A note on the existence of cyclic and 1-rotational kite systems

Aijun Yu[1]
[1]Department II, Suqian College
Suqian 223800, P. R. China

Lingye Wang[2], Su Wang[2] and Jinhua Wang[2]
[2]School of Sciences, Nantong University
Nantong 226007, P. R. China
Email: jhwang@ntu.edu.cn(J. Wang)

Abstract

A kite graph is a graph obtained from a 3-cycle (or triple) by adding a pendent edge to a vertex of the 3-cycle. A kite system of order v is a pair $(X, \mathcal{B})$, where $\mathcal{B}$ is an edge disjoint collection of kite graphs which partitions the edge set of K_v. A kite system of order v is cyclic if it admits an automorphism of order v, and 1-rotational if it admits an automorphism containing one fixed point and a cycle of length $v-1$. In this paper, we show that there exists a cyclic kite system of order v if and only if $v \equiv 1 \pmod 8$, and there exists a 1-rotational kite system of order v if and only if $v \equiv 0 \pmod 8$.

Keywords : Kite system; Skolem sequence; Cycle kite system; Rotational kite system

MSC 2010: 05B05;05B15

1 Introduction

Let G, H be simple graphs, and K_v a complete graph of order v. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. A *decomposition* $\mathcal{B}$ of G is a collection of edge disjoint subgraphs (called *blocks*) $B_1, B_2, \ldots, B_b$ of G such that every edge of G belongs to exactly one B_i for $i = 1, 2, \ldots, b$, i.e., $\bigcup_{i=1}^{b} E(B_i) = E(G)$ and $E(B_i) \bigcap E(B_j) = \emptyset, 1 \le i < j \le b$. The pair $(X, \mathcal{B})$ is called an H-*decomposition* of G if each member of $\mathcal{B}$ is isomorphic to the graph H, where $X = V(G)$. An H-decomposition of K_v is called an *H-design* of order v. When H is a complete graph of order k, K_k, an H-design of order v is better known as a *balanced incomplete block design* of order v with block size k and index $1((v,k,1)$-BIBD). For $k = 3$, a $(v,3,1)$-BIBD is a *Steiner triple system* (STS(v)). It is well known that there exists an STS(v) if and only if $v \equiv 1,3 \pmod 6$ [5]. When H is a kite graph, an H-design of order v is called a *kite system* of order v (denoted by KS(v)), where the kite graph is a graph obtained from a 3-cycle (or triple) by adding a pendent edge to a vertex of the 3-cycle. In 1977, Bermond and Schönheim [1] showed that there exists a KS(v) if and only if $v \equiv 0,1 \pmod 8$.

Let Θ be an automorphism group of an H-design of order v, $(X, \mathcal{B})$, that is a group of permutations on the vertex set X of v points such that the collection $\mathcal{B}$ of H graphs is invariant. If there is an automorphism of order v, then the design is said to be *cyclic*. If there is an automorphism consisting of a single fixed point and one cycle of length $v-1$, then the design is said to be *1-rotational.*

The spectrum problem for cyclic STS(v)s and 1-rotational STS(v)s was completely settled [5].

Theorem 1.1 *A cyclic STS(v) exists if and only if* $v \equiv 1,3 \pmod 6$ *and* $v \neq 9$.

Theorem 1.2 *A 1-rotational STS(v) exists if and only if* $v \equiv 3,9 \pmod{24}$

In this note, we mainly use Skolem-type sequences to construct cyclic KS(v)s and 1-rotational KS(v)s, and establish the spectra of the two classes of designs.

2 Preliminaries

In this section, We first present a number of preliminary definitions and theorems on some special Skolem-type sequences. we use the definitions from the Handbook of Combinatorial Designs [10], although equivalent definitions can be found in the literature (see for example [3]).

A *Skolem sequence* of order n is a sequence $S_n = (s_1, s_2, \ldots, s_{2n})$ of $2n$ positive integers that satisfies the conditions:

1. for every $k \in \{1, 2, \ldots, n\}$ there are exactly two elements $s_i, s_j \in S_n$ such that $s_i = s_j = k$, and

2. if $s_i = s_j = k, i < j$, then $j - i = k$.

As an example, $S_5 = (1,1,3,4,5,3,2,4,2,5)$ is a Skolem sequence of order 5.

An *m-extended Skolem sequence* of order n is a sequence $ES_n = (s_1, s_2, \ldots, s_{2n+1})$ of $2n+1$ non negative integers that satisfies the conditions:

1. for every $k \in \{1, 2, \dots, n\}$ there are exactly two elements $s_i, s_j \in S_n$ such that $s_i = s_j = k$;

2. if $s_i = s_j = k, i < j$, then $j - i = k$; and

3. $s_m = 0, 1 \leq m \leq 2n + 1$.

Where the null element s_m in the sequence is also called a *hook*.

As an example, $(3, 1, 1, 3, 4, 2, 0, 2, 4)$ is a 7-extended Skolem sequence of order 4.

A *hooked Skolem sequence* of order n is an extended Skolem sequence of order n with $s_{2n} = 0$, i.e., $2n$-extended Skolem sequence of order n.

As an example, $HS_6 = (1, 1, 2, 5, 2, 4, 6, 3, 5, 4, 3, 0, 6)$ is a hooked Skolem sequence of order 6.

It is known that the necessary conditions for the existence of (hooked) Skolem sequences are sufficient. For more details the reader may see [10].

Theorem 2.1 (Skolem [11]) *A Skolem sequence of order n exists if and only if $n \equiv 0, 1 \pmod 4$.*

Theorem 2.2 (O'Keefe [7]) *A hooked Skolem sequence of order n exists if and only if $n \equiv 2, 3 \pmod 4$.*

Thus the combined work of Skolem and O'Keefe showed the sufficiency of the existence of a cyclic STS($6n + 1$). In 1966, Rosa [8] introduced two types of sequences for the purpose of constructing cyclic STS($6n + 3$)s. These two types of sequences are now known as Rosa and hooked Rosa sequences, respectively.

A *Rosa sequence* of order n is an extended Skolem sequence of order n with $s_{n+1} = 0$, i.e., $n + 1$-extended Skolem sequence of order n.

As an example, $R_n = (2, 3, 2, 0, 3, 1, 1)$ is a Rosa sequence of order 3.

A *hooked Rosa sequence* of order n is a sequence $(s_1, s_2, \dots, s_{2n+2})$ of $2n + 2$ non negative integers that satisfies the conditions:

1. for every $k \in \{1, 2, \dots, n\}$ there are exactly two elements $s_i, s_j \in S_n$ such that $s_i = s_j = k$;

2. if $s_i = s_j = k, i < j$, then $j - i = k$; and

3. $s_{n+1} = s_{2n+1} = 0$.

As an example, $HR_n = (2, 3, 2, 4, 3, 0, 5, 4, 1, 1, 0, 5)$ is a hooked Rosa sequence of order 5.

Theorem 2.3 (Rosa [8]) *A Rosa sequence of order n exists if and only if $n \equiv 0, 3 \pmod 4$.*

Theorem 2.4 (Rosa [8]) *A hooked Rosa sequence of order n exists if and only if $n \equiv 1, 2 \pmod 4$.*

The existence of Rosa and hooked Rosa sequences for all admissible orders showed the sufficiency for the existence of cyclic STS($6n+3$)s.

A number of authors, for example, Billington [2], Colbourn and Jiang [4], and Fu, Lin and Mishima[6] considered generalizations of such sequences for the purpose of constructing various types of designs and codes. In 2003, Shalaby considered a generalization of Rosa sequences.

Let m, n be positive integers, $m \leq n$. A *near-Rosa (or m-near-Rosa) sequence* of order n and defect m is a sequence $NR_n = (s_1, s_2, \ldots, s_{2n-1})$ of integers $s_i \in \{1, 2, \ldots, m-1, m+1, \ldots, n\}$ which satisfy the following conditions:

1. for every $k \in \{1, 2, \ldots, m-1, m+1, \ldots, n\}$ there are exactly two elements $s_i, s_j \in NR_n$ such that $s_i = s_j = k$;
2. if $s_i = s_j = k, i < j$, then $j - i = k$; and
3. $s_n = 0$.

As an example, $NR_8 = (8,4,2,7,2,4,5,0,8,3,7,5,3,1,1)$ is a 6-near Rosa sequence of order 8 .

Shalaby shown that the necessary condition for the existence of $(n-2)$-near-Rosa sequence of order n is sufficient, with one definite exception.

Theorem 2.5 (Shalaby [9]) *For* $n \equiv 0, 1 \pmod 4$, *there exists an* $(n-2)$*-near-Rosa sequence with the exception of* $n = 4$.

3 Cyclic Kite Systems

For a cyclic kite system of order v $(X, \mathcal{B})$, the set X of v points can be identified with Z_v, i.e., the residue group of integers modulo v. In this case, the design has an automorphism $\sigma : i \mapsto i+1 \pmod v$ which is also represented by $\sigma = (0, 1, \ldots, v-1)$. Let B be a kite block of a cyclic kite system. For brevity, we will use the notation $[a, b, c; d]$ to denote the kite block obtained from a 3-cycle (a, b, c) by adding a pendent edge $\{c, d\}$. A *block orbit* $Orb(B)$ of $B = [a, b, c; d]$ is defined by $\{B + i = [a+i, b+i, c+i; d+i] | i \in Z_v\}$. The length of a block orbit is its cardinality. A block orbit of length v is said to be *full*, otherwise *short*. A *base block* of a block orbit O is a block $B \in O$ which is chosen arbitrarily. Any cyclic kite system should be generated from base blocks.

Lemma 3.1 *There exists a cyclic KS*(v) *for* $v \equiv 1 \pmod{24}$.

Proof. Let $v = 24t + 1$ and $n = 4t$. By Theorem2.1, there exists a Skolem sequence of order n. Let $S_n = (s_1, s_2, \ldots, s_{2n})$ be a Skolem sequence of order n, and $\{(a_i, b_i) | 1 \leq i \leq n\}$ the pairs of positions in S_n such that $b_i - a_i = i$ and $s_{a_i} = s_{b_i} = i$. Hence, the base kite blocks $[a_{3i+1}+n, b_{3i+1}+n, 0; a_{3t+1+i}+n]$,$[a_{3i+2}+n, b_{3i+2}+n, 0; b_{3t+1+i}+n]$,$[a_{3i+3}+$

$n, b_{3i+3}+n, 0; b_{3t+1+i}-a_{3t+1+i}]$, $0 \le i \le t-1$, will generate a cyclic KS(v). □

Example 3.2 *The Skolem sequence* $S_4 = (1,1,3,4,2,3,2,4)$ *gives the kites* $[5,6,0;8]$, $[9,11,0;12]$, $[7,10,0;4]$. *These kites yield the base blocks for a cyclic KS(25).*

Lemma 3.3 *There exists a cyclic KS(v) for* $v \equiv 9 \pmod{24}$.

Proof. Let $v = 24t+9$ and $n = 4t+1$. By Theorem 2.4, there exists a hooked Rosa sequence of order n. Let $HR_n = (s_1, s_2, \ldots, s_{2n+2})$ be hooked Rosa sequence of order n, and $\{(a_i, b_i) | 1 \le i \le n\}$ the pairs of positions in HR_n such that $b_i - a_i = i$ and $s_{a_i} = s_{b_i} = i$. Hence, the base kite blocks $[a_{3i+1}+n, b_{3i+1}+n, 0; a_{3t+1+i}+n]$, $[a_{3i+2}+n, b_{3i+2}+n, 0; b_{3t+1+i}+n]$, $[a_{3i+3}+n, b_{3i+3}+n, 0; b_{3t+1+i}-a_{3t+1+i}]$, $0 \le i \le t-1$, and $[a_{4t+1}+n, b_{4t+1}+n, 0; 2n+1]$ will generate a cyclic KS(v). □

Example 3.4 *The hooked Rosa sequence* $HR_5 = (2,3,2,4,3,0,5,4,1,1,0,5)$ *gives the kites* $[14,15,0;9]$, $[6,8,0;13]$, $[7,10,0;4]$, $[12,17,0;11]$. *These kites yield the base blocks for a cyclic KS(33).*

Lemma 3.5 *There exists a cyclic KS(v) for* $v \equiv 17 \pmod{24}$.

Proof. Let $v = 24t+17$ and $n = 4t+2$. By Theorem 2.4, there exists a hooked Rosa sequence of order n. Let $HR_n = (s_1, s_2, \ldots, s_{2n+2})$ be hooked Rosa sequence of order n and $\{(a_i, b_i) | 1 \le i \le n\}$ the pairs of positions in HR_n such that $b_i - a_i = i$ and $s_{a_i} = s_{b_i} = i$. Hence, the base kite blocks $[a_{3i+1}+n, b_{3i+1}+n, 0; a_{3t+1+i}+n]$, $[a_{3i+2}+n, b_{3i+2}+n, 0; b_{3t+1+i}+n]$, $[a_{3i+3}+n, b_{3i+3}+n, 0; b_{3t+1+i}-a_{3t+1+i}]$, $0 \le i \le t-1$ and $[a_{4t+1}+n, b_{4t+1}+n, 0; 2n+1]$, $[a_{4t+2}+n, b_{4t+2}+n, 0; 3n+1]$ will generate a cyclic KS(v). □

Example 3.6 *The hooked Rosa sequence* $HR_6 = (2,4,2,6,3,4,0,3,5,6,1,1,0,5)$ *gives the kites* $[17,18,0;8]$, $[7,9,0;12]$, $[11,14,0;4]$, $[15,20,0;13]$, $[10,16,0;19]$. *These kites yield the base blocks for a cyclic KS(41).*

Now we are in a position to give our first main theorem.

Theorem 3.7 *A cyclic KS(v) exists if and only if* $v \equiv 1 \pmod{8}$.

Proof. For necessity, let $(Z_v, \mathcal{B})$ be a cyclic KS(v), then the number of the kite blocks in $\mathcal{B}$ is $v(v-1)/8$. For any block $B = [a,b,c;d] \in \mathcal{B}$, let l be the length of the block orbit $Orb(B)$, then l is a divisor of v and $[a,b,c;d] = [a+l, b+l, c+l; d+l]$. Note that the vertices a, b, c, d of the kite graph B have degrees $2,2,3,1$, respectively. We can assert that $c+l \equiv c$

(mod v) and $d+l \equiv d \pmod{v}$. From $c+l \equiv c \pmod{v}$, we have $l \equiv 0 \pmod{v}$, that is $l = v$. Thus all kite orbits are full and $\mathcal{B}$ is partitioned into $(v-1)/8$ kite orbits. So $v \equiv 1 \pmod{8}$. The sufficiency follows from Lemmas 3.1, 3.3 and 3.5. This completes the proof.

4 1-Rotational Kite Systems

For a 1-rotational kite system of order v, the automorphism can be represented by $\pi : i \mapsto i+1 \pmod{v-1}$) and $\infty \mapsto \infty$ which is also represented by $\pi = (\infty)(0,1,\ldots,v-2)$ on the point-set $X = \{\infty\} \cup Z_{v-1}$. A block orbit of a 1-rotational kite system is defined similarly to that of a cyclic kite system, but under the automorphism π. Any 1-rotational kite system should be generated from base blocks.

Lemma 4.1 *There exists a 1-rotational KS(v) for* $v \equiv 0 \pmod{24}$.

Proof. For $v = 24$, we directly construct three kite base bolcks as follows:

$$[1,11,0;4],[2,9,0;6],[3,8,0;\infty].$$

For $v = 24t, t \geq 2$. Let $n = 4t$, by Theorem 2.5, there exists an $(n-2)$-near Rosa sequence of order n. Let $NR_n = (s_1, s_2, \ldots, s_{2n-1})$ be an $(n-2)$-near Rosa sequence of order n, and $\{(a_i, b_i) | 1 \leq i \leq n, i \neq n-2\}$ the pairs of positions in NR_n such that $b_i - a_i = i$ and $s_{a_i} = s_{b_i} = i$. Hence, the base kite blocks $[a_{3i+1}+n, b_{3i+1}+n, 0; a_{3(t-1)+1+i}+n]$, $[a_{3i+2}+n, b_{3i+2}+n, 0; b_{3(t-1)+1+i}+n]$, $[a_{3i+3}+n, b_{3i+3}+n, 0; b_{3(t-1)+1+i}-a_{3(t-1)+1+i}]$, $0 \leq i \leq t-2$, and $[a_{4t-3}+n, b_{4t-3}+n, 0; \infty]$, $[a_{4t-1}+n, b_{4t-1}+n, 0; 2n]$, $[a_{4t}+n, b_{4t}+n, 0; n-2]$ will generate a 1-rotational KS(v). □

Example 4.2 *The* 6*-near Rosa sequence* $NR_8 = (8,4,2,7,2,4,5,0,8,3,7,5,3,1,1)$ *gives the kites* $[22,23,0;10]$, $[11,13,0;14]$, $[18,21,,0;4]$, $[15,20,0;\infty]$, $[12,19,0;16]$, $[9,17,0;6]$. *These kites yield the base blocks for a* 1*-rotational KS*(48).

Lemma 4.3 *There exists a* 1*-rotational KS(v) for* $v \equiv 8 \pmod{24}$.

Proof. Let $v = 24t+8$ and $n = 4t+1$. By Theorem 2.1, there exists a Skolem sequence of order n. Let $S_n = (s_1, s_2, \ldots, s_{2n})$ be a Skolem sequence of order n, and $\{(a_i, b_i) | 1 \leq i \leq n\}$ the pairs of positions in S_n such that $b_i - a_i = i$ and $s_{a_i} = s_{b_i} = i$. Hence, the base kite blocks $[a_{3i+1}+n, b_{3i+1}+n, 0; a_{3t+1+i}+n]$, $[a_{3i+2}+n, b_{3i+2}+n, 0; b_{3t+1+i}+n]$, $[a_{3i+3}+n, b_{3i+3}+n, 0; b_{3t+1+i}-a_{3t+1+i}]$, $0 \leq i \leq t-1$, and $[a_{4t+1}+n, b_{4t+1}+n, 0; \infty]$ will generate a 1-rotational KS(v). □

Example 4.4 *The Skolem sequence* $S_5 = (1,1,3,4,5,3,2,4,2,5)$ *gives the kites* $[6,7,0;9]$, $[12,14,0;13]$, $[8,11,0;4]$, $[10,15,0;\infty]$. *These kites yield the base blocks for a* 1*-rotational* $KS(32)$.

Lemma 4.5 *There exists a* 1*-rotational* $KS(v)$ *for* $v \equiv 16 \pmod{24}$.

Proof. Let $v = 24t + 16$ and $n = 4t + 2$. By Theorem 2.2, there exists a hooked Skolem sequence of order n. Let $HS_n = (s_1, s_2, \ldots, s_{2n+1})$ be a hooked Skolem sequence of order n, and $\{(a_i, b_i) | 1 \le i \le n\}$ the pairs of positions in HS_n such that $b_i - a_i = i$ and $s_{a_i} = s_{b_i} = i$. Hence, the base kite blocks $[a_{3i+1} + n, b_{3i+1} + n, 0; a_{3t+1+i} + n]$, $[a_{3i+2} + n, b_{3i+2} + n, 0; b_{3t+1+i} + n]$, $[a_{3i+3} + n, b_{3i+3} + n, 0; b_{3t+1+i} - a_{3t+1+i}]$, $0 \le i \le t-1$, and $[a_{4t+1} + n, b_{4t+1} + n, 0; \infty]$,$[a_{4t+2} + n, b_{4t+2} + n, 0; 3n]$ will generate a 1-rotational KS(v). □

Example 4.6 *The hooked Skolem sequence* $HS_6 = (1,1,2,5,2,4,\ 6,3,5,\ 4,3,0,\ 6)$ *gives the kites* $[7,8,0;12]$, $[9,11,0;16]$, $[14,17,0;4]$, $[10,15,0;\infty]$, $[13,19,0;18]$. *These kites yield the base blocks for a* 1*-rotational* $KS(40)$.

Now we are in a position to give our another main theorem.

Theorem 4.7 *A* 1*-rotational* $KS(v)$ *exists if and only if* $v \equiv 0 \pmod 8$.

Proof. The proof of the necessity of Theorem 4.7 is very similar to that of Theorem 3.7. Let $(Z_{v-1} \cup \{\infty\}, \mathcal{B})$ be a 1-rotational KS(v), then the number of the kite blocks in $\mathcal{B}$ is $v(v-1)/8$. Note that all kite orbits are full and the length of any block orbit is $v - 1$, thus $\mathcal{B}$ is partitioned into $v/8$ kite orbits. So $v \equiv 0 \pmod 8$. The sufficiency follows from Lemmas 4.1, 4.3 and 4.5. This completes the proof.

Acknowledgements
The authors would like to thank the referee for his/her helpful comments and suggestions. This research is supported by NSFC under Grant No. 11371207 and the Application Research Plan Project of Nantong under Grant No.BK2014060.

References

[1] J. C. Bermond and J. Schönheim, G-decomposition of K_n, where G has four vertices or less, Discrete Math 19 (1977), 113-120.

[2] E. Billington, Cyclic balanced ternary designs with block size three and index two, Ars Combin 23(1987), 215-232.

[3] M. Buratti, 1-rotational Steiner triple systems over arbitrary groups, J Combin Des 9(2001), 215-226.

[4] C. Colbourn and Z. Jiang, The spectrum for rotational Steiner triple systems, J Combin Des 4 (1996), 105-217.

[5] C. Colbourn and A. Rosa, Triple Systems, Oxford Science Publications, Oxford, 1999.

[6] H. L. Fu, Y. H. Lin and M. Mishima, Optimal conflict-avoiding codes of even length and weight 3, IEEE Trans Inf Theory 56(11)(2010), 5747-5756.

[7] E. S. O'Keefe, Verification of a conjecture of Th. Skolem, Math Scand 9(1961), 80-82.

[8] A. Rosa, Poznamka o cyklikch Steinerovych systemoch trojic (a note on cyclic Steiner triple systems(Slovak)), Mat Fyz Casopis 16(1966), 285-290.

[9] N. Shalaby, The existence of near-Rosa and hooked near-Rosa sequences, Discrete Math 261(2003), 435-450.

[10] N. Shalaby, Skolem and Langford sequences. In: Colbourn C.J., Dinitz, J.H. (eds.), The CRC Handbook of Combinatorial Designs, CRC Press, Boca Raton (2006), 612-616.

[11] Th. Skolem, On certain distributions ofintegers in pairs with given diKerences, Math Scand 5(1957), 57-68.