b_{r2}(G) \leq \max\{12, \Delta(G) + 5\}$.

Proof. First we show that $b_{r2}(G) \leq 12$. By Corollary 1 and our assumption, we have $\delta(G) \leq 4$. Let $X = \{v \in V(G) \mid \deg(v) \leq 4\}$ and $Y = V(G) - X$.

First suppose that there exists a vertex $v \in Y$ which has three neighbors w_1, w_2, w_3 in X. Let F be the set of all edges incident to w_1, w_2 or w_3. Then $|F| \leq 12$ and the vertices w_1, w_2, w_3 are isolated in $G - F$. If f is a $\gamma_{r2}(G - F)$-function, then clearly $|f(w_1)| = |f(w_2)| = |f(w_3)| = 1$ and the function g defined on $V(G)$ by $g(v) = \{1, 2\}, g(w_1) = g(w_2) = g(w_3) = \emptyset$ and $g(w) = f(w)$ otherwise, is a 2RDF of G with weight less than $\omega(f)$ and hence $b_{r2}(G) \leq 12$. Thus we may assume that each vertex in Y has at most two neighbors in X.

Assume that $G[Y]$ is the subgraph induced by Y. If there exists a vertex $u \in Y$ of degree at most 7 in $G[Y]$, and u has exactly two neighbors w_1, w_2 in X, then consider the path $w_1 u w_2$. It follows from Proposition F that $b_{r2}(G) \leq 12$. Thus we may assume that each vertex of degree at most 7 in $G[Y]$ has at most one neighbor in X.

Since the subgraph $G[Y]$ is a planar graph, we deduce from Corollary 1 that $\delta(G[Y]) \leq 5$. If $\delta(G[Y]) \leq 4$ and v is a vertex of $G[Y]$ of degree $\delta(G[Y])$, then we obtain $\deg_G(v) \leq 4$ since v has at most one neighbor in X and G has no vertex of degree 5, contradicting the definition of X. Therefore, $\delta(G[Y]) = 5$. It follows from Proposition B that there is an edge xy in $G[Y]$ such that $\deg_{G[Y]}(x) = 5$ and $\deg_{G[Y]}(y) \in \{5, 6\}$. Since G has no vertex of degree 5, we conclude that x has exactly one neighbor in X and y has at most one neighbor in X. Hence $\deg_G(x) = 6$ and $\deg_G(y) \in \{6, 7\}$. Let z be the neighbor of x in X and consider the path zxy. Proposition F implies that $b_{r2}(G) \leq 12$ as desired.

Now we show that $b_{r2}(G) \leq \Delta(G) + 5$. Since $b_{r2}(G) \leq 12$, we may assume that $\Delta(G) \leq 7$. As above, we have $\delta(G) \leq 4$. Let $X = \{v \in V(G) \mid \deg(v) \leq 4\} = \{v_1, v_2, \ldots, v_k\}$. Suppose on the contrary that $b_{r2}(G) \geq \Delta + 6$. Since $b_{r2}(G) \leq 12$, we have $\Delta(G) \leq 6$. By Proposition F, we deduce that for any two distinct vertices $u, v \in X$, $d(u, v) \geq 3$. Define $H_0 = G$ and $H_i = H_{i-1} + S_i$ for $1 \leq i \leq k$, where S_i is a subset of $E_{v_i} = \{xy \mid x, y \in N(v_i), xy \notin E(H_{i-1})\}$ such that $H_{i-1} + S_i$ is still a planar graph and $H_i[N(v_i)]$ is 2-connected when $\deg_G(v_i) \geq 3$. Now let $v \in X$ and $y \in N_G(v)$. Since $b_{r2}(G) \geq \Delta + 6$, we deduce from Proposition F that $\deg_G(v) \geq 3$. If $\deg_G(v) = 3$, then by Proposition F, we obtain

$$\Delta + 6 \leq b_{r2}(G) \leq \deg_G(v) + \deg_G(y) + \Delta - 3 = \deg_G(y) + \Delta.$$

The inequality chain and the fact $\Delta(G) \leq 6$, leads to $\deg(y) = 6$ and thus $\deg_{H_k}(y) \geq 8$. Assume next that $\deg(v) = 4$. By Proposition F and the fact $\Delta(G) \leq 6$, we obtain

$$\Delta + 6 \leq b_{r2}(G) \leq \deg_G(v) + \deg_G(y) + \Delta - 3 = \deg_G(y) + \Delta + 1 .$$

This implies that $\deg_G(y) \geq 5$. Since G has no vertex of degree 5, we obtain $\deg_G(y) = 6 = \Delta(G)$ and so $\deg_{H_k}(y) \geq 8$. Obviously, H_k is planar. Since

$d(u,v) \geq 3$ for any two distinct vertices $u, v \in X$, we see that $H_k - X$ is a planar graph with minimum degree at least 6 which contradicts Corollary 1. This complete the proof. □

Theorem 5. For any connected planar graph G of order $n \geq 3$, $b_{r2}(G) \leq \{15, \Delta(G)+6\}$.

Proof. By Proposition A, we need only to show that $b_{r2}(G) \leq \Delta(G)+6$. Let $X_{\leq 3} = \{v \in V(G) \mid \deg(v) \leq 3\}$, $X_i = \{v \in V(G) \mid \deg(v) = i\}$ for $i = 4, 5$. If $X_5 = \emptyset$, then the result is immediate by Theorem 4. Henceforth, we assume that $X_5 \neq \emptyset$. Assume, to the contrary, that $b_{r2}(G) \geq \Delta + 7$. Since $b_{r2}(G) \leq 15$, we have $\Delta(G) \leq 8$. Proposition F implies that $d(x,y) \geq 3$, if either $x, y \in X_3 \cup X_4$ or $x \in X_3 \cup X_4$ and $y \in X_5$. Moreover, if $x \in X_{\leq 3}$ and $y \in N_G(x)$, then $\deg(y) \geq 7$. Suppose that $I \subseteq X_5$ is an independent set such that $|I|$ is maximum. Then $X_5 \subseteq I \cup N(I)$ and $N(X_4) \cap N(I) = \emptyset$. Now let $X_4 \cup I = \{v_1, v_2, \ldots, v_k\}$ and $H = G - X_{\leq 3}$. Define $H_0 = H$ and $H_i = H_{i-1} + S_i$ for $1 \leq i \leq k$, where S_i is a subset of $E_{v_i} = \{xy \mid x, y \in N(v_i),\ xy \notin E(H_{i-1})\}$ such that $H_{i-1} + S_i$ is still a planar graph and $H_i[N(v_i)]$ is 2-connected. We consider the following facts.

Fact 1. If $X_4 \neq \emptyset$, then $\deg_{H_k}(y) \geq 7$ for each vertex $y \in N_G(X_4)$.
Let $x \in X_4$ and $y \in N_G(x)$. By Proposition F, we have

$$\Delta + 7 \leq b_{r2}(G) \leq \deg_G(x) + \deg_G(y) + \Delta - 3 = \deg_G(y) + \Delta + 1 \ ,$$

implying that $\deg_G(y) \geq 6$, and so $\deg_{H_k}(y) \geq 7$.

Fact 2. For each vertex $y \in N_G(I)$, $\deg_{H_k}(y) \geq 7$.
Let $x \in I$ and $y \in N_G(x)$. It follows from Proposition F that

$$\Delta + 7 \leq b_{r2}(G) \leq \deg_G(x) + \deg_G(y) + \Delta - 3 = \deg_G(y) + \Delta + 2 \ .$$

This implies that $\deg_G(y) \geq 5$, and so $\deg_{H_k}(y) \geq 7$.

By Facts 1 and 2, we see that $G^* = H_k - X_4$ is a planar graph such that (i) the minimum degree of G^* is 5, (ii) $I = \{v \in V(G^*) \mid d_{G^*}(v) = 5\}$ is an independent set in G^* and (iii) $\deg_{G^*}(v) \geq 7$ for each vertex $v \in N_{G^*}(I) = N_G(I)$. Let B be the bipartite graph with partite sets I and $N(I)$ and the edge set $\{uv \in E(G^*) \mid u \in I,\ v \in N(I)\}$. Then B is a bipartite planar graph with exactly $5|I|$ edges. Using Proposition C and the fact $g(B) \geq 4$, we obtain $5|I| \leq 2|I| + 2|N(I)| - 4$ (note that this bound remains valid if $g = \infty$ that means that B is a forest) and therefore $|N(I)| \geq \frac{3}{2}|I| + 2$.

Therefore

$$\begin{aligned}
|E(G^*)| &= \frac{1}{2}\sum_{v\in V(G^*)} d_{G^*}(v) \\
&\geq \frac{1}{2}(5|I| + 7|N(I)| + 6(|V(G^*)| - |I| - |N(I)|)) \\
&= 3|V(G^*)| + \frac{1}{2}|N(I)| - \frac{1}{2}|I| \\
&\geq 3|V(G^*)| + \frac{1}{4}|I| + 1 > 3|V(G^*)| - 6,
\end{aligned}$$

a contradiction with Proposition C, and the proof is complete. □

For a graph G, let $n_i(G) = n_i$ be the number of vertices of degree i and $\tau_i(G) = \tau_i$ be the number of vertices of degree at least i for $i = 1, 2, \ldots, \Delta$.

Theorem 6. Let G be a connected planar graph of order $n \geq 3$. If $g(G) \geq 4$, then $b_{r2}(G) \leq \max\{15, \Delta + 4\}$.

Proof. By Proposition A, we need only to show that $b_{r2}(G) \leq \Delta + 4$. If G is a tree or $\Delta \leq 4$, then the result follows from Proposition J and Theorem 3. Assume that $4 \leq g(G) < \infty$ and $\Delta \geq 5$. Since $g(G) \geq 4$, Corollary 1 implies that $\delta \leq 3$. Since $n(G) = n_1 + n_2 + \ldots + n_\Delta$ and $2m(G) = n_1 + 2n_2 + \ldots + \Delta n_\Delta$, we conclude from Proposition C that

$$2m = n_1 + 2n_2 + \ldots + \Delta n_\Delta \leq 4n - 8 = 4n_1 + 4(n_2 + n_3 + \ldots + n_\Delta) - 8,$$

and thus

$$3n_1 + 2n_2 + n_3 \geq n_5 + 2n_6 + 3n_7 + 4n_8 + \ldots + (\Delta - 4)n_\Delta + 8. \quad (3)$$

We consider the following cases.

Case 1. $\Delta = 5$.
If $\delta \leq 2$, then the results follows from Proposition G. If $\delta = 3$ and there exist a vertex u of degree 3 and a vertex v with $d(u, v) \leq 2$ and $\deg(v) \leq 4$, then Proposition F leads to $b_{r2}(G) \leq \Delta + 4$. Assume that $\delta = 3$ and that all neighbors of each vertex of degree 3 has degree 5. Then $n_5 \geq 3n_3$, a contradiction with (3).

Case 2. $\Delta = 6$.
If $\delta = 1$, then the result is immediate by Proposition G. If there exists a vertex u of degree r and a vertex v with $d(u, v) \leq 2$ and $\deg(v) \leq 7 - r$ for $r = 2, 3$, then it follows from Proposition F that $b_{r2}(G) \leq \Delta + 4$. In the remaining cases, we observe that $\tau_5 \geq 2n_2 + 3n_3$. Now (3) leads to the following contradiction

$$2n_2 + n_3 \geq n_5 + n_6 + 8 = \tau_5 + 8 \geq 2n_2 + 3n_3 + 8.$$

Case 3. $\Delta \geq 7$.
If there exists a vertex u of degree r and a vertex v with $d(u,v) \leq 2$ and $\deg(v) \leq 7 - r$ for $r = 1,2,3$, then it follows from Proposition F that $b_{r2}(G) \leq \Delta + 4$. In the remaining cases, we observe that $\tau_5 \geq 2n_2 + 3n_3$ and $\tau_7 \geq n_1$. Now (3) leads to the following contradiction

$$3n_1 + 2n_2 + n_3 \geq \tau_5 + 2\tau_7 + 8 \geq 3n_1 + 2n_2 + 3n_3 + 8.$$

This completes the proof. □

Theorem 7. Let G be a connected planar graph of order $n \geq 3$. If $g(G) \geq 5$, then $b_{r2}(G) \leq \{15, \Delta + 3\}$.

Proof. By Proposition A, we need only to show that $b_{r2}(G) \leq \Delta + 3$. If G is a tree or $\Delta \leq 3$, then the result follows from Proposition J and Theorem 3. Assume that $4 \leq g(G) < \infty$ and $\Delta(G) \geq 5$. By Corollary 1 and the assumption $g(G) \geq 5$, we have $\delta(G) \leq 3$. Using Proposition C and an argument similar to that described in the proof of Theorem 6, we obtain

$$6m = 3n_1 + 6n_2 + 9n_3 + \ldots + 3\Delta n_\Delta \leq 10(n-2)$$

and hence

$$7n_1 + 4n_2 + n_3 \geq 2n_4 + 5n_5 + 8n_6 + \ldots + (3\Delta - 10)n_\Delta + 20. \qquad (4)$$

First let $\Delta(G) = 4$. If $\delta(G) \leq 2$, then the result is immediate by Proposition G. If $\delta(G) = 3$ and there exist a vertex u of degree 3 and a vertex v with $d(u,v) \leq 2$ and $\deg(v) \leq 3$, then Proposition F leads to $b_{r2}(G) \leq \Delta(G)+3$. In the remaining case, we have $n_4 \geq 3n_3$, a contradiction to (4).

Now let $\Delta(G) = 5$. If $\delta = 1$, then the result follows from Proposition G. Suppose that $2 \leq \delta \leq 3$. If there exist a vertex u of degree r and a vertex v with $d(u,v) \leq 2$ and $\deg(v) \leq 6 - r$ for $r = 2,3$, then Proposition F yields to the desired bound. In the remaining cases, we observe that $\tau_4 \geq 2n_2 + 3n_3$, a contradiction to (4).

Finally let $\Delta \geq 6$. Proposition F yields $b_{r2}(G) \leq \Delta+3$, when there exist a vertex u of degree r and a vertex v with $d(u,v) \leq 2$ and $\deg(v) \leq 6 - r$ for $r = 1,2,3$. In the remaining cases, we observe that $\tau_4 \geq n_1 + 2n_2 + 3n_3$ and $\tau_6 \geq n_1$. Applying this inequality and (4), we obtain the contradiction $7n_1 + 4n_2 + n_3 \geq 2\tau_4 + 6\tau_6 + 20 \geq 8n_1 + 4n_2 + 6n_3 + 20$. □

Theorem 8. Let G be a connected planar graph of order $n \geq 3$ with $g(G) \geq 6$. Then $b_{r2}(G) \leq \max\{15, \Delta + 2\}$.

Proof. By Proposition A, we need only to show that $b_{r2}(G) \leq \Delta + 2$. If G is a tree, then by Proposition J we have $b_{r2}(G) \leq 2$ as desired. It follows from Corollary 1 and the assumption $g(G) \geq 6$ that $\delta \leq 2$. If $\Delta \leq 3$, then Proposition G leads to $b_{r2}(G) \leq \delta + 2\Delta - 3 \leq \Delta + 2$. Thus we may assume that $6 \leq g(G) < \infty$ and $\Delta \geq 4$. Using Proposition C and an argument similar to that described in the proof of Theorem 6, we obtain

$$2n_1 + n_2 \geq n_4 + 2n_5 + 3n_6 + \ldots + (\Delta - 3)n_\Delta + 6. \tag{5}$$

First let $\Delta = 4$. If $\delta = 1$, then the result follows from Proposition G. If $\delta = 2$ and there exist a vertex u of degree 2 and a vertex v with $d(u, v) \leq 2$ and $\deg(v) \leq 3$, then Proposition F leads to $b_{r2}(G) \leq \Delta + 2$. In the remaining case, we have $n_4 \geq 2n_2$, a contradiction to (5). Now let $\Delta \geq 5$. Theorem F yields $b_{r2}(G) \leq \Delta + 2$, if there exist a vertex u of degree r and a vertex v with $d(u, v) \leq 2$ and $\deg(v) \leq 5 - r$ for $r = 1, 2$. In the remaining cases, we observe that $\tau_4 \geq n_1 + 2n_2$ and $\tau_5 \geq n_1$. Using this inequality and (5), we obtain the contradiction $2n_1 + n_2 \geq \tau_4 + \tau_5 + 6 \geq 2n_1 + 2n_2 + 6$. □

Theorem 9. Let G be a connected planar graph of order $n \geq 3$. If $g(G) \geq 8$, then $b_{r2}(G) \leq \{15, \Delta + 1\}$.

Proof. By Proposition A, we need only to show that $b_{r2}(G) \leq \Delta + 1$. If G be a tree, then result is immediate by Proposition J. Since $g(G) \geq 8$, Proposition 1 implies that $\delta \leq 2$. If $\Delta \leq 2$, then Proposition G leads to $b_{r2}(G) \leq \delta + 2\Delta - 3 \leq \Delta + 1$. Thus we may assume that $8 \leq g(G) < \infty$ and $\Delta \geq 3$. By Proposition C,

$$5n_1 + 2n_2 - c \geq n_3 + 4n_4 + 7n_5 + \ldots + (3\Delta - 8)n_\Delta + 16,$$

where c is the number of cut-edges in the graph G. Since $c \geq n_1$, we have $4n_1 + 2n_2 \geq 5n_1 + 2n_2 - c$ implying that

$$4n_1 + 2n_2 \geq n_3 + 4n_4 + 7n_5 + \ldots + (3\Delta - 8)n_\Delta + 16. \tag{6}$$

First let $\Delta = 3$. If $\delta = 1$, then the result is immediate by Proposition G. If $\delta = 2$ and there exist a vertex u of degree 2 and a vertex v with $d(u, v) \leq 2$ and $\deg(v) \leq 2$, then Proposition F leads to $b_{r2}(G) \leq \Delta + 1$. In the remaining case, we have $n_3 \geq 2n_2$, a contradiction to (6). Assume now that $\Delta \geq 4$. Proposition F yields $b_{r2}(G) \leq \Delta + 1$, when there exist a vertex u of degree r and a vertex v with $d(u, v) \leq 2$ and $\deg(v) \leq 4 - r$ for $r = 1, 2$. In the remaining cases, we observe that $\tau_3 \geq n_1 + 2n_2$ and $\tau_4 \geq n_1$. Using this inequality with (7), we obtain the contradiction $4n_1 + 2n_2 \geq \tau_3 + 3\tau_4 + 16 \geq 4n_1 + 2n_2 + 16$. □

Acknowledgment This work has been supported by Research Office of Azarbaijan Shahid Madani University.

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