A Note on Roman Bondage Number of Graphs

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Abstract

A *Roman dominating function*, (or simply RDF) on a graph $G = (V(G), E(G))$ is a labeling $f : V(G) \to \{0,1,2\}$ satisfying the condition that every vertex with label 0 has at least a neighbor with label 2. The *Roman domination number*, $\gamma_R(G)$ of G, is the minimum of $\sum_{v \in V(G)} f(v)$ over such functions. The *Roman bondage number* $b_R(G)$ of a graph G with maximum degree at least two is the minimum cardinality among all sets $E \subseteq E(G)$ for which $\gamma_R(G - E) > \gamma_R(G)$. It was conjectured that if G is a graph of order n with maximum degree at least two, then $b_R(G) \leq n-1$. In this paper we settle this conjecture. More precisely, we prove that for every connected graph of order $n \geq 3$, $b_R(G) \leq \min\{n-1, n-\gamma_R(G)+5\}$.

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1 Introduction

In this paper all graphs are simple. For a graph G, $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. For a vertex u in $V(G)$,

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$N(u)$ denotes the set of its neighbors and we write $d(u) = |N(u)|$.

A subset $D \subseteq V(G)$ of the vertices of a graph G is a *dominating set* if every vertex of $G - D$ has a neighbor in D. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. A labeling $f : V(G) \to \{0, 1, 2\}$ is a *Roman dominating function* (or simply RDF), if every vertex u with $f(u) = 0$, has a neighbor v with $f(v) = 2$. Let (V_0, V_1, V_2) be an ordered partition of $V(G)$, where $V_i = \{v \in V(G) : f(v) = i\}$, for $i = 0, 1, 2$. There is a one to one correspondence between all Roman domination functions and all ordered partitions (V_0, V_1, V_2) of $V(G)$ with this property that each vertex of V_0, has a neighbor in V_2, and we write $f = (V_0, V_1, V_2)$. The *weight* of a Roman domination function f, denoted by $w(f)$, is the value $\sum_{v \in V(G)} f(v)$, and the *Roman domination number* of G, denoted by $\gamma_R(G)$, is the minimum weight of a Roman domination function and such function is called a $\gamma_R(G)$*-function.*

The *bondage number*, denoted by $b(G)$, is the minimum cardinality among all sets $E \subseteq E(G)$ for which $\gamma(G-E) > \gamma(G)$. The *Roman bondage number* $b_R(G)$ of a graph G with maximum degree at least two, is the minimum cardinality of all sets $E \subseteq E(G)$ for which $\gamma_R(G-E) > \gamma_R(G)$. This concept introduced and studied for the first time in [3]. In [3], the authors introduced some upper bounds for $b_R(G)$ and proved that for any tree, T, then $b_R(T) \leq 3$. These authors obtained some new upper bounds for the Roman bondage number of planar graphs too, see [4].

Fink et al [2] proved that for every graph G of order n, $b(G) \leq n - 1$. Ebadi and PushpaLatha [1] conjectured that if G is a graph of order n with maximum degree at least two, then $b_R(G) \leq n - 1$. In this paper it is shown that this conjecture is true. Also, we prove that for every connected graph of order $n \geq 3$, $b_R(G) \leq n - \gamma_R(G) + 5$.

2 Results

Theorem 1 *If G is a connected graph of order $n \geq 3$, then*

$$b_R(G) \leq \min\{n-1, n-\gamma_R(G)+5\}.$$

Proof. Since G is connected and $n \geq 3$, there are three vertices $u, u_1, u_2 \in V(G)$ such that $u_1, u_2 \in N(u)$. Let E_u denote the set of all edges of G incidence with u. We have

$$\gamma_R(G - E_u) = \gamma_R(G-u) + 1 \geq \gamma_R(G).$$

If $\gamma_R(G) < \gamma_R(G-u)+1$, then $b_R(G) \leq d(u) \leq n-1$. On the other hand, if $V_0 = N(u)$, $V_1 = V(G) \setminus (N(u) \cup \{u\})$ and $V_2 = \{u\}$, then $f = (V_0, V_1, V_2)$ is an RDF for G and so $\gamma_R(G) \leq w(f) = n - |E_u| + 1$. Thus, $b_R(G) \leq n - \gamma_R(G) + 1$ and $b_R(G) \leq \min\{n-1, n-\gamma_R(G)+1\}$.

So assume that

$$\gamma_R(G - E_u) = \gamma_R(G-u) + 1 = \gamma_R(G).$$

Let

$$D = \bigcup\{V_2, f = (V_0, V_1, V_2) \text{ is a } \gamma_R(G-u)\text{-function}\}.$$

We claim that $D \cap N(u) = \emptyset$. Toward a contradiction, let $w \in D \cap N(u)$. Since $w \in D$, there exists a $\gamma_R(G-u)$-function $f = (V_0, V_1, V_2)$ in which $f(w) = 2$. Thus $(V_0 \cup \{u\}, V_1, V_2)$ is an RDF for G, a contradiction.

Let E_1 denote the set of all edges of $G-u$ between u_1 and D. Since $D \cap N(u) = \emptyset$, $|E_1 \cup E_u| \leq n-1$. On the other hand, if $V_0 = (N(u) \cup N(u_1)) \setminus \{u, u_1\}$, $V_1 = V(G) \setminus (N(u) \cup N(u_1))$ and $V_2 = \{u, u_1\}$, then $f = (V_0, V_1, V_2)$ is an RDF for G and $\gamma_R(G) \leq w(f) \leq 4 + (n - |E_1 \cup E_u| - 1)$. Thus, $|E_1 \cup E_u| \leq n - \gamma_R(G) + 3$ and $|E_1 \cup E_u| \leq \min\{n-1, n-\gamma_R(G)+3\}$.

If $\gamma_R(G-u) < \gamma_R(G-u-E_1)$, then since $\gamma_R(G-E_u-E_1) = \gamma_R(G-$

$u - E_1) + 1$, we find that

$$\begin{aligned}\gamma_R(G) &= \gamma_R(G - E_u) \\ &= \gamma_R(G - u) + 1 \\ &< \gamma_R(G - u - E_1) + 1 \\ &= \gamma_R(G - E_u - E_1),\end{aligned}$$

and therefore $b_R(G) \leq \min\{n - 1, n - \gamma_R(G) + 3\}$.

So we can assume that $\gamma_R(G - u) = \gamma_R(G - u - E_1)$. Since every $\gamma_R(G-u-E_1)$-function is an RDF for $G-u$ and $\gamma_R(G-u) = \gamma_R(G-u-E_1)$, every $\gamma_R(G - u - E_1)$-function is a $\gamma_R(G - u)$-function. We claim that for every $\gamma_R(G - u - E_1)$-function f, $f(u_1) = 1$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_R(G - u - E_1)$-function. By the above argument and the definition of D, f is a $\gamma_R(G-u)$-function and $V_2 \subseteq D$. Since $D \cap N(u) = \emptyset$, we conclude that $f(u_1) \neq 2$. If $f(u_1) = 0$, then since f is a $\gamma_R(G - u)$-function, u_1 should be adjacent to a vertex of D in $G - u - E_1$, a contradiction. Therefore $f(u_1) = 1$.

Now, let

$$D' = \bigcup\{V_2, f = (V_0, V_1, V_2) \text{ is a } \gamma_R(G - u - E_1)\text{-function}\}.$$

Since every $\gamma_R(G - u - E_1)$-function is a $\gamma_R(G - u)$-function, we have $D' \subseteq D$.

Let E_2 denote the set of all edges of $G-u-E_1$ between u_2 and D'. We claim that there is no $z \in D'$ such that $\{u_1, u_2\} \subseteq N(z)$ in $G-u$. Toward a contradiction, assume that there is a vertex $z \in N(u_1) \cap N(u_2) \cap D'$ in $G-u$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_R(G-u-E_1)$-function such that $f(z) = 2$. Since f is a $\gamma_R(G-u-E_1)$-function, by the previous paragraph $f(u_1) = 1$. Now, we conclude that $(V_0 \cup \{u_1\}, V_1 - \{u_1\}, V_2)$ is an RDF for $G - u$ of weight $\gamma_R(G)-2$ and this is a contradiction. This shows that $|E_1 \cup E_2 \cup E_u| \leq n-1$. On the other hand, if $V_0 = (N(u) \cup N(u_1) \cup N(u_2)) \setminus \{u, u_1, u_2\}$, $V_1 = V(G) \setminus (N(u) \cup N(u_1) \cup N(u_2))$ and $V_2 = \{u, u_1, u_2\}$, then $f = (V_0, V_1, V_2)$ is an RDF for G and so $\gamma_R(G) \leq w(f) \leq 6 + (n - |E_1 \cup E_2 \cup E_u| - 1)$. Thus,

$|E_1 \cup E_2 \cup E_u| \leq n - \gamma_R(G) + 5$ and this implies that $|E_1 \cup E_2 \cup E_u| \leq \min\{n-1, n-\gamma_R(G)+5\}$.

We claim that $\gamma_R(G-u-E_1) < \gamma_R(G-u-(E_1 \cup E_2))$. By the contrary, suppose that $\gamma_R(G-u-E_1) = \gamma_R(G-u-(E_1 \cup E_2)) = \gamma_R(G)-1$. Similarly, as we did before, every $\gamma_R(G-u-(E_1 \cup E_2))$-function, is a $\gamma_R(G-u-E_1)$-function and so it is a $\gamma_R(G-u)$-function. Let $f = (V_0, V_1, V_2)$ be a $\gamma_R(G-u-(E_1 \cup E_2))$-function. Since f is a $\gamma_R(G-u-E_1)$-function, $f(u_1) = 1$ and f is a $\gamma_R(G-u)$-function, $f(u_2) \neq 2$. If $f(u_2) = 1$, then $(V_0 \cup \{u_1, u_2\}, V_1 - \{u_1, u_2\}, V_2 \cup \{u\})$ is an RDF for G of weight $\gamma_R(G)-1$, a contradiction. If $f(u_2) = 0$, then u_2 should be adjacent to a vertex u' of V_2 in $G-u-(E_1 \cup E_2)$. On the other hand, f is a $\gamma_R(G-u-E_1)$-function. So $u' \in D'$ and by definition of E_2, $u_2u' \in E_2 \cap E(G-u-(E_1 \cup E_2))$, a contradiction. So the claim is proved.

We have $\gamma_R(G-E_u-(E_1 \cup E_2)) = \gamma_R(G-u-(E_1 \cup E_2)) + 1$ and so

$$\begin{aligned}
\gamma_R(G) &= \gamma_R(G-E_u) \\
&= \gamma_R(G-u)+1 \\
&= \gamma_R(G-u-E_1)+1 \\
&< \gamma_R(G-u-(E_1 \cup E_2))+1 \\
&= \gamma_R(G-E_u-(E_1 \cup E_2)).
\end{aligned}$$

Thus $b_R(G) \leq \min\{n-1, n-\gamma_R(G)+5\}$. The proof is complete. □

Corollary 1 *If G is a graph of order n with maximum degree at least two, then $b_R(G) \leq n-1$.*

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