Signed (j, k)-domatic numbers of graphs

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Abstract

Let G be a finite and simple graph with vertex set $V(G)$, and let $f : V(G) \to \{-1, 1\}$ be a two-valued function. If $k \geq 1$ is an integer and $\sum_{x \in N[v]} f(x) \geq k$ for each $v \in V(G)$, where $N[v]$ is the closed neighborhood of v, then f is a signed k-dominating function on G. A set $\{f_1, f_2, \dots, f_d\}$ of distinct signed k-dominating functions on G with the property that $\sum_{i=1}^{d} f_i(x) \leq j$ for each $x \in V(G)$, is called a signed (j, k)-dominating family (of functions) on G, where $j \geq 1$ is an integer. The maximum number of functions in a signed (j, k)-dominating family on G is the signed (j, k)-domatic number on G, denoted by $d_{jkS}(G)$.

In this paper we initiate the study of the signed (j, k)-domatic number, and present different bounds on $d_{jkS}(G)$. Some of our results are extensions of well-known properties of different other signed domatic numbers.

Keywords: Signed domatic number, Signed (j, k)-domatic number, Signed k-domination number, Signed k-dominating function

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1 Terminology and introduction

We consider finite, undirected and simple graphs G with vertex set $V(G) = V$ and edge set $E(G) = E$. The cardinality of the vertex set of a graph G is called the *order* of G and is denoted by $n(G) = n$. If $v \in V(G)$, then $N_G(v) = N(v)$ is the *open neighborhood* of v, i.e., the set of all vertices adjacent to v. The *closed neighborhood* $N_G[v] = N[v]$ of a vertex v consists of the vertex set $N(v) \cup \{v\}$. The number $d_G(v) = d(v) = |N(v)|$ is the *degree* of the vertex v. The *minimum* and *maximum degree* of a graph G are denoted by $\delta(G)$ and $\Delta(G)$. The *complement* of a graph G is denoted by $\overline{G}$. We write K_n for the *complete graph* of order n and C_n for a *cycle* of length n. A *fan* is a graph obtained from a path by adding a new vertex and edges joining it to all the vertices of the path. If $A \subseteq V(G)$ and f is a mapping from $V(G)$ into some set of numbers, then $f(A) = \sum_{x\in A} f(x)$.

If $k \geq 1$ is an integer, then the *signed k-dominating function* (SkD function) is defined in [13] as a two-valued function $f : V(G) \rightarrow \{-1, 1\}$ such that $\sum_{x\in N[v]} f(x) \geq k$ for each $v \in V(G)$. The sum $f(V(G))$ is called the weight $w(f)$ of f. The minimum of weights $w(f)$, taken over all signed k-dominating functions f on G, is called the *signed k-domination number* of G, denoted by $\gamma_{kS}(G)$. A $\gamma_{kS}(G)$*-function* is SkD-function on G of weight $\gamma_{kS}(G)$. As the assumption $\delta(G) \geq k - 1$ is necessary, we always assume that when we discuss $\gamma_{kS}(G)$, all graphs involved satisfy $\delta(G) \geq k - 1$ and thus $n(G) \geq k$. The function assigning $+1$ to every vertex of G is a SkD function, called the function ϵ, of weight n. Thus $\gamma_{kS}(G) \leq n$ for every graph of order n with $\delta \geq k - 1$. Moreover, the weight of every SkD function different from ϵ is at most $n - 2$ and more generally, $\gamma_{kS}(G) \equiv n \pmod 2$. Hence $\gamma_{kS}(G) = n$ if and only if ϵ is the unique SkD function of G. The special case $k = 1$ was defined and investigated in [2], and has been studied by several authors (see for example [1, 3]). Further information on $\gamma_{1S}(G) = \gamma_S(G)$ can be found in the monographs [5] and [6] by Haynes, Hedetniemi, and Slater. We make use of the following result.

Proposition A. *([4])* Let G be a graph of order n and minimum degree $\delta \geq k - 1$. Then $\gamma_{kS}(G) = n$ if and only if for each $v \in V$, there exists a vertex $u \in N[v]$ such that $d(u) = k - 1$ or $d(u) = k$ (this condition implies $\delta \leq k$).

Rall [7] has defined a variant of the domatic number of G, namely the fractional domatic number of G, using functions on $V(G)$. Analogous to the fractional domatic number we may define the signed (j, k)-domatic number.

Let $j \geq 1$ be an integer. A set $\{f_1, f_2, \ldots, f_d\}$ of distinct signed k-dominating functions on G with the property that $\sum_{i=1}^{d} f_i(x) \leq j$ for each $x \in V(G)$, is called a *signed (j,k)-dominating family* on G. The maximum number of functions in a signed (j, k)-dominating family on G is the *signed*

(j,k)-*domatic number* of G, denoted by $d_{jkS}(G)$. The signed (j,k)-domatic number is well-defined and $d_{jkS}(G) \geq 1$ for all graphs G, since the set consisting of any SkD function, for instance the function ϵ, forms a signed (j,k)-dominating family of G. A $d_{jkS}(G)$-*family* of a graph G is a signed (j,k)-dominating family containing $d_{jkS}(G)$ SkD functions.

Observation 1. Let G be a graph of order n. If $\gamma_{kS}(G) = n$, then ϵ is the unique SkD function of G and so $d_{jkS}(G) = 1$.

The following observations are consequences of Observations 1 and Proposition A.

Observation 2. If G is a graph of order n and $k = n$, then G is the complete graph and thus $\gamma_{kS}(G) = n$ and $d_{jkS}(G) = 1$.

Observation 3. If G is a graph of order $n \geq 2$ and $k = n-1$, then $\gamma_{kS}(G) = n$ and so $d_{jkS}(G) = 1$.

Observation 4. If G is an r-regular graph and $k = r+1$ or r, then $\gamma_{kS}(G) = n$ and $d_{jkS}(G) = 1$.

Observation 5. Let $k \geq 2$ be an integer, and let $r = k-1$. If G is a graph such that $r \leq d_G(x) \leq r+1$ for each $x \in V(G)$, then $\gamma_{kS}(G) = n$ and $d_{jkS}(G) = 1$.

Corollary 6. If P_n is a path of order n, then $\gamma_{2S}(P_n) = n$ and so $d_{j2S}(P_n) = 1$.

First we study basic properties of $d_{jkS}(G)$. Some of them are extensions of well-known results on the signed domatic number $d_S(G) = d_{11S}(G)$ (cf. [8], [10], [11], [12]), the signed k-domatic number $d_{kS}(G) = d_{1kS}(G)$ (cf. [4]) and the signed (k,k)-domatic number $d_{kkS}(G)$ (cf. [9]).

Let C_n be a cycle of length n. Volkmann and Zelinka [12] have shown that $d_S(C_n) = 3$ when n is divisible by 3 and $d_S(C_n) = 1$ otherwise. For $k = 2, 3$, Observation 4 leads immediately to the next result.

Corollary 7. If C_n is a cycle of length n, then $\gamma_{2S}(C_n) = \gamma_{3S}(C_n) = n$ and thus $d_{j2S}(C_n) = d_{j3S}(C_n) = 1$.

The case $k = 1$ seems to be complicated. For example, $d_{j1S}(C_3) = 3$ for each $j \geq 1$, $d_{j1S}(C_4) = 4$ for $j \geq 2$ and $d_{j1S}(C_5) = 2$ for $j = 2$ and $d_{j1S}(C_5) = 5$ for $j \geq 3$.

Proposition 8. If G is a graph of order $n \geq 4$ and $k = n-2$, then

$$d_{jkS}(G) = \begin{cases} 1 & \text{if } \delta(G) = n-3 \text{ or } \delta(G) = n-2, \\ n+1 & \text{if } \delta(G) = n-1 \text{ and } j \geq n-1, \\ n & \text{if } \delta(G) = n-1 \text{ and } j = n-2, \\ j & \text{if } \delta(G) = n-1 \text{ and } j < n-2. \end{cases}$$

Proof. If $\delta(G) = n-3$ or $\delta(G) = n-2$, then it is easy to see that for each $v \in V$, there exists a vertex $u \in N[v]$ such that $d(u) = k-1$ or $d(u) = k$. It follows from Proposition A that $\gamma_{kS}(G) = n$, and hence Observation 1 implies that $d_{jkS}(G) = 1$.

Now let $\delta(G) = n-1$. Then G is the complete graph of order n and obviously $\gamma_{kS}(G) = n-2$. Let $\{v_1, v_2, \ldots, v_n\}$ be the vertex set of G. Then the functions ϵ and $f_i : V(G) \to \{-1, 1\}$ defined by $f_i(v_i) = -1$ and $f_i(x) = 1$ for each vertex $x \in V(G) \setminus \{v_i\}$ and each $i \in \{1, 2, \ldots, n\}$ are the set of all signed k-dominating functions of G. If $j \geq n-1$, then clearly $\{\epsilon, f_i \mid 1 \leq i \leq n\}$ is a signed (j,k)-dominating family of G and so $d_{jkS}(G) = n+1$. If $j = n-2$, then obviously $\{f_i \mid 1 \leq i \leq n\}$ is a signed (j,k)-dominating family of G and so $d_{jkS}(G) \geq n$. Since $\{\epsilon, f_i \mid 1 \leq i \leq n\}$ is not a signed (j,k)-dominating family of G, we deduce that $d_{jkS}(G) = n$.

Finally assume that $j < n-2$. Obviously $\{f_i \mid 1 \leq i \leq j\}$ is a signed (j,k)-dominating family of G and so $d_{jkS}(G) \geq j$. If $\{g_1, g_2, \ldots, g_\ell\}$ is a signed (j,k)-dominating family of G with $\ell > j$, then we observe that there exists a vertex $v \in V(G)$ such that $g_j(v) = 1$ for each j or $\{g_1, g_2, \ldots, g_\ell\} = \{\epsilon, f_i \mid 1 \leq i \leq n\}$. In both cases we obtain $\sum_{j=1}^{\ell} g_j(v) > j$ which is a contradiction. Thus $d_{jkS}(G) = j$. □

An *independent set* in a graph G is a set of pairwise nonadjacent vertices, and the *independence number*, denoted by $\alpha(G)$, is the maximum size of an independent set of vertices.

Proposition 9. If G is a graph of order $n \geq 4$ and $k = n-3$, then

$$d_{jkS}(G) = \begin{cases} 1 & \text{if } \alpha(G) = 3 \text{ or } \alpha(G) = 4, \\ 1 & \text{if } \alpha(G) = 2 \text{ and there exists an } \alpha(G)\text{-set } \{x,y\} \\ & \quad \text{such that } \max\{d(x), d(y)\} \leq n-3, \\ 1 & \text{if } \alpha(G) = 2 \text{ and there exist two adjacent vertices} \\ & \quad x, y \text{ such that } \max\{d(x), d(y)\} \leq n-3, \\ \min\{j, 3\} & \text{if } \alpha(G) = 2,\ \delta(G) = n-3 \text{ and for each two vert-} \\ & \quad \text{ices } x, y \text{ with } \min\{d(x), d(y)\} = n-3 \text{ we have} \\ & \quad \max\{d(x), d(y)\} \geq n-2, \\ \min\{j, 4\} & \text{if } \alpha(G) = 2,\ \delta(G) = n-4 \text{ and for each two vert-} \\ & \quad \text{ices } x, y \text{ with } \min\{d(x), d(y)\} = n-4 \text{ we have} \\ & \quad \max\{d(x), d(y)\} \geq n-2, \\ n+1 & \text{if } \delta(G) \geq n-2 \text{ and } j \geq n-1, \\ n & \text{if } \delta(G) \geq n-2 \text{ and } j = n-2, \\ j & \text{if } \delta(G) \geq n-2 \text{ and } j < n-2. \end{cases}$$

Proof. Since $\delta(G) \geq k-1 = n-4$, it follows that $\alpha(G) \leq 4$. If $\alpha(G) = 4$, then $\gamma_{kS}(G) = n$, and Observation 1 implies that $d_{jkS}(G) = 1$. If $\alpha(G) = 3$, then it is easy to see that for each $v \in V$, there exists a vertex $u \in N[v]$

such that $d(u) = k-1$ or $d(u) = k$. It follows from Proposition A that $\gamma_{kS}(G) = n$, and hence Observation 1 implies that $d_{jkS}(G) = 1$.

Let $\alpha(G) = 2$ and assume that there exists an $\alpha(G)$-set $\{x, y\}$ such that $\max\{d(x), d(y)\} \leq n-3$. Then each vertex $v \in V(G) - \{x, y\}$ is adjacent to x or y. It follows from Proposition A that $\gamma_{kS}(G) = n$, and hence Observation 1 implies that $d_{jkS}(G) = 1$.

Let $\alpha(G) = 2$ and assume that there exist two adjacent vertices x, y such that $\max\{d(x), d(y)\} \leq n-3$. If $d(z) \leq n-3$ for some vertex $z \in V(G) - N[x]$, then the result follows from part two of this theorem. If $d(z) = n-2$ for each $z \in V(G) - N[x]$, then each vertex is adjacent to x or y and hence ϵ is the unique signed k-dominating function of G and so $d_{jkS}(G) = 1$.

Assume that $\alpha(G) = 2$, $\delta(G) = n-3$ and for each two vertices x, y with $\min\{d(x), d(y)\} = n-3$ we have $\max\{d(x), d(y)\} \geq n-2$. Let x be a vertex of minimum degree $n-3$ and let $V(G) - N[x] = \{v_1, v_2\}$. Clearly the functions ϵ and $f_i : V(G) \rightarrow \{-1, 1\}$ defined by $f_i(v_i) = -1$ and $f_i(w) = 1$ for each vertex $w \in V(G) \setminus \{v_i\}$ and each $i \in \{1, 2\}$ are the set of all signed k-dominating functions of G. Now it is easy to see that $d_{jkS}(G) = \min\{j, 3\}$.

If $\alpha(G) = 2$, $\delta(G) = n-4$ and for each two vertices x, y with $\min\{d(x), d(y)\} = n-4$ we have $\max\{d(x), d(y)\} \geq n-2$, then the result follows as above.

Now let $\delta(G) \geq n-2$. Then obviously $\gamma_{kS}(G) = n-2$. If $\{v_1, v_2, \ldots, v_n\}$ is the vertex set of G, then the functions ϵ and $f_i : V(G) \rightarrow \{-1, 1\}$ defined by $f_i(v_i) = -1$ and $f_i(x) = 1$ for each vertex $x \in V(G) \setminus \{v_i\}$ and each $i \in \{1, 2, \ldots, n\}$ are the set of all signed k-dominating functions of G. Now an argument similar to that described in the proof of Proposition 8 proves the result. □

2 Properties of the signed (j, k)-domatic number

In this section we present basic properties of $d_{jkS}(G)$ and sharp bounds on the signed (j, k)-domatic number of a graph.

Theorem 10. If G is a graph of order n with minimum degree $\delta(G) \geq k-1$, then

$$\gamma_{kS}(G) \cdot d_{jkS}(G) \leq j \cdot n.$$

Moreover, if $\gamma_{kS}(G) \cdot d_{jkS}(G) = j \cdot n$, then for each $d_{jkS}(G)$-family $\{f_1, f_2, \ldots, f_d\}$ with $d = d_{jkS}(G)$ on G, each function f_i is a $\gamma_{kS}(G)$-function and $\sum_{i=1}^{d} f_i(x) = j$ for all $x \in V(G)$.

Proof. If $\{f_1, f_2, \ldots, f_d\}$ is a signed (j,k)-dominating family on G such that $d = d_{jkS}(G)$, then the definitions imply

$$\begin{aligned} d \cdot \gamma_{kS}(G) &= \sum_{i=1}^{d} \gamma_{kS}(G) \le \sum_{i=1}^{d} \sum_{x \in V(G)} f_i(x) \\ &= \sum_{x \in V(G)} \sum_{i=1}^{d} f_i(x) \le \sum_{x \in V(G)} j = j \cdot n. \end{aligned}$$

If $\gamma_{kS}(G) \cdot d_{jkS}(G) = j \cdot n$, then the two inequalities occurring in the proof become equalities. Hence for the $d_{jkS}(G)$-family $\{f_1, f_2, \ldots, f_d\}$ on G and for each i, $\sum_{x \in V(G)} f_i(x) = \gamma_{kS}(G)$, and thus each function f_i is a $\gamma_{kS}(G)$-function and $\sum_{i=1}^{d} f_i(x) = j$ for all $x \in V(G)$. □

Theorem 11. If G is a graph with minimum degree $\delta(G) \ge k-1$, then

$$d_{jkS}(G) \le \frac{j(\delta(G)+1)}{k}.$$

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a signed (j,k)-dominating family on G such that $d = d_{jkS}(G)$. If $v \in V(G)$ is a vertex of minimum degree $\delta(G)$, then it follows that

$$\begin{aligned} d \cdot k &= \sum_{i=1}^{d} k \le \sum_{i=1}^{d} \sum_{x \in N[v]} f_i(x) \\ &= \sum_{x \in N[v]} \sum_{i=1}^{d} f_i(x) \\ &\le \sum_{x \in N[v]} j = j(\delta(G)+1), \end{aligned}$$

and this implies the desired upper bound on the signed (j,k)-domatic number. □

The special cases $j = k = 1$ or $j = 1$ or $j = k$ of Theorems 10 and 11 can be found in [12] or [4] or [9]. The upper bound on the product $\gamma_{kS}(G) \cdot d_{jkS}(G)$ leads to a bound on the sum.

Corollary 12. If G is a graph of order n with minimum degree $\delta(G) \ge k-1$, then

$$\gamma_{kS}(G) + d_{jkS}(G) \le jn + 1.$$

Proof. According to Theorem 10, we have

$$\gamma_{kS}(G) + d_{jkS}(G) \le \frac{jn}{d_{jkS}(G)} + d_{jkS}(G).$$

Theorem 11 implies that $1 \le d_{jkS}(G) \le \frac{j(\delta(G)+1)}{k} \le \frac{jn}{k}$. Using these inequalities, and the fact that the function $g(x) = x + (jn)/x$ is decreasing for $1 \le x \le \sqrt{jn}$ and increasing for $\sqrt{jn} \le (jn)/k$, we deduce that

$$\gamma_{kS}(G) + d_{jkS}(G) \le \max\left\{jn+1, k + \frac{jn}{k}\right\} = jn + 1,$$

and the proof is complete. □

Theorem 13. If a graph G contains a vertex v such that $d(v) \le k$, then $d_{jkS}(G) \le j$.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a signed (j,k)-dominating family on G such that $d = d_{jkS}(G)$. Since $\sum_{x \in N[v]} f_i(x) \ge k$ and $|N[v]| \le k+1$, we deduce that $f_i(x) = 1$ for each $x \in N[v]$ and each $i \in \{1, 2, \ldots, d\}$. In particular, $f_i(v) = 1$ for each $i \in \{1, 2, \ldots, d\}$. It follows that

$$d_{jkS}(G) = d = \sum_{i=1}^{d} f_i(v) \le j,$$

and this is the desired upper bound. □

Let $j \ge 1$ be an integer, and let $n = j + 4$. If $P_n = x_1 x_2 \ldots x_n$ is a path of order n, then define for $3 \le t \le n-2$ the function $f_t : V(P_n) \to \{-1, 1\}$ by $f_t(x_t) = -1$ and $f_t(x) = 1$ for $x \in V(P_n) \setminus \{x_t\}$. Then it easy to see that $\{f_3, f_4, \ldots, f_{n-2}\}$ is signed $(j,1)$-dominating family on P_n. Therefore Theorem 13 implies that $d_{j1S}(P_{j+4}) = j$.

Let $j \ge 1$ be an integer, and let $n = j + 5$. Now let F_n be a fan with vertex set $\{x_1, x_2, \ldots, x_n\}$ such that $x_1 x_2 \ldots x_n x_1$ is a cycle of length n and x_n is adjacent to x_i for each $i = 2, 3, \ldots, n-2$. For $3 \le t \le n-3$ define $f_t : V(F_n) \to \{-1, 1\}$ by $f_t(x_t) = -1$ and $f_t(x) = 1$ for $x \in V(F_n) \setminus \{x_t\}$. Then it easy to see that $\{f_3, f_4, \ldots, f_{n-3}\}$ is signed $(j,2)$-dominating family on F_n. Therefore Theorem 13 implies that $d_{j2S}(F_{j+5}) = j$.

These two examples demonstrate that Theorem 13 is sharp.

Corollary 14. Let $1 \le k \le 2$ be an integer. If T is a nontrivial tree, then $d_{jkS}(T) \le j$, and if the diameter of T is at most three, then $d_{jkS}(T) = 1$.

Proof. Theorem 13 implies $d_{jkS}(T) \le j$ for $k = 1, 2$. Now let f be a SkD function of T. If the diameter of T is at most three, then each vertex of T is a leaf or a neighbor of a leaf and thus $f(x) = 1$ for every vertex $x \in V(T)$. This shows that $d_{jkS}(T) = 1$. □

The path P_n with $n = j+4$ in the example above shows that the bound $d_{j1S}(T) \leq j$ in Corollary 14 is sharp.

Let $j \geq 2$ be an integer, and let $P = x_1x_2\ldots x_j$ be a path of order j. For $1 \leq i \leq j$ let $P_i = u_iu_i'u_i''$ and $P_i' = v_iv_i'v_i''$ be two paths of order 3. Now let T be the disjoint union of P, P_i and P_i' such that x_i is adjacent to u_i' and v_i' for $1 \leq i \leq j$. Define for $1 \leq t \leq j$ the function $f_t : V(T) \to \{-1, 1\}$ by $f_t(x_t) = -1$ and $f_t(x) = 1$ for $x \in V(T) \setminus \{x_t\}$. Then it is easy to see that $\{f_1, f_2, \ldots, f_j\}$ is signed $(j, 2)$-dominating family on T. Therefore Theorem 13 implies that $d_{j2S}(T) = j$.

This example demonstrates that the inequality $d_{j2S}(T) \leq j$ in Corollary 14 is sharp too.

As an application of Theorem 11, we will prove the following Nordhaus-Gaddum type result.

Theorem 15. If $j, k \geq 1$ are integers and G a graph of order n such that $\delta(G) \geq k-1$ and $\delta(\overline{G}) \geq k-1$, then

$$d_{jkS}(G) + d_{jkS}(\overline{G}) \leq \frac{j(n+1)}{k}.$$

Moreover, if $d_{jkS}(G) + d_{jkS}(\overline{G}) = \frac{j(n+1)}{k}$, then G is regular.

Proof. Since $\delta(G) \geq k-1$ and $\delta(\overline{G}) \geq k-1$, it follows from Theorem 11 that

$$\begin{aligned} d_{jkS}(G) + d_{jkS}(\overline{G}) &\leq \frac{j(\delta(G)+1)}{k} + \frac{j(\delta(\overline{G})+1)}{k} \\ &= \frac{j}{k}(\delta(G) + \delta(\overline{G}) + 2) \\ &= \frac{j}{k}(\delta(G) + (n - \Delta(G) - 1) + 2) \\ &\leq \frac{j}{k}(n+1), \end{aligned}$$

and this is the desired Nordhaus-Gaddum inequality. If G is not regular, then $\Delta(G) - \delta(G) \geq 1$, and the above inequality chain leads to the better bound $d_{jkS}(G) + d_{jkS}(\overline{G}) \leq \frac{jn}{k}$. This completes the proof. □

Theorem 16. If v is a vertex of a graph G such that $d(v)$ is odd and k is odd or $d(v)$ is even and k is even, then

$$d_{jkS}(G) \leq \frac{j}{k+1}(d(v)+1).$$

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a signed (j,k)-dominating family on G such that $d = d_{jkS}(G)$. Assume first that $d(v)$ and k are odd. The definition yields to $\sum_{x\in N[v]} f_i(x) \geq k$ for each $i \in \{1,2,\ldots,d\}$. On the left-hand side of this inequality a sum of an even number of odd summands occurs. Therefore it is an even number, and as k is odd, we obtain $\sum_{x\in N[v]} f_i(x) \geq k+1$ for each $i \in \{1,2,\ldots,d\}$. It follows that

$$\begin{aligned} j(d(v)+1) &= \sum_{x\in N[v]} j \geq \sum_{x\in N[v]} \sum_{i=1}^{d} f_i(x) \\ &= \sum_{i=1}^{d} \sum_{x\in N[v]} f_i(x) \\ &\geq \sum_{i=1}^{d}(k+1) = d(k+1), \end{aligned}$$

and this leads to the desired bound.

Assume next that $d(v)$ and k are even. Note that $\sum_{x\in N[v]} f_i(x) \geq k$ for each $i \in \{1,2,\ldots,d\}$. On the left-hand side of this inequality a sum of an odd number of odd summands occurs. Therefore it is an odd number, and as k is even, we obtain $\sum_{x\in N[v]} f_i(x) \geq k+1$ for each $i \in \{1,2,\ldots,d\}$. Now the desired bound follows as above, and the proof is complete. □

The next result is an immediate consequence of Theorem 16.

Corollary 17. If G is a graph such that $\delta(G)$ and k are odd or $\delta(G)$ and k are even, then

$$d_{jkS}(G) \leq \frac{j}{k+1}(\delta(G)+1).$$

As an Application of Corollary 17 we will improve the Nordhaus-Gaddum bound in Theorem 15 for some cases.

Theorem 18. Let $k \geq 1$ be an integer, and let G be a graph of order n such that $\delta(G) \geq k-1$ and $\delta(\overline{G}) \geq k-1$. If $\Delta(G) - \delta(G) \geq 1$ or k is even or k and $\delta(G)$ are odd or k is odd and $\delta(G)$ and n are even, then

$$d_{jkS}(G) + d_{jkS}(\overline{G}) < \frac{j(n+1)}{k}.$$

Proof. If $\Delta(G) - \delta(G) \geq 1$, then Theorem 15 implies the desired bound. Thus assume now that G is $\delta(G)$-regular.

Case 1: Assume that k is even. If $\delta(G)$ is even, then it follows from Theorem 11 and Corollary 17 that

$$\begin{aligned} d_{jkS}(G) + d_{jkS}(\overline{G}) &\leq \frac{j}{k+1}(\delta(G)+1) + \frac{j}{k}(\delta(\overline{G})+1) \\ &= \frac{j}{k+1}(\delta(G)+1) + \frac{j}{k}(n-\delta(G)-1+1) \\ &< \frac{j(n+1)}{k}. \end{aligned}$$

If $\delta(G)$ is odd, then n is even and thus $\delta(\overline{G}) = n - \delta(G) - 1$ is even. Combining Theorem 11 and Corollary 17, we find that

$$\begin{aligned} d_{jkS}(G) + d_{jkS}(\overline{G}) &\leq \frac{j}{k}(\delta(G)+1) + \frac{j}{k+1}(\delta(\overline{G})+1) \\ &= \frac{j}{k}(n-\delta(\overline{G})) + \frac{j}{k+1}(\delta(\overline{G})+1) \\ &< \frac{j(n+1)}{k}, \end{aligned}$$

and this completes the proof of Case 1.

Case 2: Assume that k is odd. If $\delta(G)$ is odd, then it follows from Theorem 11 and Corollary 17 that

$$d_{jkS}(G) + d_{jkS}(\overline{G}) \leq \frac{j}{k+1}(\delta(G)+1) + \frac{j}{k}(n-\delta(G)) < \frac{j(n+1)}{k}.$$

If $\delta(G)$ is even and n is even, then $\delta(\overline{G}) = n - \delta(G) - 1$ is odd, and we obtain the desired bound as above. □

Theorem 19. If G is a graph such that $d_{jkS}(G)$ is even for some odd j or $d_{jkS}(G)$ is odd for some even j, then

$$d_{jkS}(G) \leq \frac{j-1}{k}(\delta(G)+1).$$

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a signed (j,k)-dominating family on G such that $d = d_{jkS}(G)$. Assume first that j is odd and d is even. If $x \in V(G)$ is an arbitrary vertex, then $\sum_{i=1}^{d} f_i(x) \leq j$. On the left-hand side of this inequality a sum of an even number of odd summands occurs. Therefore it is an even number, and as j is odd, we obtain $\sum_{i=1}^{d} f_i(x) \leq j-1$ for each

$x \in V(G)$. If v is a vertex of minimum degree, then it follows that

$$\begin{aligned} d \cdot k &= \sum_{i=1}^{d} k \leq \sum_{i=1}^{d} \sum_{x \in N[v]} f_i(x) \\ &= \sum_{x \in N[v]} \sum_{i=1}^{d} f_i(x) \\ &\leq \sum_{x \in N[v]} (j-1) \\ &= (\delta(G)+1)(j-1), \end{aligned}$$

and this yields to the desired bound. Assume second that j is even and d is odd. If $x \in V(G)$ is an arbitrary vertex, then $\sum_{i=1}^{d} f_i(x) \leq j$. On the left-hand side of this inequality a sum of an odd number of odd summands occurs. Therefore it is an odd number, and as j is even, we obtain $\sum_{i=1}^{d} f_i(x) \leq j-1$ for each $x \in V(G)$. Now the desired bound follows as above, and the proof is complete. □

If we suppose in the case $j = 1$ that $d_{1kS}(G) = d_{kS}(G)$ is an even integer, then Theorem 19 leads to the contradiction $d_{kS}(G) \leq 0$. Consequently, we obtain the next known result.

Corollary 20. *([4])* The signed k-domatic number $d_{kS}(G)$ is an odd integer.

The special case $k = 1$ in Corollary 20 can be found in [12].

Theorem 21. Let $j \geq 2$ and $k \geq 1$ be integers, and let G be a graph with minimum degree $\delta(G) \geq k-1$. Then $d_{jkS}(G) = 1$ if and only if for every vertex $v \in V(G)$ the closed neighborhood $N[v]$ contains a vertex of degree at most k.

Proof. Assume that $N[v]$ contains a vertex of degree at most k for every vertex $v \in V(G)$, and let f be a signed k-dominating function on G. If $d(v) \leq k$, then it follows that $f(v) = 1$. If $d(x) \leq k$ for a neighbor x of v, then we observe $f(v) = 1$ too. Hence $f(v) = 1$ for each $v \in V(G)$ and thus $d_{jkS}(G) = 1$.

Conversely, assume that $d_{jkS}(G) = 1$. If G contains a vertex w such $d(x) \geq k+1$ for each $x \in N[w]$ then for $i = 1, 2$, the functions $f_i : V(G) \to \{-1, 1\}$ such that $f_1(x) = 1$ for each $x \in V(G)$ and $f_2(w) = -1$ and $f_2(x) = 1$ for each vertex $x \in V(G) \setminus \{w\}$ are signed k-dominating functions on G such that $f_1(x) + f_2(x) \leq 2 \leq j$ for each vertex $x \in V(G)$. Thus $\{f_1, f_2\}$ is a signed (j,k)-dominating family on G, a contradiction to $d_{jkS}(G) = 1$. □

Next we present a lower bound on the signed (j,k)-domatic number.

Theorem 22. Let $j, k \geq 1$ be integers such that $j \leq k+1$, and let G be a graph with minimum degree $\delta(G) \geq k-1$. If G contains a vertex $v \in V(G)$ such that all vertices of $N[N[v]]$ have degree at least $k+1$, then $d_{jkS}(G) \geq j$.

Proof. Let $\{u_1, u_2, \ldots, u_j\} \subseteq N(v)$. The hypothesis that all vertices of $N[N[v]]$ have degree at least $k+1$ implies that the functions $f_i : V(G) \to \{-1, 1\}$ such that $f_i(u_i) = -1$ and $f_i(x) = 1$ for each vertex $x \in V(G) \setminus \{u_i\}$ are signed k-dominating functions on G for $i \in \{1, 2, \ldots, j\}$. Since $f_1(x) + f_2(x) + \ldots + f_j(x) \leq j$ for each vertex $x \in V(G)$, we observe that $\{f_1, f_2, \ldots, f_j\}$ is a signed (j,k)-dominating family on G, and Theorem 22 is proved. □

Corollary 23. Let $j, k \geq 1$ be integers such that $j \leq k+1$. If G is a graph of minimum degree $\delta(G) \geq k+1$, then $d_{jkS}(G) \geq j$.

Theorem 24. Let $j, k \geq 1$ be integers such that $j < k$. If G is a $(k+1)$-regular graph of order n, then $d_{jkS}(G) = j$.

Proof. Let f be an arbitrary signed k-dominating function on G. If we define the sets $P = \{v \in V(G) \,|\, f(v) = 1\}$ and $M = \{v \in V(G) \,|\, f(v) = -1\}$, then we firstly show that

$$|P| \geq \left\lceil \frac{n(k+1)}{k+2} \right\rceil \tag{1}$$

Because of $\sum_{x \in N[y]} f(x) \geq k$ for each vertex $y \in V(G)$, the $(k+1)$-regularity of G implies that each vertex $u \in P$ is adjacent to at most one vertex in M and each vertex $v \in M$ is adjacent to exactly $k+1$ vertices in P. Therefore we obtain

$$|P| \geq |M|(k+1) = (n - |P|)(k+1),$$

and this leads to (1) immediately.

Now let $\{f_1, f_2, \ldots, f_d\}$ be a signed (j,k)-dominating family on G with $d = d_{jkS}(G)$. Since $\sum_{i=1}^{d} f_i(u) \leq j$ for every vertex $u \in V(G)$, each of these sums contains at least $\lceil (d-j)/2 \rceil$ summands of value -1. Using this and inequality (1), we see that the sum

$$\sum_{x \in V(G)} \sum_{i=1}^{d} f_i(x) = \sum_{i=1}^{d} \sum_{x \in V(G)} f_i(x) \tag{2}$$

contains at least $n\lceil (d-j)/2 \rceil$ summands of value -1 and at least $d\lceil n(k+1)/(k+2) \rceil$ summands of value 1. As the sum (2) consists of exactly dn summands, it follows that

$$n\left\lceil \frac{d-j}{2} \right\rceil + d\left\lceil \frac{n(k+1)}{k+2} \right\rceil \leq dn. \tag{3}$$

and thus (3) leads to

$$\frac{n(d-j)}{2} + \frac{dn(k+1)}{k+2} \leq dn.$$

Since $j < k$, a simple calculation shows that this inequality implies $d < j+2$ and so $d \leq j+1$. If we suppose that $d = j+1$, then we observe that d and j are of different parity. Applying Theorem 19, we obtain the contradiction

$$j+1 = d \leq \frac{j-1}{k}(k+2) < j+1.$$

Therefore $d \leq j$, and Corollary 23 yields to the desired result $d = j$. □

On the one hand Theorem 24 demonstrates that the bound in Corollary 23 is sharp, on the other hand Proposition 8 with $\delta(G) = n-1$ and $j = k = n-2$ shows that Theorem 24 is not valid in general when $j = k$.

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