SOME IDENTITIES OF SYMMETRY FOR q-BERNOULLI POLYNOMIALS UNDER SYMMETRIC GROUP OF DEGREE n

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ABSTRACT. In this paper, we give some new identities of symmetry for q-Bernoulli polynomials under the symmetric group of degree n arising from p-adic q-integrals on $\mathbb{Z}_p$.

1. INTRODUCTION

Let p be a fixed prime number. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. The p-adic norm is normalized as $|p|_p = \frac{1}{p}$. Let q be an indeterminate in $\mathbb{C}_p$ such that $|1-q|_p < p^{-\frac{1}{p-1}}$. The q-analogue of the number x is defined as $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q\to 1}[x]_q = x$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim as

$$(1.1) \quad I_q(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x, \quad (\text{see [7]}).$$

From (1.1), we have

(1.2)

$$qI_q(f_1) - I_q(f) = (q-1) f(0) + \frac{q-1}{\log q} f'(0), \quad \text{where } f_1(x) = f(x+1).$$

As is well known, the Bernoulli numbers are defined by

$$B_0 = 1, \quad (B+1)^n - B_n = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases}$$

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with the usual convention about replacing B^n by B_n (see [1–12]). The Bernoulli polynomials are given by

$$(1.3) \qquad B_n(x) = \sum_{l=0}^{n} \binom{n}{l} B_l x^{n-l}, \quad (n \geq 0), \quad (\text{see [12]}).$$

In [3], L. Carlitz considered the q-analogue of Bernoulli numbers as follows:

$$(1.4) \qquad \beta_{0,q} = 1, \quad q(q\beta_q + 1)^n - \beta_{n,q} = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases}$$

with the usual convention about replacing β_q^n by $\beta_{n,q}$.

He also defined q-Bernoulli polynomials as follows:

$$(1.5) \qquad \beta_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} q^{lx} [x]_q^{n-l} \beta_{l,q} \quad (\text{see [1–3, 9]}).$$

In [7], Kim proved the following integral representation related to Carlitz q-Bernoulli polynomials:

$$(1.6) \qquad \beta_{n,q}(x) = \int_{\mathbb{Z}_p} [x+y]_q^n \, d\mu_q(x), \quad (n \geq 0).$$

From (1.2), we note that

$$(1.7) \qquad q \int_{\mathbb{Z}_p} [x+1]_q^n \, d\mu_q(x) - \int_{\mathbb{Z}_p} [x]_q^n \, d\mu_q(x) = \begin{cases} q-1 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}.$$

By (1.7), we get

$$\beta_{0,q} = 1, \quad q\beta_{n,q}(1) - \beta_{n,q} = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}.$$

The purpose of this paper is to give identities of symmetry for Carlitz's q-Bernoulli polynomials under the symmetric group of degree n arising from p-adic q-integrals on $\mathbb{Z}_p$.

2. Symmetric identiites of $\beta_{n,q}(x)$ under S_n

For $n \in \mathbb{N}$, let $w_1, w_2, \ldots, w_n \in \mathbb{N}$. Then, we have

(2.1)

$$\int_{\mathbb{Z}_p} e^{\left[(\prod_{j=1}^{n-1} w_j) y + (\prod_{j=1}^{n} w_j) x + w_n \sum_{j=1}^{n-1} \left(\prod_{\substack{i=1 \\ i \neq j}}^{n-1} w_i \right) k_j \right]_q t} d\mu_{q^{w_1 w_2 \cdots w_{n-1}}}(y)$$

$$= \lim_{N \to \infty} \frac{1}{[p^N]_{q^{w_1 w_2 \cdots w_{n-1}}}}$$

$$\times \sum_{y=0}^{p^N-1} e^{\left[(\prod_{j=1}^{n-1} w_j)y+(\prod_{j=1}^{n} w_j)x+w_n \sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1} w_i\right)k_j\right]_q t}$$

$$\times q^{(\prod_{j=1}^{n-1} w_j)y}.$$

$$= \lim_{N\to\infty} \frac{1}{[w_n p^N]_{q^{w_1 w_2\cdots w_{n-1}}}} \sum_{m=0}^{w_n-1}\sum_{y=0}^{p^N-1}$$

$$\times e^{\left[(\prod_{j=1}^{n-1} w_j)(m+w_n y)+(\prod_{j=1}^{n} w_j)x+w_n \sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1} w_i\right)k_j\right]_q t}$$

$$\times q^{w_1 w_2\cdots w_{n-1}(m+w_n y)}.$$

Thus, by 2.1, we get

(2.2)

$$\frac{1}{\left[\prod_{l=1}^{n-1} w_l\right]_q} \prod_{l=1}^{n-1}\sum_{k_l=0}^{w_l-1} q^{w_n \sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1} w_i\right)k_j}$$

$$\times \int_{\mathbb{Z}_p} e^{\left[(\prod_{j=1}^{n-1} w_j)y+\prod_{j=1}^{n} w_j x+w_n \sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1} w_i\right)k_j\right]_q t} d\mu_{q^{w_1 w_2\cdots w_{n-1}}}(y)$$

$$= \lim_{N\to\infty} \frac{1}{[\prod_{l=1}^{n} w_l p^N]_q}$$

$$\times \prod_{l=1}^{n-1}\sum_{k_l=0}^{w_l-1}\sum_{m=0}^{w_n-1}\sum_{y=0}^{p^N-1} q^{(\prod_{j=1}^{n-1} w_j)(m+w_n y)+\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1} w_i\right)k_j w_n}$$

$$\times e^{\left[(\prod_{j=1}^{n-1} w_j)(m+w_n y)+(\prod_{j=1}^{n} w_j)x+w_n \sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1} w_i\right)k_j\right]_q t}.$$

We note that (2.2) is invariant under any permutation $\sigma \in S_n$. Therefore, by (2.2), we obtain the following theorem.

Theorem 2.1. *For $w_1, w_2, \ldots, w_n \in \mathbb{N}$, the following expressions*

$$\frac{1}{\left[\prod_{l=1}^{n-1} w_{\sigma(l)}\right]_q} \prod_{l=1}^{n-1}\sum_{k_l=0}^{w_{\sigma(l)}-1} q^{w_{\sigma(n)} \sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1} w_{\sigma(i)}\right)k_j}$$

$$\times \int_{\mathbb{Z}_p} e^{\left[(\prod_{j=1}^{n-1} w_{\sigma(j)})y+\prod_{j=1}^{n} w_j x+w_{\sigma(n)} \sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1} w_{\sigma(i)}\right)k_j\right]_q t}$$

$$\times \, d\mu_{q^{w_{\sigma(1)}\cdots w_{\sigma(n-1)}}}(y)$$

are the same for any $\sigma \in S_n$.

We observe that

$$\text{(2.3)} \qquad \left[\left(\prod_{j=1}^{n-1} w_j\right) y + \left(\prod_{j=1}^{n} w_j\right) x + w_n \sum_{j=1}^{n-1} \left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right) k_j\right]_q$$

$$= \left[\prod_{j=1}^{n-1} w_j\right]_q \left[y + w_n x + \frac{w_n}{w_1}k_1 + \cdots + \frac{w_n}{w_{n-1}}k_{n-1}\right]_{q^{w_1\cdots w_{n-1}}}$$

$$= \left[\sum_{j=1}^{n-1} w_j\right]_q \left[y + w_n x + \sum_{j=1}^{n-1} \frac{w_n}{w_j}k_j\right]_{q^{w_1\cdots w_{n-1}}}.$$

Thus, by (2.3), we get

(2.4)

$$\int_{\mathbb{Z}_p} e^{\left[(\prod_{j=1}^{n-1} w_j)y + (\prod_{j=1}^{n} w_j)x + w_n \sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j\right]_q t} d\mu_{q^{w_1\cdots w_{n-1}}}(y)$$

$$= \sum_{m=0}^{\infty} \left[\prod_{j=1}^{n-1} w_j\right]_q^m \int_{\mathbb{Z}_p} \left[y + w_n x + w_n \sum_{j=1}^{n-1} \frac{k_j}{w_j}\right]_{q^{w_1\cdots w_{n-1}}}^m d\mu_{q^{w_1\cdots w_{n-1}}}(y)\frac{t^m}{m!}$$

$$= \sum_{m=0}^{\infty} \left[\prod_{j=1}^{n-1} w_j\right]_q^m \beta_{m,q^{w_1\cdots w_{n-1}}}\left(w_n x + \sum_{j=1}^{n-1} \frac{w_n}{w_j}k_j\right)\frac{t^m}{m!}.$$

For $m \geq 0$, from (2.4), we have

$$\text{(2.5)} \qquad \int_{\mathbb{Z}_p} \left[\left(\prod_{j=1}^{n-1} w_j\right) y + \left(\prod_{j=1}^{n} w_j\right) x + w_n \sum_{j=1}^{n-1} \left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right) k_j\right]_q^m$$

$$\times d\mu_{q^{w_1\cdots w_{n-1}}}(y)$$

$$= \left[\prod_{j=1}^{n-1} w_j\right]_q^m \beta_{m,q^{w_1\cdots w_{n-1}}}\left(w_n x + \sum_{j=1}^{n-1} \frac{w_n}{w_j}k_j\right).$$

Therefore, by Theorem 2.1 and (2.5), we obtain the following theorem.

Theorem 2.2. *For $m \geq 0$, $w_1, \ldots, w_n \in \mathbb{N}$, the following expressions*

$$\left[\prod_{j=1}^{n-1} w_{\sigma(j)}\right]_q^{m-1} \prod_{l=1}^{n-1} \sum_{k_l=0}^{w_{\sigma(l)}-1} q^{\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_{\sigma(i)}\right) k_j w_{\sigma(n)}}$$

$$\times \beta_{m,q^{w_{\sigma(1)}\cdots w_{\sigma(n-1)}}}\left(w_{\sigma(n)}x + w_{\sigma(n)}\sum_{j=1}^{n-1}\frac{k_j}{w_{\sigma(j)}}\right)$$

are the same for any $\sigma \in S_n$.

It is easy to show that

$$\left[y + w_n x + w_n \sum_{j=1}^{n-1}\frac{k_j}{w_j}\right]_{q^{w_1\cdots w_{n-1}}} \tag{2.6}$$

$$= \frac{[w_n]_q}{\left[\prod_{j=1}^{n-1} w_j\right]_q}\left[\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right) k_j\right]_{q^{w_n}}$$

$$+ q^{w_n \sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right) k_j} [y + w_n x]_{q^{w_1\cdots w_{n-1}}}.$$

From (2.6), we can derive the following equation:

$$\int_{\mathbb{Z}_p}\left[y + w_n x + w_n \sum_{j=1}^{n-1}\frac{k_j}{w_j}\right]_{q^{w_1\cdots w_{n-1}}}^m d\mu_{q^{w_1\cdots w_{n-1}}}(y) \tag{2.7}$$

$$= \sum_{l=0}^{m}\binom{m}{l}\left(\frac{[w_n]_q}{\left[\prod_{j=1}^{n-1} w_j\right]_q}\right)^{m-l}\left[\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right) k_j\right]_{q^{w_n}}^{m-l}$$

$$\times q^{l w_n \sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right) k_j}$$

$$\times \int_{\mathbb{Z}_p} [y + w_n x]_{q^{w_1\cdots w_{n-1}}}^l \, d\mu_{q^{w_1\cdots w_{n-1}}}(y)$$

$$= \sum_{l=0}^{m}\binom{m}{l}\left(\frac{[w_n]_q}{\left[\prod_{j=1}^{n-1} w_j\right]_q}\right)^{m-l}\left[\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right) k_j\right]_{q^{w_n}}^{m-l}$$

$$\times q^{l w_n \sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right) k_j} \beta_{l,q^{w_1\cdots w_{n-1}}}(w_n x).$$

Thus, by (2.7), we get

$$\left[\prod_{j=1}^{n-1} w_j\right]_q^{m-1} \prod_{l=1}^{n-1}\sum_{k_l=0}^{w_l-1} q^{w_n \sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right) k_j}$$

$$\times \int_{\mathbb{Z}_p} \left[y + w_n x + w_n \sum_{j=1}^{n-1} \frac{k_j}{w_j}\right]_{q^{w_1\cdots w_{n-1}}}^{m} d\mu_{q^{w_1\cdots w_{n-1}}}(y)$$

$$= \sum_{l=0}^{m} \binom{m}{l} \left[\prod_{j=1}^{n-1} w_j\right]_q^{l-1} [w_n]_q^{m-l} \beta_{l,q^{w_1\cdots w_{n-1}}}(w_n x)$$

$$\times \prod_{s=1}^{n-1}\sum_{k_s=0}^{w_s-1} q^{\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right) k_j w_n (l+1)} \left[\prod_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right) k_j\right]_{q^{w_n}}^{m-l}$$

$$= \sum_{l=0}^{m} \binom{m}{l} \left[\prod_{j=1}^{n-1} w_j\right]_q^{l-1} [w_n]_q^{m-l} \beta_{l,q^{w_1\cdots w_{n-1}}}(w_n x)$$

$$\times T_{m,q^{w_n}}(w_1, w_2, \ldots, w_{n-1} \mid l),$$

where

$$T_{m,q}(w_1, w_2, \ldots, w_{n-1} \mid l)$$

$$= \prod_{s=1}^{n-1}\sum_{k_s=0}^{w_s-1} q^{(l+1)\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right) k_j} \left[\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right) k_j\right]_q^{m-l}.$$

As this expression is invariant under any permutation in S_n, we have the following theorem.

Theorem 2.3. *For $m \geq 0$, $n, w_1, \ldots, w_n \in \mathbb{N}$, the following expressions*

$$\sum_{l=0}^{m} \binom{m}{l} \left[\prod_{j=1}^{n-1} w_{\sigma(j)}\right]_q^{l-1} [w_{\sigma(n)}]_q^{m-l} \beta_{l,q^{w_{\sigma(1)}\cdots w_{\sigma(n-1)}}}(w_{\sigma(n)} x)$$

$$\times T_{m,q^{w_{\sigma(n)}}}(w_{\sigma(1)}, \ldots, w_{\sigma(n-1)} \mid l)$$

are all the same for $\sigma \in S_n$.

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