On independent domination critical graphs and k-factor critical

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Abstract

A subset S of $V(G)$ is an independent dominating set for G if S is independent and each vertex of G is either in S or adjacent to some vertex of S. Let $i(G)$ denote the minimum cardinality of an independent dominating set for G. For a positive integer t, a graph G is t-i-critical if $i(G) = t$, but $i(G + uv) < t$ for any pair of non-adjacent vertices u and v of G. Further, for a positive integer k, a graph G is k-factor-critical if for every $S \subseteq V(G)$ with $|S| = k$, $G - S$ has a perfect matching. In this paper, we provide sufficient conditions for connected 3-i-critical graphs to be k-factor-critical in terms of connectivity and minimum degree.

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1 Introduction

Let G denote a finite simple undirected graph with vertex set $V(G)$ and edge set $E(G)$. The complement of G is denoted by $\overline{G}$. For $S \subseteq V(G)$, S is independent if no two vertices of S are adjacent. The number of components of G and the number of odd components of G are denoted by $\omega(G)$ and $\omega_0(G)$, respectively. For a vertex v of G, the neighbour of v in G, denoted by $N_G(v)$, is $\{u \in V(G)|u$ is adjacent to $v\}$ and $deg_G(v) = |N_G(v)|$. The minimum degree of G, denoted by $\delta(G)$, is min $\{deg_G(v)|v \in V(G)\}$. For $S \subseteq V(G)$, $N_G(v) \cap S$ is denoted by $N_S(v)$. For a positive integer k, a graph G is said to be k-connected if $|V(G)| > k$ and $G - X$ is connected for every $X \subseteq V(G)$ if $|X| < k$. Further, a graph G is *k-factor-critical* if for every $S \subseteq V(G)$ with $|S| = k$, $G - S$ has a perfect matching. 1-factor-critical and 2-factor-critical graphs are also called *factor-critical* and *bicritical*, respectively.

For $S \subseteq V(G)$, S is a dominating set for G if each vertex of G either belongs to S or is adjacent to a vertex of S. A dominating set which is also independent is called *independent dominating set.* The minimum cardinality of an independent dominating set for G is called the *independent domination number* of G and denoted by $i(G)$.

In 1994, Ao [6] introduced the concept of "independent domination critical". For a positive integer t, a graph G is t-i-critical if $i(G) = t$, but $i(G + uv) < t$ for any pair of non-adjacent vertices u and v of G. It is easy to see that the only 1-i-critical graphs are K_n for some positive integer n. Ao [6] proved that G is 2-i-critical if and only if $\overline{G} \cong \bigcup_{i=1}^{n} K_{1,r_i}$ for some positive integers r_i and n. For $t \geq 3$, there are very few known results concerning connected t-i-critical graphs. Some properties of connected 3-i-critical graphs were established by Ao [6] and Ananchuen and Ananchuen [1, 2]. However, none of them concerns k-factor-critical property. In this paper, we provide sufficient conditions for connected 3-i-critical graphs to be k-factor-critical, for a positive integer k, in terms of the connectivity and minimum degree. In fact, we prove the following theorem in Section 3.

Theorem 1.1. *For a positive integer r, let G be a t-connected 3-i-critical graph of order n where $n \equiv r(\mathrm{mod}\ 2)$ and*

$$t = \begin{cases} r+1, & \text{for } 1 \leq r \leq 12, \\ \lfloor \frac{7r+1}{6} \rfloor, & \text{for } r \geq 13. \end{cases}$$

Further, for $1 \leq r \leq 12$, $\delta(G) \geq r + 2$. Then G is r-factor-critical. □

We conclude this section by pointing out that sufficient conditions for critical graphs with respect to ordinary domination (see [3] and [4]) and connected domination (see [5]) to be k-factor-critical were investigated. All of these results concern only $1 \leq k \leq 3$.

2 Preliminary results

In this section, we state some results that we make use of in establishing our main results. We begin with some terminology. For a pair of non-adjacent vertices u

and v of G, I_{uv} denotes a minimum independent dominating set for $G + uv$. Our first two results follow immediately from the definition of k-i-critical graphs.

Lemma 2.1. *Let G be a connected k-i-critical graph and let u and v be non-adjacent vertices of G. Then $|I_{uv}| = k - 1$ and $|I_{uv} \cap \{u, v\}| = 1$.* □

Lemma 2.2. *Let G be a connected 3-i-critical graph and let u and v be non-adjacent vertices of G. Then*

1. *$I_{uv} = \{u, w\}$ or $I_{uv} = \{v, w\}$ for some $w \in V(G) - \{u, v\}$.*
2. *If $\{w\} = I_{uv} - \{u, v\}$, then $\{u, v, w\}$ is independent.* □

The next two results provide necessary and sufficient condition for a graph to contain a perfect matching and to be k-factor-critical.

Theorem 2.3. *(Tutte's Theorem)(see Page 76 in [7]) A nontrivial graph G has a perfect matching if and only if, for every proper subset S of $V(G)$, $\omega_0(G - S) \leq |S|$.* □

Theorem 2.4. *[8] A graph G is k-factor-critical if and only if $\omega_0(G-S) \leq |S|-k$, for every $S \subseteq V(G)$ and $|S| \geq k$.* □

Ao [6] provided an upper bound on $\omega(G - S)$ for connected 3-i-critical graphs G with a vertex cutset S as follows.

Theorem 2.5. *[6] Let G be a connected 3-i-critical graph and S a vertex cutset. Then $\omega(G - S) \leq |S| + 1$.* □

The bound in Theorem 2.5 is improved considerably by Ananchuen and Ananchuen [2].

Theorem 2.6. *[2] Let G be a connected 3-i-critical graph and S a vertex cutset. If $|S| \geq 3$, then $\omega(G - S) \leq \frac{1+\sqrt{8|S|+1}}{2}$.* □

As a consequence of Theorems 2.5 and 2.6, we have:

Corollary 2.7. *Let G be a connected 3-i-critical graph and S a vertex cutset. Then*

$$\omega(G - S) \leq \begin{cases} |S| - 2, & \text{for } |S| \geq 6, \\ 3, & \text{for } |S| \leq 5. \end{cases}$$ □

Our last result concerns the minimum degree of connected 3-i-critical graphs G with a cutset of small size.

Lemma 2.8. *[2] Let G be a connected 3-i-critical graph and S a minimum cutset where $2 \leq |S| \leq 3$. If $\omega(G - S) = 3$, then $\delta(G) = |S|$.* □

3 The main results

In this section, we establish the proof of Theorem 1.1. We begin with some lemmas. Our first result follows immediately from Theorems 2.3 and 2.5.

Lemma 3.1. *If G is a connected 3-i-critical graph of even order, then G has a perfect matching.* □

Lemma 3.2. *Let G be a 3-connected 3-i-critical graph of even order. If $\delta(G) \geq 4$, then G is bicritical.*

Proof. Suppose to the contrary that G is not bicritical. By Theorem 2.4, there exists $S \subseteq V(G)$ where $|S| \geq 2$ and $\omega_0(G-S) > |S|-2$. Theorem 2.3 and Lemma 3.1 together with the fact that G is of even order implies that $\omega_0(G-S) = |S|$. Consequently, $|S| \leq 3$ by Corollary 2.7. Then, by the connectivity of G, $|S| = 3$ and thus $\omega(G-S) = \omega_0(G-S) = |S| = 3$. By Lemma 2.8, $\delta(G) = 3$, contradicting the minimum degree of G. This completes the proof of our lemma. □

Lemma 3.3. *Let G be a connected 3-i-critical graph and S a cutset of G. Let x_1, x_2, y_1, y_2 be four distinct vertices of $G-S$ where $x_iy_i \notin E(G)$ for $1 \leq i \leq 2$ and these four vertices belong to at least two different components of $G-S$. Further, let $\{z_i\} = I_{x_iy_i} - \{x_i, y_i\}$, $1 \leq i \leq 2$. If either $\omega(G-S) = 3$ but no components of $G-S$ is a singleton or $\omega(G-S) \geq 4$, then $\{z_1, z_2\} \subseteq S$ and $z_1 \neq z_2$.*

Proof. By our hypothesis, it is easy to see that, for $1 \leq i \leq 2$, there are at least two components of $G - (S \cup \{x_i, y_i\})$ which are dominated by z_i. Then $z_i \in S$ and thus $\{z_1, z_2\} \subseteq S$ as required. Let C be a component of $G-S$ containing x_1. We first suppose that $y_1 \in V(C) - \{x_1\}$. Then $x_2 \notin V(C)$ or $y_2 \notin V(C)$ and thus $z_1x_2 \in E(G)$ or $z_1y_2 \in E(G)$ since $\{z_1\} = I_{x_1y_1} - \{x_1, y_1\}$. Hence, $z_1 \neq z_2$ because $\{x_2, y_2, z_2\}$ is independent by Lemma 2.2(2).

We now assume that $y_1 \notin V(C)$. Since $\{x_2, y_2\} \subseteq V(G) - S$, it follows that $|N_G(x_2) \cap \{x_1, y_1\}| \leq 1$ and $|N_G(y_2) \cap \{x_1, y_1\}| \leq 1$. Thus $x_1 \in N_G(z_2)$ or $y_1 \in N_G(z_2)$ since $\{z_2\} = I_{x_2y_2} - \{x_2, y_2\}$. Consequently, $z_1 \neq z_2$ because $\{x_1, y_1, z_1\}$ is independent by Lemma 2.2(2). This completes the proof of our lemma. □

Lemma 3.4. *Let G be a connected 3-i-critical graph and S a cutset of G. Let x_1, x_2 and x_3 be three distinct vertices of $G-S$ and no two vertices of $\{x_1, x_2, x_3\}$ belong to the same component of $G-S$. Further, let $\{z_1\} = I_{x_1x_2} - \{x_1, x_2\}$ and $\{z_2\} = I_{x_1x_3} - \{x_1, x_3\}$. If either $\omega(G-S) = 3$ but no components of $G-S$ is a singleton or $\omega(G-S) \geq 4$, then $\{z_1, z_2\} \subseteq S$ and $z_1 \neq z_2$.*

Proof. By similar arguments as in the proof of Lemma 3.3, $\{z_1, z_2\} \subseteq S$. By Lemma 2.2(2), $\{x_1, x_2, z_1\}$ and $\{x_1, x_3, z_2\}$ are independent. Since x_3 does not belong to the same component containing x_1 or x_2, $z_1x_3 \in E(G)$ because $\{z_1\} = I_{x_1x_2} - \{x_1, x_2\}$. By Lemma 2.2(2), $z_1 \neq z_2$. This completes the proof of our lemma. □

Theorem 3.5. *For a positive integer k, let G be a t-connected 3-i-critical graph of even order where*

$$t = \begin{cases} 2k+1, & for\ 1 \le k \le 6, \\ \lfloor \frac{7k}{3} \rfloor, & for\ k \ge 7. \end{cases}$$

Further, for $1 \le k \le 6$, $\delta(G) \ge 2k+2$. *Then* G *is* $2k$*-factor-critical.*

Proof. We shall prove by mathematical induction. Clearly, our result holds for $k = 1$ by Lemma 3.2. We now suppose that our result holds for $k-1$ where $k \ge 2$. Suppose to the contrary that G satisfies our hypothesis but G is not $2k$-factor-critical. By Theorem 2.4, there is a vertex cutset S where $|S| \ge 2k$ and $\omega_0(G-S) > |S| - 2k$. By our induction hypothesis, G is $2(k-1)$-factor-critical and thus, by Theorem 2.4, $\omega_0(G-S) \le |S| - 2(k-1)$. Since G is of even order, $\omega_0(G-S) = |S| - 2k + 2 \ge 2$. Then $|S| \ge t \ge 2k+1$ and thus $\omega_0(G-S) \ge 3$. Because $\delta(G) \ge 2k+2$, if $\omega_0(G-S) = 3$ (that is $|S| = 2k+1$), then no components of $G-S$ is a singleton.

For $1 \le i \le |S| - 2k + 2$, let C_i be an odd component of $G-S$. Choose $x_i \in V(C_i)$. Then, for $1 \le i \ne j \le |S| - 2k + 2$, the only vertex of $I_{x_i x_j} - \{x_i, x_j\}$ must be in S by Lemmas 3.3 and 3.4. Thus $|S| \ge \binom{|S|-2k+2}{2}$. It also follows by Lemma 2.2(2) that x_i is not adjacent to the only vertex of $I_{x_i x_j} - \{x_i, x_j\}$ which is in S. Then $|N_S(x_i)| \le |S| - (|S| - 2k + 1) = 2k - 1$. Since $\delta(G) \ge 2k+2$, $2k+2 \le deg_G(x_i) \le (|V(C_i)| - 1) + |N_S(x_i)| \le |V(C_i)| + 2k - 2$. Hence, for $1 \le i \le |S| - 2k + 2$, $|V(C_i)| \ge 5$ because C_i is odd.

For $1 \le m \ne n \le |S| - 2k + 2$, let $x_{m_1}, \cdots, x_{m_5} \in V(C_m)$ and $x_{n_1}, \cdots, x_{n_5} \in V(C_n)$ where $x_{m_i} \ne x_{m_j}$ and $x_{n_i} \ne x_{n_j}$ for $1 \le i \ne j \le 5$. Consider $G + x_{m_i}x_{n_i}$ for $1 \le i \le 5$. Put $\{z_{x_{m_i}x_{n_i}}\} = I_{x_{m_i}x_{n_i}} - \{x_{m_i}, x_{n_i}\}$. Then, by Lemmas 3.3 and 3.4, $T_{mn} = \{z_{x_{m_i}x_{n_i}} | 1 \le i \le 5\} \subseteq S$, $|T_{mn}| = 5$ and for $1 \le m \ne n, m' \ne n' \le |S| - 2k + 2$, if $\{m, n\} \ne \{m', n'\}$, then $T_{mn} \cap T_{m'n'} = \emptyset$. Consequently, $|S| \ge 5\binom{|S|-2k+2}{2} = \frac{5}{2}(|S| - 2k + 2)(|S| - 2k + 1)$. Hence, $5|S|^2 + (13 - 20k)|S| + 5(4k^2 - 6k + 2) \le 0$. Consequently, $|S| \le \frac{20k-13+\sqrt{80k-31}}{10}$.

We next show that $\frac{20k-13+\sqrt{80k-31}}{10} < 2k+1$ for $1 \le k \le 6$ and $\frac{20k-13+\sqrt{80k-31}}{10} < \frac{7k}{3} - 1$ for $k \ge 7$. We first assume that $1 \le k \le 6$. Suppose to the contrary that $\frac{20k-13+\sqrt{80k-31}}{10} \ge 2k+1$ Then $\sqrt{80k-31} \ge 23$ and thus $k \ge 7$, a contradiction. Hence, $\frac{20k-13+\sqrt{80k-31}}{10} < 2k+1$ for $1 \le k \le 6$ as required. We now assume that $k \ge 7$. Again, suppose to the contrary that $\frac{20k-13+\sqrt{80k-31}}{10} \ge \frac{7k}{3} - 1$. Then $3\sqrt{80k-31} \ge 10k + 9$ and thus $0 \ge 100k^2 - 540k + 360 = (10k - 27)^2 - 369 \ge 1480$ since $k \ge 7$, a contradiction. Hence, $\frac{20k-13+\sqrt{80k-31}}{10} < \frac{7k}{3} - 1$ for $k \ge 7$. Therefore, $|S| < t$, contradicting the connectivity of G. Hence, G is $2k$-factor-critical as required. This completes the proof of our theorem. □

We now turn our attention to connected 3-i-critical graphs of odd order.

Lemma 3.6. *Let* G *be a* 2*-connected* 3*-i-critical graph of odd order. If* $\delta(G) \ge 3$, *then* G *is factor-critical.*

Proof. Suppose to the contrary that G is not factor-critical. By Theorem 2.4, there exists $S \subseteq V(G)$ where $|S| \ge 1$ and $\omega_0(G-S) > |S| - 1$. Since G is of odd order, $\omega_0(G-S) \ge |S| + 1$. This implies by Corollary 2.7 that $1 \le |S| \le 2$. Then, by the hypothesis that G is 2-connected, $|S| = 2$ and thus $\omega(G-S) =$

$\omega_0(G-S)=3$. By Lemma 2.8, $\delta(G)=2$, contradicting the minimum degree requirement of G. This completes the proof of our lemma. □

By applying similar arguments as in the proof of Theorem 3.5, we have the following results.

Theorem 3.7. *For a positive integer k, let G be a t-connected 3-i-critical graph of odd order where*

$$t=\begin{cases} 2k, & \text{for } 1\le k\le 6,\\ \lfloor \frac{7k-3}{3}\rfloor, & \text{for } k\ge 7.\end{cases}$$

Further, for $1\le k\le 6$, $\delta(G)\ge 2k+1$. Then G is $(2k-1)$-factor-critical. □

As a consequence of Theorems 3.5 and 3.7, Theorem 1.1 follows.

We conclude our paper by pointing out that the bound on minimum degree hypothesis in Theorem 1.1 is best possible. The graph H in Figure 1 is 2-connected 3-i-critical graph with minimum degree 2. Clearly, H is not factor-critical. For positive integers $r\ge 2$ and $n\ge 2r+2$, let $G=(2K_1\cup K_{r-1})+(2K_1\cup K_{n-r-3})$ be the join of graphs $2K_1\cup K_{r-1}$ and $2K_1\cup K_{n-r-3}$. It is not difficult to see that G is $(r+1)$-connected 3-i-critical with minimum degree $r+1$. But G is not r-factor-critical.

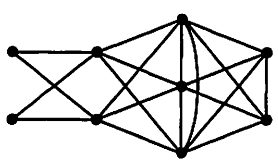

Figure 1: A 2-connected 3-i-critical graph with minimum degree 2

References

[1] Ananchuen, N., Ananchuen, W.: A characterization of independent domination critical graphs with a cutvertex, The Journal of Combinatorial Mathematics and Combinatorial Computing (to appear).

[2] Ananchuen, N., Ananchuen, W.: Some toughness results in independent domination critical graphs, Discussiones Mathematicae Graph Theory (to appear).

[3] Ananchuen, N., Plummer, M.D.: Matching properties in domination critical graphs, Discrete Mathematics, **277**, 1–13 (2004).

[4] Ananchuen, N., Plummer, M.D.: 3-factor-criticality in domination critical graphs, Discrete Mathematics, **307**, 3006–3015 (2007).

[5] Ananchuen, N., Ananchuen, W., Plummer, M.D.: Matching properties in connected domination critical graphs. Discrete Mathematics, **308**, 1260–1267 (2008)

[6] Ao, S.: Independent domination critical graphs, Master Thesis, University of Victoria, (1994)

[7] Bondy, A., Murty, U.S.: Graph Theory with Applications, Elsevier Science Publishing Co., Inc, New York (1976)

[8] Favaron, O.: On k-factor-critical graphs, Discussiones Mathematicae Graph Theory, **16**, 41–51 (1996).