Covering a Unit Hypercube with Hypercubes *

Chunxia Shen, Zhanjun Su†, Liping Yuan
suzj888@163.com
College of Mathematics and Information Science,
Hebei Normal University, Shijiazhuang, 050024, China.

Abstract

The covering and packing of a unit square (resp. cube) with squares (resp. cubes) are considered. In d-dimensional Euclidean space $\mathbf{E}^d$, the size of a d-hypercube is given by its side length and the size of a covering is the total size of the d-hypercubes used to cover the unit hypercube. Denote by $g_d(n)$ the smallest size of a minimal covering (which consisting of n hypercubes) of a d-dimensional unit hypercube. In this paper we consider the problem of covering a unit hypercube with hypercubes in $\mathbf{E}^d$ for $d \geq 4$ and determine the tight upper bound and lower bound for $g_d(n)$.
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1 Introduction and Notations

P. Erdős defined a function $f(n)$ which denotes the maximum sum of n squares that can be packed into a unit square [1]. Erdős and Soifer gave some results for $f(n)$ (see [2]). Inspired by [2], Fan and Zhang discussed the dual version, that is, a square-covering problem [6]. And in [3]-[5] they discussed the cube-covering problem and cube-packing problem. In this paper, we generalize this kind of covering problem to the case of d-dimensional hypercubes for $d \geq 4$. That is, use d-dimensional hypercubes to cover a d-dimensional unit hypercube and obtain the corresponding results.

In d-dimensional Euclidean space $\mathbf{E}^d$, for a given d-hypercube P, the size $s(P)$ of P is denoted by the side length of P. A covering $\mathbb{C}$ is given by a set of hypercubes S positioned inside a d-dimensional unit hypercube H

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†Corresponding Author.

in such a way that the d-dimensional hypercubes from $\mathcal{S}$ have sides parallel to those of H and that $0 < s(C) < 1$ for each $C \in \mathcal{S}$, and any point of H is covered by at least one of the d-dimensional hypercubes from $\mathcal{S}$.

For a covering $\mathbb{C}$ of the unit hypercube H using a set of hypercubes $\mathcal{S} = \{C_1, \ldots, C_n\}$ of n hypercubes, where $0 < s(C_i) < 1$, denote by $s(\mathbb{C})$ the size of the covering $\mathbb{C}$, which is given by $\sum_{i=1}^{n} s(C_i)$.

A covering of H is said to be *minimal* if there is no other covering of H using a set of hypercubes $\mathcal{S}'$, where $\mathcal{S}' = \{C_1, \ldots, C_{i-1}, C_{i+1}, \ldots, C_n\}$ or $\mathcal{S}' = \{C_1, \ldots, C_{i-1}, C_i', C_{i+1}, \ldots, C_n\}$ with $s(C_i') < s(C_i)$. We denote by $g_d(n)$ the smallest size of a minimal covering using a set of n hypercubes. That is, $g_d(n) = \min\{s(\mathbb{C})$: $\mathbb{C}$ is a minimal covering of the unit hypercube with n hypercubes$\}$.

Let $\mathbb{C}$ be a covering of a d-dimensional unit hypercube H using a set of hypercubes $\mathcal{S} = \{C_1, \ldots, C_n\}$. Since each corner of H has to be covered by a d-dimensional hypercube from $\mathcal{S}$ and since the size of any hypercube from $\mathcal{S}$ is less than 1, we know that every different corner of H must be covered by a different hypercube from $\mathcal{S}$. Therefore, the following proposition is true.

Proposition 1. *If $\mathbb{C}$ is a covering of the d-dimensional unit hypercube, then $\mathbb{C}$ contains at least 2^d d-dimensional hypercubes.*

For example of a covering, consider the case when $n = 2^d$. It is easy to show that $g_d(2^d) \leq 2^{d-1}$. To see this, we can use a set $\mathcal{S}$ with 2^d d-dimensional hypercubes of size $1/2$, each one positioned in a different corner of the d-dimensional unit hypercube. This covering is clearly minimal, as we cannot remove a d-dimensional hypercube from $\mathcal{S}$ or replace by a smaller hypercube to obtain a smaller covering. We will also see, in Theorem 8, that $g_d(n) \geq 2^{d-1}$ for any $n \geq 2^d$. Therefore, the following proposition is valid.

Proposition 2. $g_d(2^d) = 2^{d-1}$

By the definition of the minimal covering the following result holds.

Proposition 3. *If $\mathbb{C}$ is a minimal covering of the d-dimensional unit hypercube and $\mathbb{C}$ has n d-dimensional hypercubes, then $g_d(n) \leq s(\mathbb{C})$.*

2 The Main Results

In this section, we determine the upper and lower bounds for $g_d(n)$.

Denote by H the d-dimensional unit hypercube. We present two sets C_1 and C_2: C_1 has $2^{d-1} - 1$ d-dimensional hypercubes with size $1 - \varepsilon$ and one with size $1-\varepsilon'$, where $\varepsilon' = (n - 2^d + 1)\varepsilon$ and $0 < \varepsilon < 1/(n \cdot d^2)$; C_2 has $(d-1)$ d-dimensional hypercubes of size ε' and $(n - 2^{d-1} - d + 1)$ d-dimensional hypercubes of size ε.

We first give the following lemmas.

Lemma 4. *If $C_1 \cup C_2$ can cover H, then the total length of the edges of H that can be covered by the hypercubes from C_2 is less than 1.*

Proof. Since there exist $(d-1)$ hypercubes of size $\varepsilon' = (n - 2^d + 1)\varepsilon$ and $(n-2^{d-1}-d+1)$ hypercubes of size ε from C_2, we know that their total size is $(d-1)\varepsilon' + (n-2^{d-1}-d+1)\varepsilon = (d-1)(n-2^d+1)\varepsilon + (n-2^{d-1}-d+1)\varepsilon = (dn - 2^{d-1}(2d-1))\varepsilon < dn\varepsilon$. If each hypercube from C_2 is positioned in a corner of H, it partially covers d edges, and thus the total edge length covered by the hypercubes from C_2 is less than $d \cdot dn\varepsilon = d^2 n\varepsilon$. The proof is complete, for we have $\varepsilon < 1/(n \cdot d^2)$. □

Lemma 5. *If $C_1 \cup C_2$ can cover H, then each edge of H must be intercepted by a hypercube from C_1.*

Proof. From Lemma 4, the total edge length covered by the hypercubes from C_2 is less than 1. Since the side length of H is 1, each edge of H must be intercepted by some hypercube from C_1. □

Lemma 6. *If $C_1 \cup C_2$ can cover H, then each hypercube from C_1 must cover a different corner of H and each 2-dimensional face of H has exactly two hypercubes from C_1 covering opposite corners of this face.*

Proof. Consider a covering with the hypercubes from $C_1 \cup C_2$ and suppose (by contradiction) that there exists a hypercube $C \in C_1$ that does not cover a corner of H. As each hypercube cannot intercept more than d edges of H, we can maximize the total edge covering if we place the large hypercubes from $C_1 \backslash \{C\}$ in the corners. Moreover, each hypercube from C_1 has size less than 1, and so, the $2^{d-1} - 1$ hypercubes from $C_1 \backslash \{C\}$ cover a total edge length that is at most $d(2^{d-1} - 1)$; the hypercube C cover an edge length that is less than 1 and the hypercubes from C_2 cover a total edge length that is less than 1 (from Lemma 4). This leads to a total edge length covered that is less than $d(2^{d-1} - 1) + 1 + 1 = d \cdot 2^{d-1} - d + 2 \leq d \cdot 2^{d-1}$, which is insufficient to cover the total edge length of H that is $d \cdot 2^{d-1}$. Therefore, all hypercubes from C_1 cover a corner point of H.

Now, consider a covering with the hypercubes from $C_1 \cup C_2$ and suppose (by contradiction) that we have two hypercubes from C_1 that common cover one edge of H. This means that the 2^{d-1} hypercubes from C_1 cover a total edge length that is less than $d \cdot 2^{d-1} - 1$. And from Lemma 5, the hypercubes from C_2 can cover a total edge length of H that is less than 1. Therefore, the total edge length covered by all hypercubes from $C_1 \cup C_2$ is less than $d \cdot 2^{d-1}$, which is a contradiction. Therefore, two hypercubes from C_1 cannot cover a same edge. This leads to a configuration where in each 2-dimensional face of H, we have exactly two hypercubes from C_1 covering opposite corners of the face. □

The next theorem shows that $g_d(n)$ cannot be greater than 2^{d-1} for any $n \geq 2^d + 1$.

Theorem 7. *For $n \geq 2^d + 1$, we have $g_d(n) \leq 2^{d-1} + \delta$, where δ is a positive value that can be made as close to 0 as desired.*

We consider the non-covered space after placing the hypercubes from C_1. There are 2^{d-1} d-dimensional hypercuboids: $2^{d-1} - d$ hypercuboids with dimensions $(\varepsilon, \varepsilon, \cdots, \varepsilon)$, and d hypercuboids with dimensions $(\varepsilon, \varepsilon, \cdots, \varepsilon')$.

We can regard these non-covered hypercuboid regions as one-dimensional bins, considering the largest edge of the hypercuboid as the size of a one-dimensional bin, that must be covered by one-dimensional items of size ε or ε' (all remaining hypercubes are the hypercubes from C_2, which has $n - 2^{d-1} - d + 1$ hypercubes of size ε and $d - 1$ hypercubes of size ε').

So, the total size of these bins is $(2^{d-1} - d)\varepsilon + (n - 2^d + 1)\varepsilon + (d-1)\varepsilon' = (2^{d-1} - d)\varepsilon + \varepsilon' + (d-1)\varepsilon' = (2^{d-1} - d)\varepsilon + d\varepsilon'$. On the other hand, the total size of hypercubes from C_2 is equal to $(n - 2^{d-1} - d + 1)\varepsilon + (d-1)\varepsilon'$ which is also $(2^{d-1} - d)\varepsilon + d\varepsilon'$. So, to have a covering of these bins (non-covered hypercuboids) with the hypercubes from C_2, we have to obtain a perfectly covering of the bin size. In fact, the covering is easy to obtain, $2^{d-1} - d$ hypercubes of size ε covering $2^{d-1} - d$ hypercuboids of size ε, $d-1$ hypercubes of size ε' covering the $d-1$ hypercuboids of the size ε' and the remaining $(n - 2^d + 1)$ hypercubes of size ε covering perfectly the remaining hypercuboids of size ε'.

To see that the above covering is minimal, note that we cannot replace one hypercube from C_2 by a smaller hypercube, as the small hypercubes fit perfectly in the total length of the bins (hypercuboid largest edge). And we also cannot replace one large hypercube from C_1 by a smaller hypercube, as there is no more small hypercubes to be used to cover the new larger hypercuboid regions.

Now, consider the size of the obtained covering. The hypercubes from C_1 have total size $(2^{d-1} - 1)(1 - \varepsilon) + (1 - \varepsilon') = 2^{d-1} - 2^{d-1}\varepsilon - 1 + \varepsilon + 1 - (n - 2^d + 1)\varepsilon = 2^{d-1} - (n - 2^{d-1})\varepsilon$. The hypercubes from C_2 have total size $(d-1)\varepsilon' + (n - 2^{d-1} - d + 1)\varepsilon = (d-1)(n - 2^d + 1)\varepsilon + (n - 2^{d-1} - d + 1)\varepsilon = (dn - (2d-1)2^{d-1})\varepsilon$. So, the size of the covering is $2^{d-1} - (n - 2^{d-1})\varepsilon + (dn - (2d-1)2^{d-1})\varepsilon = 2^{d-1} + (2^{d-1}(2 - 2d) + n(d-1))\varepsilon = 2^{d-1} + (d-1)(n - 2^d)\varepsilon$. Since $n \geq 2^d + 1$ and since ε can be made as close to 0 as desired, the size of the covering can also be made as close to 2^{d-1} as desired.

In the following we give the lower bound for $g_d(n)$.

Theorem 8. *For any $n \geq 2^d$, we have that $g_d(n) \geq 2^{d-1}$.*

Proof. We shall use induction on d. For $d = 2, 3$ we know that the results are true. Suppose that the statement is valid of dimension $< d$. Let $\mathbb{C}$ be a covering of the unit d-dimensional hypercube by a set of hypercubes $\mathcal{F}$. If a $n-1$ dimensional top face of H and a hypercube $C \in \mathcal{F}$ have a common point, then C and the $n-1$ dimensional bottom face of H have no common point, because $0 < s(C) < 1$. Let $\{A_1, A_2, \cdots, A_s\}$ be the set of hypercubes from $\mathcal{F}$ which have common point with the top face of H and let $\{B_1, B_2, \cdots, B_t\}$ be the set of hypercubes from $\mathcal{F}$ which have common point with the bottom face of H, then $s+t \leq n$ and $\{A_1, \cdots, A_s\} \cap \{B_1, \cdots, B_t\} = \emptyset$.

For $i = 1, 2, \cdots, s$, the projection of A_i in the $n-1$ dimensional top face of H, is a $n-1$ dimensional hypercube which has the same length with A_i. That is, the projection of $A_1, \cdots, A_s$ in the top face leads to covering of the top face of H. By the inductive hypothesis, $n-1$ dimensional hypercube covering problem, the side length of these projections is no less than 2^{d-2}, so the total size of the hypercubes $A_1, \cdots, A_s$ is no less than 2^{d-2}. In the same way, the sum of the sizes of the hypercubes $B_1, \cdots, B_t$ in no less than 2^{d-2}.

So,

$$g_d(n) \geq 2^{d-2} + 2^{d-2} = 2^{d-1}.$$

□

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