Binomial transform of products

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Abstract. Given the binomial transforms $\{b_n\}$ and $\{c_n\}$ of the sequences $\{a_n\}$ and $\{d_n\}$ correspondingly, we compute the binomial transform of the sequence $\{a_n c_n\}$ in terms of $\{b_n\}$ and $\{d_n\}$. In particular, we compute the binomial transform of the sequences $\{n(n-1)...(n-m+1)\, a_n\}$ and $\{a_k x^k\}$ in terms of $\{b_n\}$. Further applications include new binomial identities with the binomial transforms of the products $H_n B_n$, $H_n F_n$, $H_n L_n(x)$, and $B_n F_n$, where H_n, B_n, F_n, and $L_n(x)$ are correspondingly the harmonic numbers, the Bernoulli numbers, the Fibonacci numbers, and the Laguerre polynomials.

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1. Introduction and main results

Given a sequence $\{a_n\}$, its binomial transform is the sequence $\{b_n\}$ defined by the formula

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k \tag{1.1}$$

with inversion

$$a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} b_k \, .$$

The binomial transform is related to Euler's series transformation [2] and provides numerous nice and elegant binomial identities (see [1], [4], [7]). It is a powerful instrument in the theory of special numbers [6] and in combinatorics.

Our purpose is to develop a technique that helps to generate new binomial transform identities from old. When the binomial transform (1.1) is known, we want to compute the image sequence

$$\sum_{k=0}^{n} \binom{n}{k} a_k c_k$$

($n = 0, 1, \ldots$), where the transform

$$d_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} c_k, \quad c_n = \sum_{k=0}^{n} \binom{n}{k} d_k \tag{1.2}$$

is also known. Work in this direction was started in the recent paper [1], where it was shown that

$$\sum_{k=1}^{n} \binom{n}{k} \frac{a_k}{k+\lambda} = n! \sum_{m=1}^{n} \frac{b_m}{m!\,(\lambda+m)(\lambda+m+1)\ldots(\lambda+n)}$$

for any $\lambda \geq 0$. Writing for brevity $\nabla b_n = b_n - b_{n-1}, \nabla^0 b_n = b_n$ for the backward difference, we have also the following result from [1]

$$\sum_{k=0}^{n} \binom{n}{k} k^p a_k = (n\nabla)^p b_n \tag{1.3}$$

for any $0 \leq p \leq n$. This will be needed later.

In this paper we prove the following theorem.

Theorem 1. Let $\{a_n\}$ and $\{c_n\}$ be two sequences and let $\{b_n\}$ and $\{d_n\}$ be defined by (1.1) and (1.2). Then we have the identity

$$\sum_{k=0}^{n}\binom{n}{k} a_k c_k = \sum_{m=0}^{n}\binom{n}{m} d_m \nabla^m b_n , \tag{1.4}$$

where $\nabla b_n = b_n - b_{n-1}$ with $\nabla^0 b_n = b_n$.

For the symmetric version of the transform $\{c_n\} \leftrightarrow \{d_n\}$, namely,

$$d_n = \sum_{k=0}^{n}\binom{n}{k}(-1)^k c_k, \quad c_n = \sum_{k=0}^{n}\binom{n}{k}(-1)^k d_k ,$$

equation (1.4) takes the form

$$\sum_{k=0}^{n}\binom{n}{k} a_k c_k = \sum_{m=0}^{n}\binom{n}{m}(-1)^m d_m \nabla^m b_n . \tag{1.5}$$

As we shall see, with appropriate choices of the sequences $\{a_n\}$ and $\{c_n\}$ this formula produces interesting new identities involving binomial polynomials and special numbers. In several cases the iterated differences $\nabla^m b_n$ can be computed explicitly.

The proof of the theorem is based on the lemma:

Lemma 1. Suppose the sequences $\{a_n\}$ and $\{b_n\}$ are defined from (1.1) and let $\nabla b_n = b_n - b_{n-1}$, $\nabla^0 b_n = b_n$. Then for every two integers $0 \le m \le n$ we have

$$\sum_{k=0}^{n}\binom{n}{k} k(k-1)\ldots(k-m+1)\, a_k = m!\binom{n}{m}\nabla^m b_n \tag{1.6}$$

or, in a shorter form,

$$\sum_{k=0}^{n}\binom{n}{k}\binom{k}{m} a_k = \binom{n}{m}\nabla^m b_n . \tag{1.7}$$

The case $m=1$ is (1.3) with $p=1$. It was proved in [1].

The next lemma presents one possible way to compute the RHS in the above identity.

Lemma 2. For any integers $0 \le m \le n$,

$$\binom{n}{m}\nabla^m b_n = \sum_{j=0}^{n}\binom{n}{j}\binom{j}{n-m}(-1)^{n-j} b_j . \tag{1.8}$$

This can also be put in the form

$$\nabla^m b_n = m! \sum_{j=n-m}^{n} \frac{(-1)^{n-j} b_j}{(n-j)!(j-n+m)!} .$$

The proofs are given in section 5.

Example 1. Let $a_k = 1$ for all k. Then

$$b_n = \sum_{k=0}^{n}\binom{n}{k} = 2^n ,$$

and Lemma1 gives for any $n = 0,1,2,...,$ and any $0 \le m \le n$,

$$\sum_{k=0}^{n}\binom{n}{k} k(k-1)...(k-m+1) = m!\binom{n}{m}\nabla^m 2^n = m!\binom{n}{m}2^{n-m} , \tag{1.9}$$

as by a simple computation we find $\nabla^m 2^n = 2^{n-m}$.

In the next section we apply our theorem for the case when $c_k = x^k$, and in section 3, among other things, we consider the case $a_k = (-1)^{k-1} H_k$, where H_k are the harmonic numbers. We shall compute the iterated differences $\nabla^m H_n$ and prove the identity

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{k-1} H_k c_k = (-1)^{n-1} H_n d_n + \sum_{m=0}^{n-1}\frac{(-1)^m d_m}{n-m} , \tag{1.10}$$

for any sequence $\{c_k\}$ where c_k and d_k are related by (1.2). In examples 13, 14, and 15 we apply this formula to the cases where c_k are correspondingly the Fibonacci numbers, the Bernoulli numbers, and the Lagurre polynomials.

A similar identity is proved also for the Fibonacci numbers,

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{k-1} F_k c_k = \sum_{m=0}^{n} \binom{n}{m} d_m F_{n-2m} \tag{1.11}$$

with c_k and d_k as above.

2. Binomial polynomials

For a given sequence $\{a_n\}$ we consider the polynomials

$$p_n(x) = \sum_{k=0}^{n} \binom{n}{k} a_k x^k .$$

When the binomial transform (1.1) is known, we want to compute the polynomials $p_n(x)$ explicitly in terms of the numbers $p_n(1) = b_n$. We present here two solutions to this problem. They both follow from Theorem1 with different choices of the sequences $\{a_n\}$ and $\{c_n\}$.

Corollary 1. Assuming the binomial transform (1.1) is given, we have the identity

$$\sum_{k=0}^{n} \binom{n}{k} a_k x^k = \sum_{j=0}^{n} \binom{n}{j} b_j x^j (1-x)^{n-j} \tag{2.1}$$

$$= (1-x)^n \sum_{j=0}^{n} \binom{n}{j} b_j \left(\frac{x}{1-x} \right)^j .$$

Proof. In Theorem1 we set

$$a_n = (-1)^n x^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k}(1-x)^k$$

so that in view of (1.1), $b_n = (1-x)^n$. Simple computation shows that for any z, m, and n,

$$\nabla^m z^n = z^{n-m}(z-1)^m$$

and thus

$$\nabla^m b_n = \nabla^m (1-x)^n = (1-x)^{n-m}(-x)^m.$$

From Theorem 1

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k x^k c_k = \sum_{m=0}^{n} \binom{n}{m} d_m (-x)^m (1-x)^{n-m}$$

$$= \sum_{m=0}^{n} \binom{n}{m} (-1)^m d_m x^m (1-x)^{n-m}.$$

Here we change notations in order to write this equation in terms of the sequences $\{a_n\}$ and $\{b_n\}$. Setting $a_k = (-1)^k c_k$ we have from (1.2)

$$(-1)^n d_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k c_k = \sum_{k=0}^{n} \binom{n}{k} a_k = b_n,$$

and the above equation becomes

$$\sum_{k=0}^{n} \binom{n}{k} a_k x^k = \sum_{m=0}^{n} \binom{n}{m} b_m x^m (1-x)^{n-m},$$

as needed.

Second proof, independent of Theorem 1. Using the inversion formula we can write

$$p_n(x) = \sum_{k=0}^{n} \binom{n}{k} a_k x^k = \sum_{k=0}^{n} \binom{n}{k} x^k \left\{ \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} b_j \right\}$$

$$= \sum_{j=0}^{n} (-1)^j b_j \left\{ \sum_{k=j}^{n} \binom{n}{k} \binom{k}{j} (-1)^k x^k \right\},$$

and the rest follows from the well-known identity [4], (3.11.8)

$$\sum_{k=j}^{n} \binom{n}{k} \binom{k}{j} (-1)^k x^k = (-1)^j \binom{n}{j} x^j (1-x)^{n-j}. \tag{2.2}$$

Remark 1. Identity (2.2) itself follows from Corollary 1 applied to the convolution identity

$$\sum_{k=j}^{n} \binom{n}{k} \binom{k}{j} (-1)^k = (-1)^n \delta_{nj}.$$

Here are some examples of representations like (2.1).

Example 2. The generalized Stirling numbers $S(\alpha, n)$ of the second kind are defined by the binomial formula (see [3] and the references therein)

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k k^\alpha = (-1)^n n! \, S(\alpha, n), \tag{2.3}$$

where α is any complex number with $\operatorname{Re}\alpha > 0$. According to Corollary1 we have

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k k^\alpha x^k = \sum_{j=0}^{n} \binom{n}{j} j! \, S(\alpha, j) (-1)^j x^j (1-x)^{n-j},$$

or, changing x to $-x$,

$$\sum_{k=0}^{n} \binom{n}{k} k^\alpha x^k = \sum_{j=0}^{n} \binom{n}{j} j! \, S(\alpha, j) \, x^j (1+x)^{n-j}. \tag{2.4}$$

When α is a non-negative integer, $S(\alpha, j)$ are the usual Stirling numbers of the second kind [6].

Example 3. Setting $X=2$ in (2.1) we obtain the curious identity

$$\sum_{k=0}^{n}\binom{n}{k}2^k a_k=\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}2^j b_j .$$

Example 4. With $X=\frac{1}{2}$ in (2.1),

$$\sum_{k=0}^{n}\binom{n}{k}2^{n-k} a_k=\sum_{j=0}^{n}\binom{n}{j}b_j$$

which explains the action of the iterated binomial transform.

The representation (2.1) in the above corollary is short and simple, but its RHS is not a polynomial in standard form. From Theorem 1 we obtain also a second corollary:

Corollary 2. Suppose the sequence $\{b_n\}$ is the binomial transform of the sequence $\{a_n\}$. Then

$$\sum_{k=0}^{n}\binom{n}{k}a_k X^k=\sum_{m=0}^{n}\binom{n}{m}\nabla^m b_n(X-1)^m . \tag{2.5}$$

Proof. Taking $c_k = X^k$ in Theorem 1, equation (1.4), we have the one-line proof

$$d_n=\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}X^k=(-1)^n\sum_{k=0}^{n}\binom{n}{k}(-X)^k=(X-1)^n .$$

Formula (2.5) gives, in fact, the Taylor expansion of the polynomial $p_n(X)$ centered at $X=1$. Most existing examples of binomial transforms with "X" have this format.

Example 5. With $X=2$ in (2.5) we find

$$\sum_{k=0}^{n}\binom{n}{k}2^k a_k=\sum_{m=0}^{n}\binom{n}{m}\nabla^m b_n .$$

This together with Example 3 provide

$$\sum_{m=0}^{n}\binom{n}{m}\nabla^m b_n=\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}2^j b_j . \tag{2.6}$$

Example 6. Here is one very simple demonstration how the corollary works. Let $a_n=(-1)^{n-1}$. Then we have for $n\geq 0$

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{k-1}=\begin{cases}-1, & n=0\\ 0, & n>0\end{cases}$$

and from (1.8) we find $\nabla^m b_n=0$ for $m\neq n$ and $\nabla^n b_n=(-1)^{n-1}$. Therefore, from (2.5),

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{k-1}x^k=(-1)^{n-1}(x-1)^n=-(1-x)^n . \tag{2.7}$$

Of course, this follows immediately from the binomial formula.

3. Identities with special numbers

This section contains some new identities for products of harmonic, Bernoulli and Fibonacci numbers. We start with the following lemma:

Lemma 3. For any two integers $1\leq m\leq n$,

$$\sum_{k=m}^{n}\binom{n}{k}\binom{k}{m}\frac{(-1)^k}{k}=\frac{(-1)^m}{m} . \tag{3.1}$$

The proof is given in Section 5.

Example 7. Let $a_k = \frac{(-1)^k}{k}, \ k \geq 1$. Then for any two integers $1 \leq m \leq n$ we have from the above lemma and from Lemma1,

$$\sum_{k=1}^{n} \binom{n}{k} \binom{k}{m} a_k = \binom{n}{m} \nabla^m b_n = \frac{(-1)^m}{m} . \tag{3.2}$$

Then from Theorem1, the symmetric version, equation (1.5)

$$\sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^k c_k}{k} = \sum_{m=1}^{n} \frac{d_m}{m} . \tag{3.3}$$

This property of the binomial transform was discussed in [1]. It is true for any two sequences $\{c_k\}, \{d_k\}, k \geq 1$ related by

$$d_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k c_k, \quad c_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k d_k,$$

and the factor $(-1)^k$ can be replaced by $(-1)^{k-1}$. For the transform (1.1) the property is

$$\sum_{k=1}^{n} \binom{n}{k} \frac{a_k}{k} = \sum_{m=1}^{n} \frac{b_m}{m} .$$

Example 8. This example is related to the previous one. Let

$$H_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}, \ H_0 = 0, \ n \geq 0,$$

be the harmonic numbers. Then we have (see [4])

$$\sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k-1}}{k} = H_n . \tag{3.4}$$

According to Lemma 1,

$$\sum_{k=m}^{n}\binom{n}{k}\binom{k}{m}\frac{(-1)^{k-1}}{k}=\binom{n}{m}\nabla^{m}H_{n} \tag{3.5}$$

and then from Lemma 3, for $1\leq m\leq n$,

$$\binom{n}{m}\nabla^{m}H_{n}=\frac{(-1)^{m-1}}{m} \tag{3.6}$$

and when $m=0$ we have $\binom{n}{0}\nabla^{0}H_{n}=H_{n}$.

Therefore, from (3.4) and Corollary 2, by separating the first term in the second sum,

$$\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{k-1}x^{k}}{k}=\sum_{m=0}^{n}\binom{n}{m}\nabla^{m}H_{n}(x-1)^{m}$$

$$=H_{n}+\sum_{k=1}^{n}\frac{(-1)^{k-1}(x-1)^{k}}{k},$$

that is,

$$\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{k-1}x^{k}}{k}=H_{n}-\sum_{k=1}^{n}\frac{(1-x)^{k}}{k}. \tag{3.7}$$

This equation also follows from (2.7) and (3.3).

Formula (2.5) can be used to evaluate the iterated differences $\nabla^{m}b_{n}$ when the LHS in (2.5) is known.

Example 9. By inversion in (3.4) we have

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{k-1}H_{k}=\frac{1}{n}. \tag{3.8}$$

The version with "x" was presented in [2], that is,

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{k-1}x^k H_k$$

$$=\frac{1}{n}+\frac{1-x}{n-1}+\frac{(1-x)^2}{n-2}+\ldots+\frac{(1-x)^{n-2}}{2}+\frac{(1-x)^{n-1}}{1}-(1-x)^n H_n\,.$$

Comparing this to (2.5) we conclude that

$$\binom{n}{m}\nabla^m\frac{1}{n}=\frac{(-1)^m}{n-m}\quad\text{when } 0\le m<n\,, \tag{3.9}$$

and $\nabla^n \dfrac{1}{n}=(-1)^{n-1}H_n$.

From Example 9 and Theorem1 we derive the following most interesting result:

Corollary 3. Let $\{c_k\}$ and $\{d_k\}$ be any two sequences related as in (1.2). Then

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{k-1}H_k c_k=(-1)^{n-1}H_n d_n+\sum_{m=0}^{n-1}\frac{(-1)^m d_m}{n-m}\,. \tag{3.10}$$

For the proof we take $a_k=(-1)^{k-1}H_k$ and then in view of (1.1) and (3.8) we have $b_n=\dfrac{1}{n}$. The rest follows from (3.9) and Theorem 1.

To show Corollary 3 in action we shall give several examples.

Example 10. Applying property (3.3) to equation (3.8) we find

$$\sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1}\frac{H_k}{k}=1+\frac{1}{2^2}+\ldots+\frac{1}{n^2}\,.$$

With the notation

$$H_n^{(2)}=1+\frac{1}{2^2}+\ldots+\frac{1}{n^2},\quad H_0^{(2)}=0,$$

we obtain by inversion $(n \geq 1)$

$$\frac{H_n}{n} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{k-1} H_k^{(2)} ,$$

and (3.10) yields (with $c_k = \frac{H_k}{k}$, $d_m = (-1)^{m-1} H_m^{(2)}$, related as in (1.2)),

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{k-1} \frac{H_k^2}{k} = H_n H_n^{(2)} - \sum_{m=0}^{n-1} \frac{H_m^{(2)}}{n-m} . \tag{3.11}$$

Example 11. In the same way, starting from the identity (see [1], equation (16))

$$\frac{H_n}{n+1} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{k-1} \frac{H_k}{k+1} ,$$

and taking $c_k = \frac{H_k}{k+1}$, $d_m = \frac{(-1)^{m-1} H_m}{m+1}$ in (3.10) we obtain

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{k-1} \frac{H_k^2}{k+1} = \frac{H_n^2}{n+1} - \sum_{m=0}^{n-1} \frac{H_m}{(n-m)(m+1)} . \tag{3.12}$$

Now we show identities involving other special numbers.

Example 12. By inversion in (2.3),

$$n^\alpha = \sum_{k=0}^{n} \binom{n}{k} k!\, S(\alpha, k)$$

and therefore, from (3.10) with $c_k = k^\alpha$ and $d_m = m!\, S(\alpha, m)$,

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{k-1} H_k k^\alpha = (-1)^{n-1} n!\, H_n S(\alpha, n) + \sum_{m=0}^{n-1} \frac{(-1)^m m!\, S(\alpha, m)}{n-m} . \tag{3.13}$$

Example 13. For the Fibonacci numbers F_n it is known that the following two binomial identities are true

$$F_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{k-1} F_k \, , \tag{3.14}$$

$$F_{2n} = \sum_{k=0}^{n} \binom{n}{k} F_k \, .$$

From here and (3.10) we derive two new identities involving products of harmonic and Fibonacci numbers

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{k-1} H_k F_k = H_n \, F_n - \sum_{m=0}^{n-1} \frac{F_m}{n-m} \, , \tag{3.15}$$

by taking $c_k = F_k$, $d_m = (-1)^{m-1} F_m$. Also

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{k-1} H_k F_{2k} = (-1)^{n-1} H_n \, F_n + \sum_{m=0}^{n-1} \frac{(-1)^m F_m}{n-m} \tag{3.16}$$

with $c_k = F_{2k}$, $d_m = F_m$.

Example 14. Here we use the Bernoulli numbers B_n defined by the generating function

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \; | \, t \, | < 2\pi \, .$$

For the Bernoulli numbers it is known that

$$(-1)^n B_n = \sum_{k=0}^{n} \binom{n}{k} B_k \, . \tag{3.17}$$

From (3.10) with $c_k = (-1)^k B_k$, $d_m = B_m$ we obtain the identity

$$\sum_{k=0}^{n}\binom{n}{k} H_k B_n = (-1)^n H_n B_n - \sum_{m=0}^{n-1} \frac{(-1)^m B_m}{n-m} . \qquad (3.18)$$

Example 15. In this last example related to (3.10) we use the Laguerre polynomials

$$L_n(x) = \frac{e^x}{n!}\left(\frac{d}{dx}\right)(x^n e^{-x})$$

which satisfy the identity

$$L_n(x) = \sum_{k=0}^{n}\binom{n}{k}\frac{(-x)^k}{k!} .$$

Here (3.10) provides the curious formula ($c_k = L_k(x)$ and $d_m = \dfrac{(-x)^m}{m!}$)

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k H_k L_k(x) = \frac{x^n}{n!} H_n - \sum_{m=0}^{n-1} \frac{x^m}{m!(n-m)} . \qquad (3.19)$$

Next we turn again to the sequence of Fibonacci numbers defined by the recurrence $F_n = F_{n-1} + F_{n-2}$, and starting with $F_0 = 0,\ F_1 = 1$. We can extend the sequence F_n for negative indices by using the equation $F_{n-2} = F_n - F_{n-1}$. Thus we come to the negatively indexed Fibonacci numbers, where $F_{-n} = (-1)^{n+1} F_n,\ n \geq 0$. Computing the backward differences we find

$$\nabla F_n = F_n - F_{n-1} = F_{n-2} ,$$

$$\nabla^2 F_n = F_{n-2} - F_{n-3} = F_{n-4} ,$$

etc. Obviously, $\nabla^m F_n = F_{n-2m}$ and this is true for any non-negative integer m . Now we can formulate the desired result:

Corollary 4. For any pair of sequences $\{c_k\}$ and $\{d_k\}$ as in (1.2), and every non-negative integer n we have

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{k-1}F_k c_k=\sum_{m=0}^{n}\binom{n}{m}d_m F_{n-2m}\,. \qquad (3.20)$$

For the proof we use (1.4) in Theorem 1 with $a_k=(-1)^{k-1}F_k$ and $b_n=F_n$ (see (3.14)).

Formula (3.20) can be used in the same way as (3.10) to generate various new identifies by choosing different sequences $\{c_k\}$. For illustration we provide the following example:

Example 16. Choosing $c_n=(-1)^n B_n$ and $d_k=B_k$, where B_n are the Bernoulli numbers, we obtain from (3.17) and (3.20) an identity connecting Bernoulli and Fibonacci numbers

$$\sum_{k=0}^{n}\binom{n}{k}B_k F_k=-\sum_{m=0}^{n}\binom{n}{m}B_m F_{n-2m} \qquad (3.21)$$

or,

$$\sum_{k=0}^{n}\binom{n}{k}B_k(F_k+F_{n-2k})=0\,.$$

The Lucas numbers L_n satisfy the same recurrence $L_n=L_{n-1}+L_{n-2}$ as the Fibonacci numbers and for them binomial identities like (3.14) hold too. Therefore, a property similar to (3.20) is also true for the Lucas numbers.

4. Some variations

Remark 2. If the binomial transform is defined by the formula

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k a_k=b_n\,,$$

then (2.5) takes the form

$$\sum_{k=0}^{n}\binom{n}{k}a_k(-x)^k=\sum_{m=0}^{n}\binom{n}{m}\nabla^m b_n(x-1)^m \tag{4.1}$$

or

$$\sum_{k=0}^{n}\binom{n}{k}a_k x^k=\sum_{m=0}^{n}\binom{n}{m}(-1)^m\nabla^m b_n(x+1)^m .$$

Remark 3. Another expression for the coefficients

$$C(n,m)=\binom{n}{m}\nabla^m b_n$$

can be written in terms of the Stirling numbers of the first kind $s(m, j)$. A good reference for these numbers is the book [6].

Suppose the sequence $\{b_n\}$ is the binomial transform of the sequence $\{a_n\}$ as in (1.1). Then

$$C(n,m)=\frac{1}{m!}\sum_{j=0}^{m}s(m,j)\,(n\nabla)^j b_n. \tag{4.2}$$

Proof. We have the representation

$$\binom{k}{m}=\frac{1}{m!}\sum_{j=0}^{m}s(m,j)k^j$$

(see [6]) and from here and (1.7)

$$C(n,m)=\sum_{k=0}^{n}\binom{n}{k}\binom{k}{m}a_k=\frac{1}{m!}\sum_{k=0}^{n}\binom{n}{k}\left\{\sum_{j=0}^{m}s(m,j)\,k^j a_k\right\}$$

$$=\frac{1}{m!}\sum_{j=0}^{m}s(m,j)\left\{\sum_{k=0}^{n}\binom{n}{k}k^j a_k\right\}.$$

Therefore, in view of (1.3) we come to (4.2).

5. Proofs

Here we prove the three lemmas and Theorem 1.

Proof of Lemma 1. When $n=0$ this is obviously true. Take any integers $n \geq 1$. We shall do induction on $1 \leq m \leq n$. Suppose the identity is true for some $m < n$. We shall prove it for $m+1$.

The LHS then becomes (using (1.3) with $p=1$ in the second equality)

$$\sum_{k=0}^{n}\binom{n}{k} k(k-1)\ldots(k-m+1)(k-m)\, a_k =$$

$$\sum_{k=0}^{n}\binom{n}{k} k\{k(k-1)\ldots(k-m+1)\, a_k\} - m\sum_{k=0}^{n}\binom{n}{k} k(k-1)\ldots(k-m+1)\, a_k$$

$$= n\nabla\left\{m!\binom{n}{m}\nabla^m b_n\right\} - m\left\{m!\binom{n}{m}\nabla^m b_n\right\}$$

$$= m!\left\{n\binom{n}{m}\nabla^m b_n - n\binom{n-1}{m}\nabla^m b_{n-1} - m\binom{n}{m}\nabla^m b_n\right\}$$

$$= m!\left\{(n-m)\binom{n}{m}\nabla^m b_n - n\binom{n-1}{m}\nabla^m b_{n-1}\right\}$$

$$= m!\left\{\frac{n!}{(n-m-1)!\,m!}\nabla^m b_n - \frac{n!}{(n-m-1)!\,m!}\nabla^m b_{n-1}\right\}$$

$$= m!\left\{\frac{n!}{(n-m-1)!\,m!}\nabla^{m+1} b_n\right\} = (m+1)!\binom{n}{m+1}\nabla^{m+1} b_n\,.$$

Proof of Lemma 2. From Lemma 1 and the inversion formula for the binomial transform

$$\binom{n}{m}\nabla^m b_n = \frac{1}{m!}\sum_{k=0}^{n}\binom{n}{k} k(k-1)\ldots(k-m+1)\, a_k = \sum_{k=0}^{n}\binom{n}{k}\binom{k}{m} a_k$$

$$= \sum_{k=0}^{n}\binom{n}{k}\binom{k}{m}\left\{\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j} b_j\right\}$$

$$= \sum_{j=0}^{n}(-1)^j b_j\left\{\sum_{k=0}^{n}\binom{n}{k}\binom{k}{m}\binom{k}{j}(-1)^k\right\}$$

$$= \sum_{j=0}^{n}\binom{n}{j}\binom{j}{n-m}(-1)^{n-j} b_j$$

by using the identity ([7], p.15)

$$\sum_{k=0}^{n}\binom{n}{k}\binom{k}{m}\binom{k}{j}(-1)^k = (-1)^n\binom{n}{j}\binom{j}{n-m}.$$

Proof of Theorem1.

$$\sum_{k=0}^{n}\binom{n}{k} a_k c_k = \sum_{k=0}^{n}\binom{n}{k} a_k\left\{\sum_{m=0}^{k}\binom{k}{m} d_m\right\}$$

$$= \sum_{m=0}^{n} d_m\left\{\sum_{k=0}^{n}\binom{n}{k}\binom{k}{m} a_k\right\} = \sum_{m=0}^{n}\binom{n}{m} d_m \nabla^m b_n$$

according to Lemma1.

Proof of Lemma 3.

The starting point of this proof is the identity (2.2) with $1 \le j \le n$. We divide both sides by X and integrate from 0 to 1. This yields

$$\sum_{k=j}^{n}\binom{n}{k}\binom{k}{j}\frac{(-1)^k}{k}=(-1)^j\binom{n}{j}\int_0^1 x^{j-1}(1-x)^{n-j}dx$$

$$=(-1)^j\binom{n}{j}B(n-j+1,j)=\frac{(-1)^j}{j}.$$

The evaluation of the integral is from table [5], namely, this is entry 3.191 (3). Here $B(x,y)$ is Euler's Beta function,

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

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