On Incidence Energy of Tree

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Abstract Let $G = (V, E)$ be a simple graph, $I(G)$ its incidence matrix. The incidence energy of G, denoted by $IE(G)$, is the sum of the singular values of $I(G)$. The incidence energy $IE(G)$ of a graph is recently proposed quantities. However, $IE(G)$ is closely related with the eigenvalues of the Laplacian and signless Laplacian matrices of G. The trees with the maximal, the second maximal, the third maximal, the smallest, the second smallest and the third smallest incidence energy were characterized. In this paper, the trees with the fourth and fifth smallest incidence energy are characterized by quasi-order method and Coulson integral formula, respectively. In addition, the fourth maximal incidence energy among all trees on n vertices is characterized.

1 Introduction

Let G be a simple graph with order n. Let A be the adjacency matrix of G with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

The energy of G is defined as

$$E(G) = \sum_{j=1}^{n} |\lambda_j|.$$

This quantity has a clear connection to chemical problems and for more details on graph energy see the reviews [17,19,28]. Nikiforov [27] recently extended the concept of energy to all (not necessarily square) matrices, defining the energy of a matrix M as the sum of the singular values of M. Recall that the singular values of a matrix M are equal to the square roots of the eigenvalues of the (square) matrix MM^t.

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In line with Nikiforov's idea, Jooyandeh et al. [23] introduced the incidence energy $IE(G)$ of a graph G, $IE(G)$ was defined as the energy of its incidence matrix $I(G)$. More details on incidence energy can be found in [1,2,5,8,9,15,16,21,22].

Let $D(G) = diag(d_1, d_2, \cdots, d_n)$ be the diagonal matrix of vertex degree. The Laplacian matrix $L(G)$ of G is $L(G) = D(G) - A(G)$ with eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$, and the signless Laplacian matrix $L^+(G)$ of G is $L^+(G) = D(G) + A(G)$ with eigenvalues $\mu_1{}^+ \geq \mu_2{}^+ \geq \cdots \geq \mu_n{}^+$. All eigenvalues of both $L(G)$ and $L^+(G)$ are real and non-negative.

The Laplacian-like energy $LEL(G)$ of a graph G, introduced in [20], is the sum of the square roots of eigenvalues of its Laplacian matrix, i.e.,

$$LEL(G) = \sum_{j=1}^{n} \sqrt{\mu_j}.$$

The following results are well known [11,16,25,26]:

Lemma 1.1 The spectra of $L(G)$ and $L^+(G)$ coincide if and only if the graph G is bipartite.

Lemma 1.2 If $IE(G)$ is the incidence energy of a bipartite graph G, and $\mu_1, \mu_2, \cdots, \mu_n$ are the eigenvalues of the Laplacian matrix of G, then

$$IE(G) = \sum_{j=1}^{n} \sqrt{\mu_j} = LEL(G).$$

In other words, for bipartite graphs the incidence energy IE and the Laplacian-like energy LEL coincide.

Since trees are bipartite, any result on $LEL(G)$ on trees is automatically applicable for $IE(G)$.

Denote by $\psi(G, x)$ the characteristic polynomial of the Laplacian matrix of the graph G. It is known [24] that this polynomial is of the form

$$\psi(G, x) = \det(xI - L) = \sum_{j=1}^{n} (-1)^j c_j(G) x^{n-j},$$

where $c_j(G) \geq 0$.

The characteristic polynomial of the adjacent matrix A of a graph G is called the characteristic polynomial of G, denoted by

$$\phi(G, x) = \det(xI - A) = x^n + a_1 x^{n-1} + \cdots + a_n.$$

It is well-known that the characteristic polynomial of a bipartite graph G takes the form

$$\phi(G, x) = \sum_{j=1}^{\lfloor n/2 \rfloor} a_{2j} x^{n-2j} = \sum_{j=1}^{\lfloor n/2 \rfloor} (-1)^j b_{2j} x^{n-2j},$$

where $b_{2j} = (-1)^j a_{2j}$ and $b_{2j} \geq 0$ for all $j = 1, \cdots, \lfloor n/2 \rfloor$, especially

$b_0 = a_0 = 1$. Moreover, the characteristic polynomial of a tree T can be expressed as

$$\phi(T,x) = \sum_{j=1}^{\lfloor n/2 \rfloor} (-1)^j m(T,j) x^{n-2j},$$

where $m(T,j)$ is the number of j-matching of T.

For $LEL(G)$ and $E(T)$, Gutman et al. [14] obtained the Coulson integral formulas as:

$$LEL(G) = \frac{1}{\pi} \int_0^{+\infty} \ln[\sum_{j \geq 0}^{n} c_j(G) x^{2j}] \frac{dx}{x^2},$$

$$E(T) = \frac{2}{\pi} \int_0^{+\infty} \frac{1}{x^2} \ln(\sum_{j=0}^{\lfloor n/2 \rfloor} m(T,j) x^{2j}) dx.$$

Let $s(G)$ be a subdivision graph of the graph G, which is obtained by inserting an additional vertex into each edge of G. A connection between the number of j-matching of $s(T)$ and coefficients $c_j(T)$ of the polynomial $\psi(G,x)$ was the following:

Lemma 1.3[6] Let T be a tree on n vertices. Then $m(s(T);j) = c_j(T)$ for $0 \leq j \leq n$.

For two trees T_1 and T_2 of the same order n, by using the Coulson integral formula and the above results, we have that if $m(T_1,j) \leq m(T_2,j)$ for all $j = 1, \cdots, \lfloor n/2 \rfloor$, then $E(T_1) \leq E(T_2)$, if there exists at least one index j_1, such that $m(T_1,j_1) < m(T_2,j_1)$, then $E(T_1) < E(T_2)$; if $c_j(T_1) \leq c_j(T_2)$ or $m(s(T_1),j) \leq m(s(T_2),j)$ for all $j = 1, \cdots, n-1$, then $LEL(T_1) \leq LEL(T_2)$, thus $IE(T_1) \leq IE(T_2)$; if there exists at least one index j_1 such that $c_{j_1}(T_1) < c_{j_1}(T_2)$ or $m(s(T_1),j) < m(s(T_2),j)$, then $IE(T_1) < IE(T_2)$.

Let Φ be a forest, $p(\Phi)$ be the product of the numbers of vertices in the components of Φ. Then $c_j(G)$ can be computed in the following way.

Lemma 1.4[24] The coefficients c_j of the polynomial $\psi(G,x)$ are given by the formula $c_j = \sum p(\Phi)$ for $1 \leq j \leq n$; where the summation is over all sub-forests Φ of G which have j edges.

Denote by S_{n_1,n_2} be the double star on n vertices ($n = n_1 + n_2 + 2, n_1 \geq n_2 \geq 1$) obtained by adding an edge between the centers of S_{n_1+1}, and S_{n_2+1}. Let T_{n_1,n_2,n_3} be a tree on n vertices obtained from the path $P_3 = u_1u_2u_3$ by adding n_1, n_2, n_3 pendant edges on u_1, u_2, u_3, respectively, where $n_1 + n_2 + n_3 + 3 = n$, and $n_1, n_3 \geq 1$.

Among the trees, it has been known [16] that the path P_n has maximum incidence energy and that the star S_n has minimal incidence energy. Recently Tang and Hou [29] characterized the trees with the second smallest, the third smallest, the second maximal and third maximal incidence energy among all trees on n vertices. These results can be stated as:

Lemma 1.5[16,29] If T is a tree on $n \geq 6$ vertices, and $T \neq S_n, S_{n-3,1}, S_{n-4,2}$, then $IE(T) > IE(S_{n-4,2}) > IE(S_{n-3,1}) > IE(S_n)$.

In this paper, we characterize the trees with the fourth smallest, the fifth smallest and the fourth maximal incidence energy among all trees on n vertices by quasi-order method and Coulson integral formula, respectively.

2 The fourth smallest incidence energy

The following σ-*transformation* can transform every tree which is not a star into a double star. The definition of σ-*transformation* and some known results were given in the following.

Let u_0 be a vertex of a tree T of degree $p+1$. Suppose that $u_0u_1, u_0u_2, \cdots, u_0u_p$ are pendant edges incident with u_0, and that v_0 is the neighbor of u_0 distinct from $u_1, u_2, \cdots, u_p$. Then we form a tree $T^* = \sigma(T, u_0)$ by removing the edges $u_0u_1, u_0u_2, \cdots, u_0u_p$ from T and adding p new pendant edges $v_0u_1, v_0u_2, \cdots, v_0u_p$ incident with v_0. We say that T^* is a σ-*transformation* of T.

Lemma 2.1[4] Let T is not a star and $T^* = \sigma(T, u_0)$ be a σ-transformation of a tree T of order n. Then $c_j(T) > c_j(T^*)$ for $2 \leq j \leq n-2$, and $c_j(T) = c_j(T^*)$ for $j = 0, 1, n-1, n$.

Lemma 2.2[29] If $n_1 \geq n_2 > 1$, Then $c_j(S_{n_1,n_2}) > c_j(S_{n_1+1,n_2-1})$ for $2 \leq j \leq n-2$, and $c_j(S_{n_1,n_2}) = c_j(S_{n_1+1,n_2-1})$ for $j = 0, 1, n-1, n$.

Lemma 2.3[23] For any graph G, $IE(G) = \frac{1}{2}E(s(G))$, where $E(s(G))$ is the energy of its subdivision graph $s(G)$.

Now we give the following observations which are used in the subsequent arguments.

Lemma 2.4 (1) If $n \geq 10$, then $c_j(T_{n-4,0,1}) > c_j(T_{1,n-5,1})$ for $4 \leq j \leq n-2$; $c_j(T_{n-4,0,1}) = c_j(T_{1,n-5,1})$ for $j = 0, 1, 2, 3, n-1, n$.
(2) If $n \geq 10$, then $c_j(S_{n-5,3}) \geq c_j(T_{1,n-5,1})$ and there exists at least one index j_1 such that $c_{j_1}(S_{n-5,3}) > c_{j_1}(T_{1,n-5,1})$.

Proof. By lemma 1.4, we have

$$c_j(T_{n-4,0,1}) = (j+1)\binom{n-4}{j} + (j+1)\binom{n-4}{j-1} + 2j\binom{n-4}{j-1}$$
$$+2j\binom{n-4}{j-1} + (j+1)\binom{n-4}{j-2} + 2j\binom{n-4}{j-2}$$

$$+ 3(j-1)\binom{n-4}{j-2} + (j+1)\binom{n-4}{j-3} = (j+1)\binom{n-4}{j}$$
$$+ (5j+1)\binom{n-4}{j-1} + (6j-2)\binom{n-4}{j-2} + (j+1)\binom{n-4}{j-3},$$

$$c_j(T_{1,n-5,1}) = (j+1)\binom{n-5}{j} + (j+1)\binom{n-5}{j-1} + 2j\binom{n-5}{j-1}$$
$$+ 2j\binom{n-5}{j-1} + (j+1)\binom{n-5}{j-1} + (j+1)\binom{n-5}{j-2} + 2j\binom{n-5}{j-2}$$
$$+ 4(j-1)\binom{n-5}{j-2} + (j+1)\binom{n-5}{j-2} + 2j\binom{n-5}{j-2}$$
$$+ (j+1)\binom{n-5}{j-2} + (j+1)\binom{n-5}{j-3} + 2j\binom{n-5}{j-3} + 2j\binom{n-5}{j-3}$$
$$+ (j+1)\binom{n-5}{j-3} + (j+1)\binom{n-5}{j-4}$$
$$= (j+1)\binom{n-5}{j} + (6j+2)\binom{n-5}{j-1} + (11j-1)\binom{n-5}{j-2}$$
$$+ (6j+2)\binom{n-5}{j-3} + (j+1)\binom{n-5}{j-4},$$

$$c_j(S_{n-5,3}) = (j+1)\binom{n-5}{j} + (j+1)\binom{n-5}{j-1} + 3\times 2j\binom{n-5}{j-1}$$
$$+ 3(j+1)\binom{n-5}{j-2} + 3\times 3(j-1)\binom{n-5}{j-2} + 3(j+1)\binom{n-5}{j-3}$$
$$+ 4(j-2)\binom{n-5}{j-3} + (j+1)\binom{n-5}{j-4}$$
$$= (j+1)\binom{n-5}{j} + (7j+1)\binom{n-5}{j-1} + (12j-6)\binom{n-5}{j-2}$$
$$+ (7j-5)\binom{n-5}{j-3} + (j+1)\binom{n-5}{j-4}.$$

So $c_j(T_{n-4,0,1}) - c_j(T_{1,n-5,1}) = (j-3)\binom{n-5}{j-3}$,

$$c_j(S_{n-5,3}) - c_j(T_{1,n-5,1}) = (j-1)\binom{n-5}{j-1} + (j-5)\binom{n-5}{j-2}$$
$$+ (j-7)\binom{n-5}{j-3}.$$

Thus we have that if $n \geq 10$, then $c_j(T_{n-4,0,1}) > c_j(T_{1,n-5,1})$ for $4 \leq j \leq n-2$; $c_j(T_{n-4,0,1}) = c_j(T_{1,n-5,1})$ for $j = 0, 1, 2, 3, n-1, n$. And for $n \geq 10$,

$$c_2(S_{n-5,3}) - c_2(T_{1,n-5,1}) = \binom{n-5}{1} - 3\binom{n-5}{0} > 0;$$

$$c_3(S_{n-5,3})-c_3(T_{1,n-5,1}) =2\binom{n-5}{2}-2\binom{n-5}{1}-4\binom{n-5}{0}>0;$$
$$c_4(S_{n-5,3})-c_4(T_{1,n-5,1}) =3\binom{n-5}{3}-\binom{n-5}{2}-3\binom{n-5}{1}>0;$$
$$c_5(S_{n-5,3})-c_5(T_{1,n-5,1}) =4\binom{n-5}{4}-2\binom{n-5}{2}\geq 0;$$
$$c_6(S_{n-5,3})-c_6(T_{1,n-5,1}) =5\binom{n-5}{5}+\binom{n-5}{4}-\binom{n-5}{3}\geq 0;$$
$c_j(S_{n-5,3})-c_j(T_{1,n-5,1}) > 0$ for $7 \leq j \leq n-2$ and $c_j(S_{n-5,3})-c_j(T_{1,n-5,1}) = 0$ for $j = 0, 1, n-1, n$.

Thus we have that if $n \geq 10$, then $c_j(S_{n-5,3}) \geq c_j(T_{1,n-5,1})$ and there exists at least one index j_1 such that $c_{j_1}(S_{n-5,3}) > c_{j_1}(T_{1,n-5,1})$.

The proof now is complete.

Combine Lemmas 1.1, 1.2 and 2.4, we have the following result.

Theorem 2.5 If $n \geq 10$, then $IE(T_{n-4,0,1}) > IE(T_{1,n-5,1})$ and $IE(S_{n-5,3}) > IE(T_{1,n-5,1})$.

Lemma 2.6 If $n \geq 8$ and $n_1 \geq n_2 > 0$, then $m(s(T_{n_2,n_1,n_3});j) > m(s(T_{n_2-1,n_1+1,n_3});j)$ for $2 \leq j \leq n_1+n_2+n_3+1$, and $m(s(T_{n_2,n_1,n_3});j) = m(s(T_{n_2-1,n_1+1,n_3});j)$ for $j = 0, 1, n_1+n_2+n_3+2$.

Proof. Suppose Γ is a graph with a vertex v_1 of degree 1, and let v_2 be the vertex adjacent to v_1. Let Γ_1 be the induced subgraph obtained by removing v_1, and Γ_{12} the induced subgraph obtained by removing $\{v_1, v_2\}$. Obiviously $m(\Gamma;j) = m(\Gamma_1;j) + m(\Gamma_{12};j-1)$. Thus from Fig.1, we have $m(s(T_{n_2,n_1,n_3});j) - m(s(T_{n_2-1,n_1+1,n_3});j) = m(G_9;i-1) - m(G_{10};j-1)$. By some computations,
$$m(G_9;j-1) = \binom{n_3+n_1+n_2-1+1}{j-1} + \binom{n_3+n_1+n_2-1}{j-2}$$
$$+n_1\binom{n_3+n_1-1+n_2-1+1}{j-2} + n_3\binom{n_3-1+n_1+n_2-1+1}{j-2}$$
$$+ n_3n_1\binom{n_3-1+n_1-1+n_2-1}{j-3},$$
$$m(G_{10};j-1) = \binom{n_3+n_1+n_2-1+1}{j-1}$$
$$+(n_2-1)\binom{n_3+n_1+n_2-2+1}{j-2} + n_3\binom{n_3-1+n_1+n_2-1}{j-2}$$
$$+ n_3(n_2-1)\binom{n_3-1+n_1-1+n_2-1}{j-3},$$
$m(s(T_{n_2,n_1,n_3};j)) - m(s(T_{n_2-1,n_1+1,n_3};j))$

$$=(n_1-n_2+2)\binom{n_1+n_2+n_3-1}{j-2}+n_3\binom{n_1+n_2+n_3-2}{j-3}$$
$$+(n_1-n_2+1)n_3\binom{n_1+n_2+n_3-3}{j-3}.$$

So $m(s(T_{n_2,n_1,n_3});j) > m(s(T_{n_2-1,n_1+1,n_3});j)$ for $2 \le j \le n_1+n_2+n_3+1$ and $m(s(T_{n_2,n_1,n_3});j) = m(s(T_{n_2-1,n_1+1,n_3});j)$ for $j = 0, 1, n_1+n_2+n_3+2$.

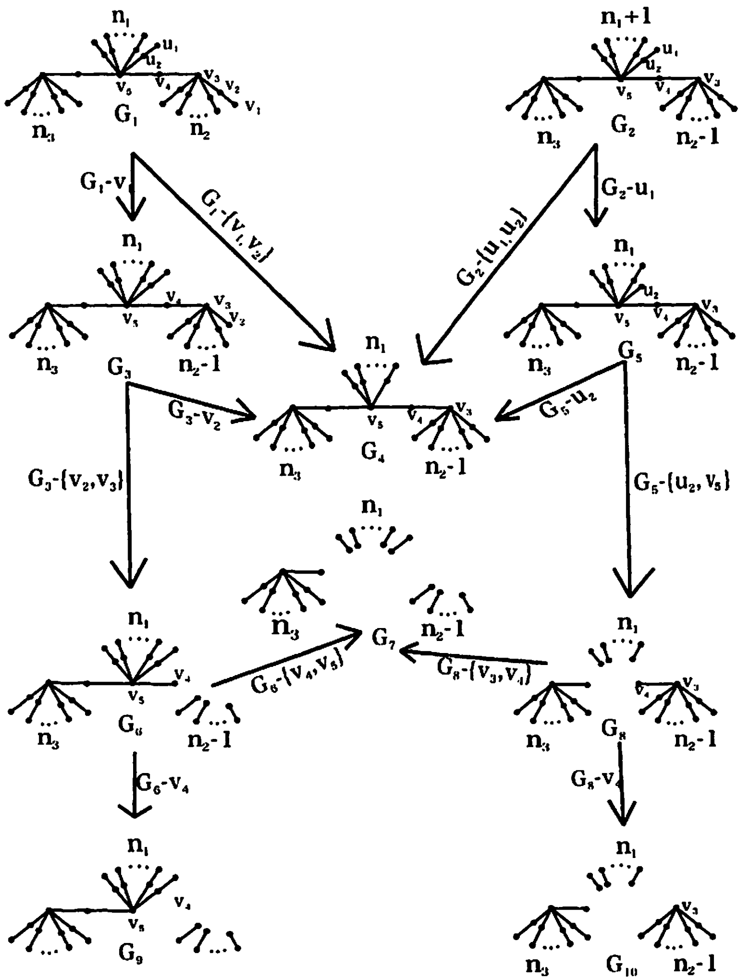

Fig.1

Combine Lemmas 1.1, 1.3 and 2.6, we have the following result.

Theorem 2.7 If $n \geq 8$ and $n_1 \geq n_2 > 0$, then
$IE(T_{n_2,n_1,n_3}) > IE(T_{n_2-1,n_1+1,n_3})$.

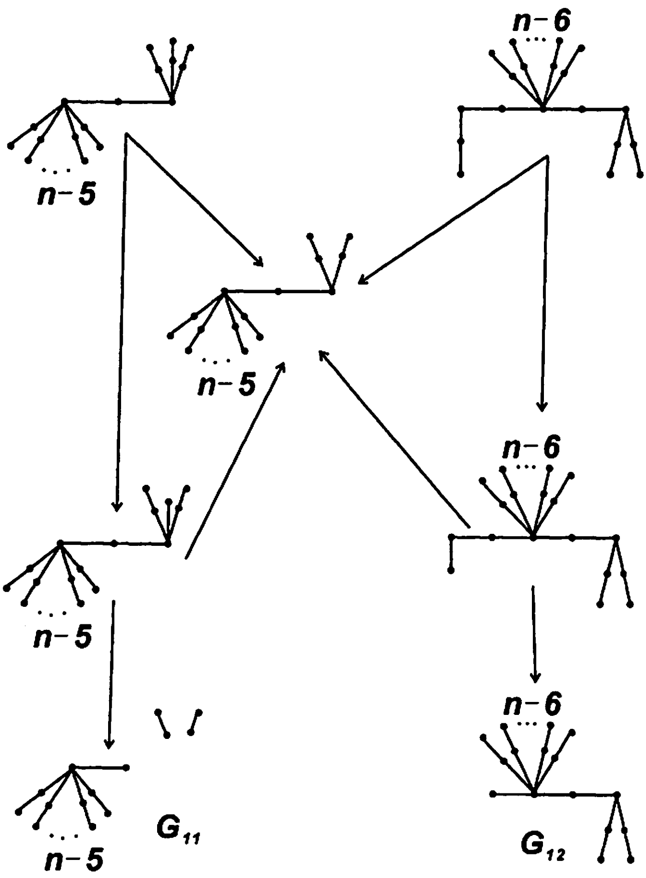

Fig.2

Lemma 2.8 If $n \geq 8$, then $m(s(S_{n-5,3});j) < m(s(T_{1,n-6,2});j)$ for $2 \leq j \leq n-2$ and $m(s(S_{n-5,3});j) = m(s(T_{1,n-6,2});j)$ for $j = 0, 1, n-1, n$.

Proof. By similar argument as in Lemma 2.8, from Fig.2, we have
$m(s(T_{1,n-6,2});j) - m(s(S_{n-5,3});j) = m(G_{12};j-1) - m(G_{11};j-1)$.
Since $m(G_{12};j-1) = \binom{n-2}{j-1} + \binom{n-4}{j-2} + (n-6)\binom{n-4}{j-2}$
$+2\binom{n-4}{j-2} + 2\binom{n-5}{j-3} + 2(n-6)\binom{n-6}{j-3}$,

and $m(G_{11};j-1)=\left(\begin{array}{c} n-2 \\ j-1 \end{array}\right)+(n-5)\left(\begin{array}{c} n-4 \\ j-2 \end{array}\right)$,

then we have $m(s(T_{1,n-6,2});j)-m(s(S_{n-5,3});j)=2\left(\begin{array}{c} n-4 \\ j-2 \end{array}\right)$
$+2\left(\begin{array}{c} n-5 \\ j-3 \end{array}\right)+2(n-6)\left(\begin{array}{c} n-6 \\ j-3 \end{array}\right)$.

Thus if $n \geq 8$, then $m(s(S_{n-5,3});j) < m(s(T_{1,n-6,2});j)$ for $2 \leq j \leq n-2$, and $m(s(S_{n-5,3});j) = m(s(T_{1,n-6,2});j)$ for $j=0,1,n-1,n$. This complete the proof of the lemma.

By Lemmas 2.3 and 2.8, we can get the following result.

Theorem 2.9 If $n \geq 8$, then $IE(S_{n-5,3}) < IE(T_{1,n-6,2})$.

Any tree with the diameter greater than 4 can be transformed to the tree with the diameter equal to 4 by *σ-transformation*. By Lemma 1.1 and Lemma 2.1, the diameter of the fourth smallest and the fifth smallest incidence energy among all trees on n vertices would be no more than 4. Thus we have the following theorem which is our main result in this section.

Theorem 2.10 If T is a tree on $n \geq 10$, and $T \neq S_n, S_{n-3,1}, S_{n-4,2}, T_{1,n-5,1}$, then $IE(S_n) < IE(S_{n-3,1}) < IE(S_{n-4,2}) < IE(T_{1,n-5,1}) < IE(T)$, i. e., $T_{1,n-5,1}$ is the unique tree with fourth smallest incidence energy among all trees on n vertices. The fifth smallest incidence energy among all trees on n vertices must be one of the trees $T_{n-4,0,1}$ and $S_{n-5,3}$.

For $T_{n-4,0,1}$ and $S_{n-5,3}$, if $n \geq 9$, then $c_2(S_{n-5,3})-c_2(T_{n-4,0,1}) = n-8 > 0$, but $c_{n-2}(S_{n-5,3})-c_{(n-2)}(T_{n-4,0,1}) = -4 < 0$. Thus the fifth smallest incidence energy among all trees on n vertices can not be determined by means of the above quasi-order method.

3 The fifth smallest incidence energy

Recently this quasi-order-incomparable problem was solved by means of the Coulson integral formula combined by methods of real analysis, algebra and combinatorics[3]. In this section, we will employ this method to get the fifth smallest incidence energy.

The following lemmas are well-known results which will be found in [3,6,10].

Lemma 3.1[10] Let G be a forest and $e=uv$ be an edge of G. The characteristic polynomial of G satisfies $\phi(G,x)=\phi(G-e,x)-\phi(G-u-v,x)$.

Lemma 3.2[3] If G_1 and G_2 are two graphs with the same number of vertices, then $E(G_1) - E(G_2) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \log \frac{\phi(G_1, ix)}{\phi(G_2, ix)} dx$.

Lemma 3.3[6] If G is a graph with n vertices and m edges then $\phi(S(G), x) = x^{m-n}\psi(G, x^2)$.

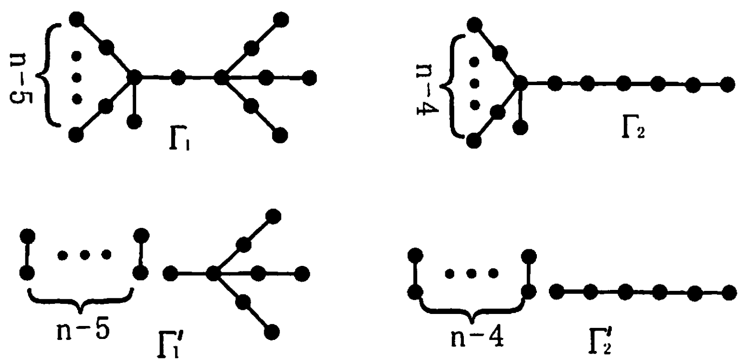

Fig.3

By some calculation, their characteristic polynomials can be expressed as follows.

Lemma 3.4 $\phi(S(T_{n-4,3}), x) = (x^2 - 1)\phi(S(T_{n-5,3}), x)$ $-x(x^2-1)^{n-4}(x^6 - 6x^4 + 6x^2 - 1)$ and $\phi(S(T_{n-3,0,1}), x)$ $= (x^2 - 1)\phi(S(T_{n-4,0,1}), x) - x(x^2-1)^{n-4}(x^6 - 5x^4 + 6x^2 - 1)$.

Proof. By Lemma 3.1 and Fig.3, we have that $\phi(S(T_{n-4,3}), x) = x\phi(\Gamma_1, x) -$ $\phi(S(T_{n-5,3}), x)$, and $\phi(\Gamma_1, x) = x\phi(S(T_{n-5,3}), x) - \phi(\Gamma'_1, x)$.
So $\phi(S(T_{n-4,3}), x) = (x^2 - 1)\phi(S(T_{n-5,3}), x) - x\phi(\Gamma'_1, x)$ and $\phi(S(T_{n-3,0,1}), x) = (x^2 - 1)\phi(S(T_{n-4,0,1}), x) - x\phi(\Gamma'_2, x)$.

By some simple calculation, $\phi(\Gamma'_1, x) = (x^2-1)^{n-4}(x^6 - 6x^4 + 6x^2 - 1)$ and $\phi(\Gamma'_2, x) = (x^2-1)^{n-4}(x^6 - 5x^4 + 6x^2 - 1)$.

Thus we have the above result.

Before showing the main result, we give some useful lemmas. For brevity, we let
$f_1(n, x) = x^{2n-1} + c_1(T_{n-4,0,1})x^{2n-3} + c_2(T_{n-4,0,1})x^{2n-5} + \cdots$
$+c_{n-1}(T_{n-4,0,1})x = \sum_{j=0}^{n} c_j(T_{n-4,0,1})x^{2n-2j-1}$;

$f_2(n,x) = x^{2n-1} + c_1(S_{n-5,3})x^{2n-3} + c_2(S_{n-5,3})x^{2n-5} + \cdots$
$+c_{n-1}(S_{n-5,3})x = \sum_{j=0}^{n} c_j(S_{n-5,3})x^{2n-2j-1}.$

It is easy to verify that $\frac{\phi(S(T_{n-5,3}),ix)}{\phi(S(T_{n-4,0,1}),ix)} = \frac{f_1(n,x)}{f_2(n,x)}$;

$$\frac{\phi(S(T_{n-4,3}),ix)}{\phi(S(T_{n-3,0,1}),ix)}$$
$$= \frac{(x^2+1)f_1(n,x) + x(x^2+1)^{n-4}(x^6+6x^4+6x^2+1)}{(x^2+1)f_2(n,x) + x(x^2+1)^{n-4}(x^6+5x^4+6x^2+1)}.$$

Lemma 3.5 For $n > 7$ and $x > 0$, we have
$\frac{\phi(S(T_{n-4,3}),ix)}{\phi(S(T_{n-3,0,1}),ix)} > \frac{\phi(S(T_{n-5,3}),ix)}{\phi(S(T_{n-4,0,1}),ix)}$, that is, $\frac{f_1(n+1,x)}{f_2(n+1,x)} > \frac{f_1(n,x)}{f_2(n,x)}$.

Proof. To prove $\frac{\phi(S(T_{n-4,3}),ix)}{\phi(S(T_{n-3,0,1}),ix)} > \frac{\phi(S(T_{n-5,3}),ix)}{\phi(S(T_{n-4,0,1}),ix)}$, we only to prove $(x^6+6x^4+6x^2+1)f_2(n,x) > (x^6+5x^4+6x^2+1)f_1(n,x)$.

Let $g_2(n,x) = (x^6+6x^4+6x^2+1)f_2(n,x)$ and $g_1(n,x) = (x^6+5x^4+6x^2+1)f_1(n,x)$, use k_j to denote coefficient of $x^{2n-2j-1}$ which is in g_1 and k_j' to denote coefficient of $x^{2n-2j-1}$ which is in g_2.

By the proof in Lemma 2.4 and some calculation, we have

$$k_j' = (68j+101)\binom{n-4}{j} + (47j+43)\binom{n-4}{j-1}$$
$$+(12j+10)\binom{n-4}{j-2} + (j+1)\binom{n-4}{j-3} + (42j+94)\binom{n-4}{j+1}$$
$$+(11j+34)\binom{n-4}{j+2} + (j+4)\binom{n-4}{j+3};$$
$$k_j = (68j+71)\binom{n-4}{j} + (47j+15)\binom{n-4}{j-1}$$
$$+(12j+6)\binom{n-4}{j-2} + (j+1)\binom{n-4}{j-3} + (42j+84)\binom{n-4}{j+1}$$
$$+(11j+33)\binom{n-4}{j+2} + (j+4)\binom{n-4}{j+3}.$$ Thus

$$k_j' - k_j = 30\binom{n-4}{j} + 28\binom{n-4}{j-1} + 4\binom{n-4}{j-2} + 10\binom{n-4}{j+1}$$
$$+\binom{n-4}{j+2} > 0.$$

So if $x > 0$. then $g_2(n,x) - g_1(n,x) > 0$, that is,

$$\frac{(x^2+1)f_1(n,x) + x(x^2+1)^{n-4}(x^6+6x^4+6x^2+1)}{(x^2+1)f_2(n,x) + x(x^2+1)^{n-4}(x^6+5x^4+6x^2+1)} > \frac{f_1(n,x)}{f_2(n,x)}.$$

Theorem 3.6 For $n = 8, 9, 10$, we have $IE(T_{n-5,3}) < IE(T_{n-4,0,1})$; for $n \geq 11$, we havw $IE(T_{n-5,3}) > IE(T_{n-4,0,1})$.

Proof. Let $h(n) = IE(T_{n-5,3}) - IE(T_{n-4,0,1})$, consider the following equation
$\dfrac{f_1(n,x)}{f_2(n,x)} =$
$\dfrac{x^{2n-1} + c_1(T_{n-4,0,1})x^{2n-3} + c_2(T_{n-4,0,1})x^{2n-5} + \cdots + c_{n-1}(T_{n-4,0,1})x}{x^{2n-1} + c_1(S_{n-5,3})x^{2n-3} + c_2(S_{n-5,3})x^{2n-5} + \cdots + c_{n-1}(S_{n-5,3})x}$, we have $\dfrac{f_1(n,x)}{f_2(n,x)} = \dfrac{f_1(n,-x)}{f_2(n,-x)}$. By Lemmas 2.3, 3.2 and 3.5, we have

$$\begin{aligned} h(n) = IE(T_{n-5,3}) - IE(T_{n-4,0,1}) &= \frac{1}{2}E(S(T_{n-5,3})) - \frac{1}{2}E(S(T_{n-4,0,1})) \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \log \frac{\phi(S(T_{n-5,3}), ix)}{\phi((S(T_{n-4,0,1}), ix)} dx = \frac{1}{\pi}\int_{0}^{+\infty} \log \frac{f_1(n,x)}{f_2(n,x)} dx \\ &> \frac{1}{\pi}\int_{0}^{+\infty} \log \frac{f_1(n-1,x)}{f_2(n-1,x)} dx. \end{aligned}$$

Thus we have $h(n) > h(n-1)$.

By some computer-aided calculations, we obtain that

$$h(10) = IE(T_{5,3})) - IE(T_{6,0,1}) = 11.3591 - 11.3606 = -0.0015;$$
$$h(11) = IE(T_{6,3})) - IE(T_{7,0,1}) = 12.5344 - 12.5284 = 0.006.$$

The proof is thus complete.

4 The fourth maximal incidence energy

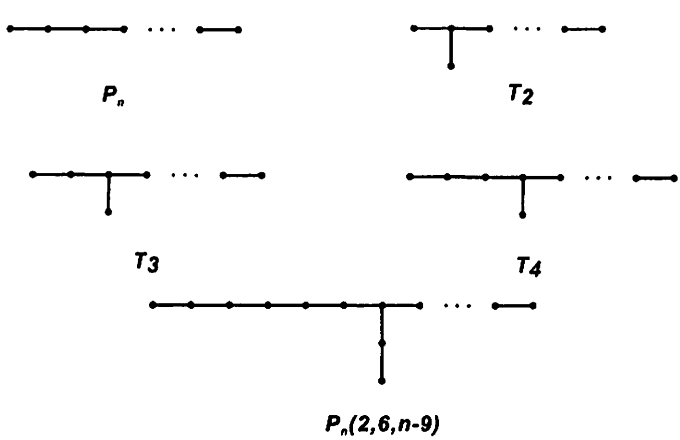

Fig.4

Let T_2, T_3 and T_4 be trees in Fig.4.

Lemma 4.1[29] If T is a tree on $n \geq 6$ vertices and $T \neq P_n, T_2, T_3$, then $IE(P_n) > IE(T_2) > IE(T_3) > IE(T)$.

Lemma 4.2[3] If $n \geq 14$, then the fourth maximal energy tree of vertices n is the tree $P_n(2, 6, n-9)$.

Combined with Lemmas 2.3, 4.1 and 4.2, we easily get the following result.

Theorem 4.3 If T is a tree on $n \geq 8$ vertices, then the fourth maximal incidence energy tree of vertices n is the tree T_4.

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