A list for vertex-primitive symmetric graphs of order 6p*

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Abstract: The aim of this paper is to classify the vertex-primitive symmetric graphs of order 6p. These works were essentially done in [1]. But in [1] there is no such situation: *G=PSL(2,13)* acting on the set Ω of cosets of subgroup $H \cong D_{14}$. Then $m = \mid \Omega \mid = 78 = 6p$, G has rank 9, and the sub-orbits of G have one of length 1, five of length 7, three of length 14 . In this paper we give a complete list of symmetric graphs of order $6p$.

Keywords: vertex-primitive, symmetric, sub-orbit, orbital graph, self-paired,

1. Introduction

Let Γ be a simple undirected graph and G a subgroup of $Aut\Gamma$. Γ is said to be $G-symmetric$, if G acts transitively on the set of ordered adjacent pairs of vertices of Γ ; Γ is said to be symmetric if it is $Aut\Gamma - symmetric$. Throughout this paper we use $V(\Gamma)$ and $E(\Gamma)$ to denote the vertex and edge sets of Γ, respectively.

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In this paper, we use the character to give a complete list for vertex-primitive symmetric graphs of order $6p$ with p being a prime. The result is the following:

Theorem 1. Let Γ be a $G-symmetric$ graph of order $6p$, where p is a prime .Assume that G acts naturally on $V(\Gamma)$ vertex-primitively, then Γ could be one of the following: $6pK_1 K_{6p}$,T_{12}, T_{12}^c, T_{13},T_{13}^c, $(M_{11})_{66}^{15}$, $(M_{11})_{66}^{20}$, $(M_{11})_{66}^{30}$, $L_2(17)_{102}^{3}$, $L_2(17)_{102}^{6}$, $L_2(17)_{102}^{8}$, $L_2(17)_{102}^{12}$, $L_2(17)_{102}^{24}$, $L_2(17)_{102}^{24'}$, $L_2(17)_{102}^{24''}$, $L_2(13)_{78}^{7}$, $L_2(13)_{78}^{7^{(1)}}$, $L_2(13)_{78}^{14}$, $L_2(13)_{78}^{14^{(1)}}$, $\overline{L}_2(13)_{78}^{14^{(i)}}$ $(0 \le i \le 2)$, $\overline{L}_2(13)_{78}^{14}$, $\overline{L}'_2(13)_{78}^{14}$, $\overline{L}_2(13)_{78}^{28}$.

The group- and graph-theoretic notation and terminology used in this paper are standard in general, the reader can refer to [1] if necessary.

Obviously, if G is doubly transitive, the non-ordinary self–paired sub-orbit Δ is isomorphic to $6pK_1$ or K_{6p} . So we san assume that G is simple vertex-primitive in this paper. Liebeek and Saxl have listed all the vertex primitive groups of order 6p where p is a prime. We will determine all the self-paired sub-orbit graphs of every vertex-primitive group order $6p$ in their table.

Check the test in [2] and moderate the vertex-primitive group table of order kp, where p is a prime, and $k \le p$. Due to a stable point in a maximal subgroup of vertex-primitive group A, in Atls [3] we get five primitive groups of degree $6p$: Case (1) A_{12} or S_{12} acting naturally on $66 = 6p$ disordered pairs of a 12-element set; Case (2) A_{13} or S_{13} acting naturally on $78 = 6p$ vertex-disordered pairs of a 13-elements set ; Case (2') M_{12} acting naturally on $78 = 6p$ cosets of the subgroup $M_{10} : 2$; Case (3) M_{11} acting naturally on $66 = 6p$ cosets of the subgroup S_5 ; Case (4) $PSL(2,17)$ acting naturally on $102 = 6p$ cosets of the subgroup S_4 ; Case (5) $PSL(2,13)$ acting naturally on $78 = 6p$ cosets of the subgroup D_{14} ; Case (6) $PSL(2,13)$ acting naturally on $78 = 6p$ cosets of the subgroup D_{28}.

2. THE ALREADY CONCLUSION

LEMMA 2.1.([1],Lemma 3.1) Let $T = soc(G) = A_n$,where $n \ge 5$, and $G \le S_n$ acting naturally on the set Ω of unordered pairs of $n-$element set.

(a) Then $:=\mid \Omega \mid = n(n-1)/2$, G has rank 3, and the subdegrees of G are $1, 2(n-2), (n-2)(n-3)/2$. All sub-orbits are self-paired.

(b) The sub-orbital graph T_n of degree $2(n-2)$, called triangular graph, is isomorphic to the linear graph $L(K_n)$ of the complete graph K_n. The sub-orbital graph of degree $(n-2)(n-3)/2$ is the complement T_n^c of T_n. T_n and T_n^c have order $n(n-1)/2$, and have the automorphism group S_n .They are the only two vertex-primitive graphs on Ω admitting G.
(C) If $T = soc(G) = A_{12}$, we get two sub-orbital graphs of order 66, with the 20-degree graph T_{12} and the 45-degree graph T_{12}^C.
(d) If $T = soc(G) = A_{12}$, we get two sub-orbital graphs of order 78, with the 22-degree graph T_{13} and the 55-degree graph T_{13}^C.

LEMMA 2.2. Let $G = M_{12}$ act naturally on the set Ω of cosets of the subgroup $H = M_{12} : 2$, then $|\Omega| = 66$, $rank(G) = 3$, and the subdegrees of G are 1,20,45. We get two self-paired sub-orbital graphs T_{12} and T_{12}^C .

Proof. M_{12} has only one conjugate block of index 66. That's to say , G acting naturally on the set of cosets of H , that is equal to G acting naturally on the set of the unordered pairs of a 12-element set . The only two vertex-primitive graphs admitting G are T_{12} and T_{12}^C . The other conclusions are obvious.

The graphs in case (3) appear in [1], but we deal with them in a different way. We will use the permutation character to prove them.

LEMMA 2.3. Let $G = M_{11}, H = S_5$ and $|G : H| = 66$. And let $G = M_{11}$ act naturally on the set Ω of cosets of the sub-group $H = S_5$, then $|\Omega| = 66$, $rank(G) = 4$,and the subdegrees of G are 1,15,22,30. We get three non-trivial sub-orbital graphs with every two of them non-isomorphic , with degrees 15,20,30, denoted $(M_{11})_{66}^{15}$, $(M_{11})_{66}^{20}$, $(M_{11})_{66}^{30}$, respectively. They are vertex-primitive symmetric graphs of order 66, and their automorphism group is M_{11}.

Proof. If π is the permutation character of G , then by Atlas [4] we get

$$\pi = \sum_{1}^{s-1} e_\lambda \chi_\lambda = 1 + \chi_2 + \chi_5 + \chi_8,$$

and $\chi_2(1) = 10$, $\chi_5(1) = 11$, $\chi_8(1) = 44$. G is double-free and has rank 4, so due to the theorem 8 in[5], all the sub-orbits of G are self-paired. Now we need to determine all the non-trivial subdegrees

$n_1, n_2, ..., n_r$. In the case of exiting a non-trivial odd number, 66 has three possible partitions: $(a)1+5+20+40$, $(b)1+5+30+30$, $(c)1+15+20+30$.

In case (a), we get $Fix_\Omega(Z_3) = 3$ by calculating the permutation character π . Assuming $Fix_\Omega(D_6) = k$, if there exits a $H-$orbit of length 20, and D_6 is its vertex-stabilizer , then $k > 1$ (in fact $k = 2$). Because $N_H(Z_3) = 2$, there exit $(3-k)/2$ $H-$orbits Δ' of length 40, and Z_3 is its vertex-stabilizer. It is contradict to the assumption. So (a) doesn't happen.

In case (b), if there exits a $H-$orbit Δ_1 of length 5, and S_4 is its vertex-stabilizer, then $k_1 =\mid Fix_\Omega(S_4) \mid= 2$ .We get $\pi z = 10$ by calculating the permutation character π on $z \in Z_2$, so $k_1 =\mid Fix_\Omega(D_4) \mid\leq 10$. $N_H(D_4) = S_4$, $\mid S_4 : D_4 \mid= 6$, then there are $(10-k_1)/6$ $H-$orbits of length 30 at least, and D_4 is its vertex-stabilizer. But $(10-k_1)/6 = 2$ is impossible , so case (b) doesn't happen.

We conclude that G has rank 4 and the sub-degrees are $1, 15, 20, 30$ from above. All the sub-orbits are self-paired as we want. In fact, the action of G on the set of cosets of H is equal to its action on the only $4-(11, 5, 1)$ designed block admitting G. Due to $Aut(M_{11}) = M_{11}$, our conclusion ends.

LEMMA 2.4. Let $G = PSL(2, 17)$ act naturally on the set Ω of cosets of the sub-group $H \cong S_4$ (in case (4)),
(a) Then $m =\mid \Omega \mid= 102$, G has rank 8, and the subdegrees are $1, 3, 6, 8, 12, 24, 24, 24$. All the sub-orbits are self-paired.
(b) Every self-paired sub-orbit gives one sub-orbital graph. Besides the graphs are symmetric and every two of them are non-isomorphic. Every symmetric graph on Ω admitting G has its sub-orbital graph with self-paired sub-orbits. The automorphism group of all the graphs is $PSL(2, 17)$.

Proof. By [4], the permutation character π is :

$$\pi = 1 + \chi_2 + \chi_3 + \chi_5 + \chi_6 + \chi_7 + \chi_8 + \chi_9,$$

where $\chi_2(1) = \chi_3(1) = 9$, $\chi_5(1) = \chi_6(1) = \chi_7(1) = 16$, $\chi_8(1) = 17$, $\chi_9(1) = 18$. So G is double-free and has rank 8. All the sub-orbits of G are self-paired by the theorem 8 in [5].

Now we need to determine the 8−partition of 102. The rank of G is 8 , and 3 is the only nontrivial odd factor of $| S_4 |$. That is to say, the 8−partition of 102 contains 3. On the other hand, it is easy to find that the 8−partition of 102 contains three 24. So the next question is to determine $2+12+12$ and $6+8+12$ which of them in the 8−partition of 102. $N_H(A_4) = A_4$, $N_G(A_4) = A_4$, so the case $2+12+12$ doesn't happen. Hence, we get only one 8−partition

$$102 = 1+3+6+8+12+24+24+24.$$

We get a symmetric graph Γ from every sub-orbit of $G = PSL(2,17)$. Assuming $A = Aut(\Gamma)$, then $A \geq G$ and A acts primitively on Ω (the set of 102 vertices). To check the table in [4], we get $PSL(2,17) \leq A \leq PGL(2,17)$. Because $PGL(2,17)$ doesn't contain the subgroup of rank 48 , $A = PSL(2,17)$. If Γ_i and Γ_j are the sub-orbital graphs adjoint with Δ_i and Δ_j , σ is a isomorphism of $\Gamma_i \mapsto \Gamma_j$, then σ belongs to the centralizers of their common automorphism group $PSL(2,17)$ in S_Ω. However, by the above tips, the group $PSL(2,17)$ in S_Ω is self-centralized. So $\sigma \in PSL(2,17)$, and $\Delta_i = \Delta_j$. We denote the graphs $L_2(17)^3_{102}$, $L_2(17)^6_{102}$,$L_2(17)^8_{102}$, $L_2(17)^{12}_{102}$,$L_2(17)^{24}_{102}$, $L_2(17)^{24'}_{102}$, $L_2(17)^{24''}_{102}$. The lemma is proved.

3.THE GRAPHS NOT IN LITERATURE [1]:

The following discussion is about the graphs which are not in the literature [1]:

LEMMA 3.1. Let $G = PSL(2,13)$ act naturally on the set Ω of cosets of subgroup $H \cong D_{14}$,

(a) Then $m = | \Omega | = 78 = 6p$, G has rank 9, and G has one sub-orbit of length 9, 5 sub-orbits of length 7, 3 sub-orbits of length 14. All the sub-orbits of length 14 and three of the sub-orbit of length 7 are self-paired.

(b) In (a), $PSL(2,13)$ exchanges two self-paired sub-orbits of length 7, the unique graph determined by the isomorphism is denoted $L_2(13)^{7^{(1)}}_{78}$. Their union is the 14−degree sub-orbital graph of $PGL(2,13)$, denoted $L_2(13)^{14^{(1)}}_{78}$. Another self-paired sub-orbit is denoted $L_2(13)^7_{78}$, it is not isomorphic to $L_2(13)^{7^{(1)}}_{78}$. The other two non-self-paired sub-orbital graphs of length 7 (Their union is the 14−degree sub-orbital graph of $PGL(2,13)$, denoted $\overline{L}'_2(13)^{14}_{78}$.) are exchanged by $PGL(2,13)$, the unique graph determined by the automorphism is denoted $L'_2(13)^7_{78}$.

(c) Two self-paired sub-orbital graphs of degree 14 (Their union is the 28−degree sub-orbital graph of $PGL(2,13)$, denoted $\overline{L}_2(13)^{28}_{78}$.) are exchanged by $PGL(2,13)$, the unique graph determined by the automorphism is denoted $L_2(13)^{14}_{78}$, and its automorphism group is $PGL(2,13)$. Another self-paired sub-orbital graph of degree 14 is not isomorphic to $L_2(13)^{14}_{78}$, denoted $L_2(13)^{14^{(1)}}_{78}$.

Proof. By Atlas [4], the permutation character π is :

$$\pi = 1 + \chi_4 + \chi_5 + \chi_6 + \chi_7 + 2\chi_8,$$

where $\chi_4(1) = \chi_5(1) = \chi_6(1) = 12$, $\chi_7(1) = 13$, $\chi_8(1) = 14$. So G is not double-free, and the rank is $r = 1+1+1+1+1+2^2 = 9$. Not all the sub-orbits of G are self-paired. Because $G = PSL(2,13)$, $G_\alpha = H = D_{14}$, we get $(G_\alpha)_\beta = Z_2$. Besides, for $z \in Z_2$, $\pi(z) = 6 = Fix_\Omega(z)$, there exits $6-1=5$ H−orbits of length 7 , and Z_2 is its stabilizer. Hence , we get that the 9−partition of 78, which is $78 = 1 + 5\times 7 + 3\times 14$. So G has five sub-orbits of length 7, $\Delta_1, ..., \Delta_5$, three sub-orbits of length 24 , $\Delta_6, \Delta_7, \Delta_8$. Now we determine which sub-orbit is self-paired. Due to $\pi(z) = 6$, so $N_G(Z_2) = D_{12} = C_G(Z_2)$. So for $z(\neq 1) \in Z_2$, there exit $\alpha \in D_{12}$ and $o(\alpha) = 6$ such that $\alpha^3 = z$. Using lemma 2.3 in [1], we find 3 sub-orbits of length 7 self-paired. Without loss of generality, we say $\Delta_1, \Delta_2, \Delta_3$ are self-paired , Δ_4, Δ_5 are not self-paired.

Assume that there are y H−orbits of length 14 self-paired. We have the following table

$\mid\Delta(\alpha)\mid$	$y\mid\Delta(\alpha)\mid$	$(G_\alpha)_\beta$	$G_{\{\alpha,\beta\}}$	$inv(\Delta)$
7	3	Z_2	D_4	2
14	y	1	Z_2	1

$N = (78-6)/2 = 36$,By lemma 2.4 in [1], we have the equation

$$\frac{12}{2\times 14}(7\times 3\times 2 + 14y) = 36$$

So $y = 3$, the other conclusions are obvious. The proof ends.

LEMMA 3.2. Let $G = PGL(2,13)$ act naturally on the set Ω of cosets of subgroup $H \cong D_{28}$,
(a) Then $m =\mid \Omega \mid= 78 = 6p$, and G has rank 7. G has one sub-orbit of length 1, two sub-orbits of length 14, one sub-orbit of length 28. All the sub-orbits are self-paired.
(b) There are three non-trivial graphs of length 7, denoted $\overline{L}_2(13)_{78}^{7(i)}$,$0 \leq i \leq 2$, respectively.
(c) There are two non-trivial graphs of degree 28(their sub-orbits are the union of the two sub-orbits of $PSL(2,13)$ with length 7 in lemma 3.1), denoted $\overline{L}_2(13)_{78}^{14}$,$\overline{L}_2'(13)_{78}^{14}$, respectively.
(d) There are one sub-orbital graph of degree 28(its sub-orbit is the union of two sub-orbits of $PSL(2,13)$ with length 14), denoted $\overline{L}_2(13)_{78}^{28}$.
(e) Every two of the graphs above are non-isomorphic, and the automorphism group of every graph is $PGL(2,13)$.

Proof. We consider $PGL(2,13)$ at first:

(1) Obviously, G has no sub-orbit of length 2 or 4, so the lengths of all the sub-orbits are ≥ 7 (by calculating the normalizers of $G_{\alpha,\beta}$ in G_α and G).

(2) Let $N = PSL(2,13)$, there is one expand $\overline{\chi}$ ($\overline{\chi} \mid_N = \chi$) to G for every $\chi \in Irr(N)$, and $\chi^G = \overline{\chi} + \lambda\overline{\chi}$, where $\lambda \mid_N = 1_N$ and $\lambda = -1_{G/N}$ from lemma 6 in [6]. We use π_G to donate the permutation group of G acting on Ω . Because N is also the transitive permutation group of Ω , by the lemma 2.3 in [7], we have $\pi_G \mid N = \pi_N$.

(3) G has one sub-orbit at least, it is the union of two sub-orbits of N .

From (1) , (2) and (3) , we have $rank(G) < rank(N) = 9$ so

$$\pi = 1 + \overline{\chi}_4 + \overline{\chi}_5 + \overline{\chi}_6 + \overline{\chi}_7 + \overline{\chi}_8 + \overline{\chi}_9,$$

where $\overline{\chi}_4(1) = \overline{\chi}_5(1) = \overline{\chi}_6(1) = 12, = \overline{\chi}_7(1) = 13, \overline{\chi}_8(1) == \overline{\chi}_9(1) = 14$. So G is double-free, and rank $r = 7$. All the sub-orbits of G are self-paired. Thus, we get a 7−partition of number 78, that is $78 = 1 + 3 \times 7 + 2 \times 14 + 1 \times 28$.

So G has one sub-orbit of length 1, three sub-orbits of length 7, two sub-orbits of length 14 and one sub-orbit of length 28. From (1),(2)and (3) , the conclusion is obvious.

The two cases are not in lemma 1 of document [1].

References

[1] C.E.Praeger and M.Y.Xu, Vertex-primitive graphs of order a product of two distinct primes, J. Conbin. Theory Ser. B59(1993), 245-266.

[2] M.W. Liebeck, and J.Saxl,Primitive permutation groups containing an element of large prime order, J. London Math. Soc. (2)31(1985), 237-249.

[3] C.E.Praeger,R.J.Wang,and M.Y.Xu,Symmetric graphs of order a product of two distinct primes, J. Conbin. Theory Ser. B58(1993), 299-318.

[4] J.H.Conway, R.T.Curtis,S.P.Norton, R.A.Parker,and R.A.Wilson, An Atlas of Finite Groups, Clarendon Press,Oxford,1985.

[5] R.W.Baddeley, Multiplicity-free and self-paired primitive permutation groups, J.Algebra 162 (1993), 482-530.

[6] B.D.Mckay and C.E.Praeger,Vertex-transitive graphs which are not Cayley Graphs, II, preprint, 1992.

[7] P.X.GallagherGroup characters and normal Hall subgroups, Nagoya Math. J.21,223-230.