Construction of super edge-connected multigraphs with prescribed degrees*

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Abstract. Let $G = (V, E)$ be a connected multigraph with order n. $\delta(G)$ and $\lambda(G)$ are the minimum degree and edge connectivity, respectively. The multigraph G is called maximally edge-connected if $\lambda(G) = \delta(G)$ and super edge-connected if every minimum edge-cut consists of edges incident with a vertex of minimum degree. A sequence $D = (d_1, d_2, \cdots, d_n)$ with $d_1 \geq d_2 \geq \cdots \geq d_n$ is called a multigraphic sequence if there is a multigraph with vertices $v_1, v_2, \cdots, v_n$ such that $d(v_i) = d_i$ for each $i = 1, 2, \cdots, n$. The multigraphic sequence D is super edge-connected if there exists a super edge-connected multigraph G with degree sequence D. In this paper, we present that a multigraphic sequence D with $d_n = 1$ is super edge-connected if and only if $\sum_{i=1}^{n} d_i \geq 2n$ and give a sufficient and necessary condition for a multigraphic sequence D with $d_n = 2$ to be super edge-connected. Moreover, we show that a multigraphic sequence D with $d_n \geq 3$ is always super edge-connected.

Keywords: Super edge-connected, degree sequence, multigraphic sequence.

1 Introduction

Let $G = (V, E)$ be a finite multigraph without loops with vertex set $V(G)$ and edge set $E(G)$. The set $N(u) = \{v : uv \in E(G)\}$ is called the *neighborhood* of the vertex u, and the set $N[u] = N(u) \cup \{u\}$ is called *closed neighborhood*. For a subset S of V, $N(S)$ denotes the neighborhood of S, i.e., $N(S) = \bigcup_{u \in S} N(u) \setminus S$, and $N[S]$ denotes the closed neighborhood of S, i.e., $N[S] = \bigcup_{u \in S} N[u]$. The degree of vertex u is $d(u) = |N(u)|$ and minimum degree of G is $\delta(G) = \min\{d(u) : u \in V(G)\}$. For two vertex sets V_1, $V_2 \subseteq V(G)$, $[V_1, V_2]_G$ is the set of edges with one end in V_1 and the other end in V_2. The *girth* $g(G)$ of the multigraph G is the length of

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its shortest cycle if G contains cycles, define $g(G) = +\infty$ otherwise. The length of a longest cycle of G is denoted by $c(G)$. We use $\lambda(G)$ to denote the edge-connectivity of G. In a multigraph G, *subdivision* of an edge uv is the operation of replacing uv with a path $u\omega v$ through a new vertex ω. The notation $\lfloor \frac{r}{s} \rfloor$ represents an integer which satisfies $\frac{r}{s} - 1 < \lfloor \frac{r}{s} \rfloor \leq \frac{r}{s}$, where r and s are integers. For example, $\lfloor \frac{5}{2} \rfloor = 2$. For notation and terminology not defined here, we refer to [4].

A network can be conveniently modeled as a graph $G = (V, E)$, with vertices representing nodes and edges representing links. A classic measure of network reliability is the connectivity $\kappa(G)$ and the edge-connectivity $\lambda(G)$. In general, the large $\kappa(G)$ (or $\lambda(G)$) is, the more reliable the network is. For $\kappa(G) \leq \lambda(G) \leq \delta(G)$, a graph G with $\kappa(G) = \delta(G)$ ($\lambda(G) = \delta(G)$) is naturally said to be *maximally connected* (*maximally edge-connected*), or *κ-optimal* (*λ-optimal*) for simplicity. As more refined indices of reliability than maximal connectivity and maximal edge-connectivity, super connectivity and super edge-connectivity were proposed in [1] [3]. A graph G is *super-connected*, *super-κ*, for short (resp. *super edge-connected*, *super-λ*, for short) if every minimum vertex-cut (resp. edge-cut) isolates a vertex of G.

A graph G is vertex-transitive if, for any pair u, v of vertices, there is an automorphism that maps u to v.

An Harary graph $H(n, k)$ has vertex set $\{1, 2, \cdots, n-1\}$. According to the parities of n and k, there are three types of Harary graph. In the following, additions are all taken modulo n.

Type 1: When k is even, suppose $k = 2r$. Two vertices i and j of $H(n, 2r)$ are adjacent if and only if $|i - j| \leq r$.

Type 2: When k is odd and n is even, suppose $k = 2r+1$. Then $H(n, k)$ is obtained from $H(n, 2r)$ by adding edges $\{(i, i+\frac{n}{2}) : i = 0, 1, \cdots, \frac{n}{2}-1\}$.

Type 3: When k and n are both odd, suppose $k = 2r + 1$. Then $H(n, k)$ is obtained from $H(n, 2r)$ by adding edges $\{(i, i + \frac{n+1}{2} : i = 0, 1, \cdots, \frac{n-3}{2}\} \cup \{(0, \frac{n-1}{2})\}$.

The first two types of Harary graph are vertex-transitive. And in this paper, all of the Harary graphs we referred are *Type* 1 or *Type* 2. For vertex-transitive graphs, the following two results are known:

Theorem 1.1 ([11]) *All connected vertex-transitive graphs are λ-optimal.*

Theorem 1.2 ([14]) *A connected vertex-transitive graph G which is neither a cycle nor a complete graph is super-λ if and only if it contains no clique K_k where k is the degree of G.*

Since the complete graph K_n is super-λ, Theorem 1.2 implies the following corollary.

Corollary 1.3 *If $k \geq 3$, then the Harary graph H(n,k) is super-λ.*

A sequence $D = (d_1, d_2, \cdots, d_n)$ of nonnegative integers is called the *degree sequence* of a graph G if the vertices of G can be labeled v_1, v_2, $\cdots$, v_n such that $d(v_i) = d_i$ for each $i = 1, 2, \cdots, n$. A sequence $D = (d_1, d_2, \cdots, d_n)$ of nonnegative integers is called graphical if it is the degree sequence of some simple graph (no loops and multiple edges). Graphical sequence was characterized by Havel [10], Erdös and Gallai [7], and Hakimi [8]. Edmonds [6] and Wang [15] proved that any graphical sequence $D = (d_1, d_2, \cdots, d_n)$ with $d_1 \geq d_2 \geq \cdots \geq d_n \geq 2$ is maximally edge-connected; and show that a graphical sequence $D = (d_1, d_2, \cdots, d_n)$ with $d_1 \geq d_2 \geq \cdots \geq d_n = 1$ is maximally edge-connected if and only if $\sum_{i=1}^{n} d_i \geq 2(n-1)$. In [13] Tian et al. studied the super edge-connectedness for a given graphical sequence D. For more results on graphs and their degree sequences, see the survey paper [9].

Multigraphic sequence was characterized by Senior [12] and Hakimi [8]. A sequence $D = (d_1, d_2, \cdots, d_n)$ of nonnegative integers is *multigraphic* if there exists a multigraph with vertices v_1, v_2, $\cdots$, v_n such that $d(v_i) = d_i$ for each $i = 1, 2, \cdots, n$.

Let $D = (d_1, d_2, \cdots, d_n)$ be a multigraphic sequence. Denote $\langle D \rangle$ the collection of all nonisomorphic multigraphs with degree sequence D. Let P be a multigraph theoretical property. We say that the multigraphic sequence D has property P if there is a multigraph $G \in \langle D \rangle$ such that G has property P. In particular, the multigraphic sequence D is said to be *maximally (super) dege-connected* if there exists a maximally(super) edge-connected multigraph $G \in \langle D \rangle$.

Chou and Frank [5] have studied that any multigraphic sequence $D = (d_1, d_2, \cdots, d_n)$ with $d_1 \geq d_2 \geq \cdots \geq d_n \geq 2$ is maximally edge-connected. In this paper, we prove that a multigraphic sequence D with $d_n = 1$ is super edge-connected if and only if $\sum_{i=1}^{n} d_i \geq 2n$ or $D = (n-1, 1, \cdots, 1)$. We also give a sufficient and necessary condition for a multigraphic sequence D with $d_n = 2$ to be super edge-connected. Furthermore, we show that a multigraphic sequence D with $d_n \geq 3$ is always super edge-connected.

2 Results

The following three results will be used frequently.

Lemma 2.1 ([8]) *A sequence $d_1 \geq d_2 \geq \cdots \geq d_n$, where $n \geq 2$, of non-negative integers is multigraphic if and only if the sum $\sum_{i=1}^{n} d_i$ is even and*

$d_1 \le d_2 + d_3 + \cdots + d_n$.

Lemma 2.2 ([2]) *Let $d_1 \ge d_2 \ge \cdots \ge d_n \ge 1$ be a sequence of non-negative integers and let $2 \le j \le n$ be an index. Then the sequence $\{d_1, d_2, \cdots, d_n\}$ is multigraphic if and only if the sequence $\{d_1 - 1, d_2, d_3, \cdots, d_{j-1}, d_j - 1, d_{j+1}, \cdots, d_n\}$ is multigraphic.*

Lemma 2.3 ([5]) *Any multigraphic sequence $D = (d_1, d_2, \cdots, d_n)$ with $d_1 \ge d_2 \ge \cdots \ge d_n \ge 2$ is maximally edge-connected.*

In this paper we consider the super edge-connectedness of the multigraphic sequence D, and get the following results.

Theorem 2.4 *Let $D = (d_1, d_2, \cdots, d_n)$ be a multigraphic sequence with $d_1 \ge d_2 \ge \cdots \ge d_n = 1$. Then D is super edge-connected if and only if $\sum_{i=1}^{n} d_i \ge 2n$ or $D = (n-1, 1, \cdots, 1)$.*

Proof. Suppose D is super edge-connected. Let $G \in \langle D \rangle$ such that G is super edge-connected. Then G is not isomorphic to a tree except $K_{1,n-1}$. Thus $\sum_{i=1}^{n} d_i \ge 2n$ or $D = (n-1, 1, \cdots, 1)$.

Now we prove the converse, suppose that D is not super edge-connected. If $D = (n-1, 1, \cdots, 1)$, then $K_{1,n-1} \in \langle D \rangle$ is a super edge-connected graph, a contradiction. If $\sum_{i=1}^{n} d_i \ge 2n$, then $G \in \langle D \rangle$ is not isomorphic to a tree. In this case we choose a connected multigraph $G_1 \in \langle D \rangle$ such that $c(G_1)$ is maximized. Let $C_1 = x_1 x_2 \cdots x_t$ be a longest cycle in G_1. Then $N[V(C_1)] \ne V(G_1)$ (for otherwise, G_1 is super edge-connected). Thus there exists a path $x_i y z$ where $x_i \in V(C_1)$, $y \in N(V(C_1)) \backslash V(C_1)$ and $z \in V(G_1) \backslash N[V(C_1)]$. Construct a connected multigraph G_2 from G_1 by deleting two edges $x_i x_{i+1}$ and yz, and adding two edges $y x_{i+1}$ and $x_i z$. It is easy to see that $G_2 \in \langle D \rangle$ and $C_2 = x_1 \cdots x_i y x_{i+1} \cdots x_t$ is a cycle of G_2, which contradicts to $c(G_1) \ge c(G_2)$. □

Theorem 2.5 *Let $D = (d_1, d_2, \cdots, d_n)$ be a multigraphic sequence with $d_1 \ge \cdots \ge d_{n-t} > d_{n-t+1} = \cdots = d_n = 2$. Then D is super edge-connected if and only if $D = (s+1, s+1, 2)$ or $D = (s+2, s, 2)$ or $\sum_{i=1}^{n-t} d_i \ge \sum_{i=n-t+1}^{n} d_i = 2t$, where s and t are integers with $s \ge 2$ and $1 \le t \le n-1$.*

Proof. Suppose that D is super edge-connected. Let $G \in \langle D \rangle$ such that G is super edge-connected. If $n = 3$, then $D = (s+1, s+1, 2)$

or $D = (s + 2, s, 2)$, where s is an integer with $s \geq 2$. If $n \geq 4$, then the set $\{v_{n-t+1}, v_{n-t+2}, \cdots, v_n\}$ is an independent set by the super edge-connectedness of G. Thus $\sum_{i=1}^{n-t} d_i \geq \sum_{i=n-t+1}^{n} d_i = 2t$ is obtained.

If $D = (s+1, s+1, 2)$ or $D = (s+2, s, 2)$, then the super edge-connected graph $G_1 \in \langle D \rangle$ or $G_2 \in \langle D \rangle$ (see Figure 1). Therefore, D is super edge-connected. We now show the sufficiency by induction on $\sum_{i=1}^{n} d_i$.

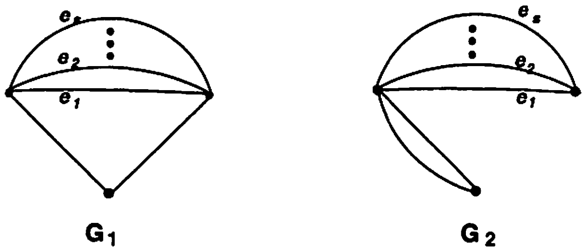

Figure 1.

If $\sum_{i=1}^{n} d_i = 4t$, then $\sum_{i=1}^{n-t} d_i = \sum_{i=n-t+1}^{n} d_i = 2t$. In the following, we construct a super edge-connected multigraph with D as its degree sequence. Let p_i be an integer such that $3 \leq d_i - 2p_i \leq 4$, $i = 1, 2, \cdots, n-t$. Let $c_i = d_i - 2p_i$.

Claim 1. The sequence $C = (c_1, c_2, \cdots, c_{n-t})$ is a multigraphic with $n - t \geq 2$.

If $n - t = 2$(i.e. $d_1 \geq d_2 > d_3 = \cdots = d_n = 2$), then $d_1 + d_2 = 2t$ and so both d_1 and d_2 are even or odd. Thus, $c_1 = c_2$.

If $n-t \geq 3$. We have $c_1 \leq 4 \leq 6 \leq 3(n-t-1) \leq \sum_{i=2}^{n-t} c_i$ as $3 \leq d_i - 2p_i = c_i \leq 4$ for $i = 1, 2, \cdots, n-t$.

Therefore, when $n - t \geq 2$, we have $c_1 \leq c_2 + \cdots + c_{n-t}$. Obviously $\sum_{i=1}^{n-t} c_i$ is even. Thus the sequence $C = (c_1, c_2, \cdots, c_{n-t})$ is multigraphic by Lemma 2.1, which completes the proof of the claim.

Claim 2. $\sum_{i=1}^{n-t} p_i < t$.

Since $3(n - t) \leq \sum_{i=1}^{n-t} (d_i - 2p_i) = \sum_{i=1}^{n-t} d_i - 2 \sum_{i=1}^{n-t} p_i = 2t - 2 \sum_{i=1}^{n-t} p_i$ and $t \leq n-1$, we have $\sum_{i=1}^{n-t} p_i \leq \frac{5t-3n}{2} < \frac{5t-3t}{2} = t$, and thus the claim is proven.

If $n - t = 1$ (i.e. $d_1 > d_2 = \cdots = d_n = 2$), then there is a super edge-connected graph $G_3 \in \langle D \rangle$ (see Figure 2).

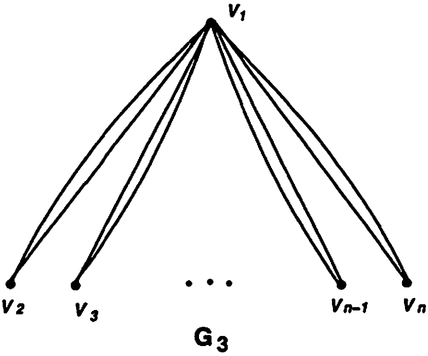

Figure 2.

If $n-t \geq 2$. By Claim 1 and Lemma 2.3, we have $C = (c_1, c_2, \cdots, c_{n-t})$ is a multigraphic and there exists a 3 edge-connected graph $G' \in \langle C \rangle$. The multigraph G'' obtained from G' by adding p_i loops on the vertex v_i for $i = 1, 2, \cdots, n-t$, and thus the sequence $D' = (d_1, d_2, \cdots, d_{n-t})$ is the degree sequence of G''. Therefore the multigraph G obtained from G'' by subdividing the $\sum_{i=1}^{n-t} p_i$(by Claim 2, we have $\sum_{i=1}^{n-t} p_i < t$) loops and any other $t - \sum_{i=1}^{n-t} p_i$ edges of G'' is super edge-connected and has D as its degree sequence.

Now we assume that it is true for all integers less than $\sum_{i=1}^{n} d_i$ ($\sum_{i=1}^{n} d_i > 4t$).

Case 1. $d_1 \geq d_2 \geq \cdots \geq d_l > 3$ $(2 \leq l \leq n-t)$.

Let $c_1 = d_1 - 1$, $c_l = d_l - 1$ and $c_i = d_i$ for $i \in \{2, \cdots, l-1, l+1, \cdots, n\}$. Then $C = (c_1, c_2, \cdots, c_n)$ is a multigraphic sequence by Lemma 2.2. Moreover, $\sum_{i=1}^{n-t} c_i = \sum_{i=1}^{n-t} d_i - 2 \geq 2t + 2 - 2 = 2t$ and $\sum_{i=1}^{n} c_i < \sum_{i=1}^{n} d_i$. By induction, there exists a multigraph $G' \in \langle C \rangle$ such that G' is super edge-connected. The multigraph G obtained from G' by adding an edge between v_1 and v_l, and G is also super edge-connected.

Case 2. $d_1 \geq 4$ and $d_2 = \cdots = d_{n-t} = 3$.

In this case, we have $n-t \geq 2$(if not, $n-t=1$, then $d_1 \geq 4$ and $d_2 = d_3 = \cdots = d_n = 2$, and so $\sum_{i=1}^{n} d_i = d_1 + (d_2 + d_3 + \cdots + d_n) \leq (d_2 + d_3 + \cdots + d_n) + (d_2 + d_3 + \cdots + d_n) = 4t$, which contradicts the hypothesis $\sum_{i=1}^{n} d_i > 4t$).

Subcase 2.1. $d_1 \leq d_2 + \cdots + d_{n-t} = 3(n-t-1)$.

Obviously, $\sum_{i=1}^{n-t} d_i$ is even. Therefore $D' = (d_1, d_2, \cdots, d_{n-t})$ is a multigraphic sequence. By Lemma 2.3, there exists a 3 edge-connected graph $G' \in \langle D' \rangle$ with at least t edges. The multigraph G obtained from G' by subdividing any t edges of G' is super edge-connected and has D as its degree sequence.

Subcase 2.2. $d_1 > d_2 + \cdots + d_{n-t} = 3(n-t-1)$.

The numbers of vertices of 2 degree and 3 degree are t and $n-t-1$, respectively. Since $d_1 + 3(n-t-1) + 2t$ is even, we have both d_1 and $3(n-t-1)$ are even or odd and so $d_1 - 3(n-t-1)$ is even.

Let $m = \frac{d_1 - 3(n-t-1)}{2}$, $p = \lfloor \frac{t-m}{3} \rfloor$ and $q = (n-t-1) - p - 1$. Therefore $t - m - 3p = 2$ or 1. We can verify that the graph G_4 or G_5 in Figure 3 is super edge-connected and has D as its degree sequence.

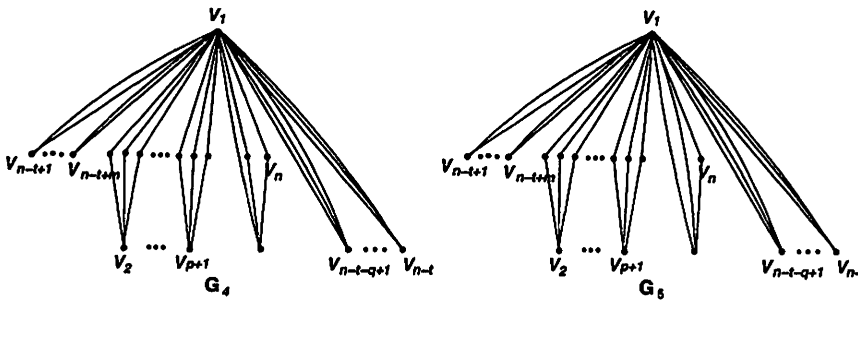

Figure 3. The proof of Theorem 2.5

Case 3. $d_1 = \cdots = d_{n-t} = 3$.

$\sum_{i=1}^{n-t} d_i = 3(n-t) = \sum_{i=1}^{n} d_i - 2t$ is even, since $D = (d_1, d_2, \cdots, d_n)$ is a multigraphic sequence and $d_{n-t+1} = \cdots = d_n = 2$. Thus the sequence $D' = (d_1, d_2, \cdots, d_{n-t})$ is multigraphic. By Lemma 2.3, there is a 3-edge-connected multigraph $G' \in \langle D' \rangle$. Then the multigraph G obtained from

G' by subdividing any t edges of G' is super edge-connected and has D as its degree sequence. □

Theorem 2.6 *Let $D = (d_1, d_2, \cdots, d_n)$ be a multigraphic sequence with $d_1 \geq d_2 \geq \cdots \geq d_n = k \geq 3$. Then D is super edge-connected.*

Proof. If $n = 2$, then the graph G which has two vertices and k multiple edges is a super edge-connected graph and has D as its degree sequence.

In the following, we assume $n \geq 3$.

By induction on $\sum_{i=1}^{n} d_i$. If $\sum_{i=1}^{n} d_i = nk$, then $H(n, k) \in \langle D \rangle$ and is super edge-connected by Corollary 1.3. Thus we now assume that it is true for all integers less than $\sum_{i=1}^{n} d_i$ ($\sum_{i=1}^{n} d_i \geq nk + 1$).

Case 1. $d_1 \geq d_2 \geq \cdots d_t > k \geq 3\,(2 \leq t \leq n-1)$.

Let $c_1 = d_1 - 1$, $c_t = d_t - 1$ and $c_i = d_i$ for $i = \{2, \cdots, t-1, t+1, \cdots, n\}$. Then the sequence $C = (c_1, c_2, \cdots, c_n)$ is a multigraphic sequence by Lemma 2.2. By induction, there is a multigraph $G' \in \langle C \rangle$ such that G' is super edge-connected. G obtained from G' by joining v_1 to v_t, is super edge-connected and $G \in \langle D \rangle$.

Case 2. $d_1 > d_2 = \cdots d_n = k \geq 3$.

Let $q = \lfloor \frac{d_1}{n-1} \rfloor$. Then $d_1 = (n-1)q + r$, $0 \leq r < n - 1$.

Subcase 2.1. $q = k$.

Then $d_1 = (n-1)k$. We can construct a multigraph by vertex set $V = \{v_1, v_2, \cdots, v_n\}$, that is we add k edges between v_1 and v_i $(i = 2, 3, \cdots, n)$. Obviously, the multigraph is super edge-connected and has D as its degree sequence.

Subcase 2.2. $q < k$.

Let $d_1 + (n-1)k = N$. Then N is even since N is a degree sum. Note that $r = d_1 - (n-1)q = N - (n-1)k - (n-1)q = N - (n-1)(k+q)$.

Subcase 2.2.1. $k - q$ is even, or $k - q$ is odd and $n - 1$ is even.

In the following we will construct a super edge-connected multigraph G with D as its degree sequence by using Harary graph $H = H(n-1, k-q)$ with vertices $\{0, 1, \cdots, n-2\}$ and an isolated vertex v_1.

First we add $(n-1)q$ edges between v_1 and H, that is adding q multiple edges between v_1 and each vertex $i \in V(H)$, $i \in \{0, 1, \cdots, n-2\}$.

If $k-q$ is even, then $k+q$ is also even, and so r is even. We delete $\frac{r}{2}$ edges $(0,1)$, $(2,3)$, $\cdots$, $(r-2, r-1)$ from $H(n-1,\ k-q)$ and add r edges between v_1 and $\{0,\ 1,\ \cdots,\ r-1\}$.

If $k-q$ is odd and $n-1$ is even, then r is even. We delete $\frac{r}{2}$ edges $(0, \frac{n-1}{2})$, $(1, \frac{n-1}{2}+1)$, $\cdots$, $(\frac{r}{2}-1, \frac{n-1}{2}+\frac{r}{2}-1)$ from $H(n-1,\ k-q)$ and add r edges between v_1 and $\{0,\ \frac{n-1}{2},\ 1,\ \frac{n-1}{2}+1, \cdots,\ \frac{r}{2}-1,\ \frac{n-1}{2}+\frac{r}{2}-1\}$.

Finally we identify each vertex i of $H(n-1,\ k-q)$ with vertex v_{i+2}, $i = 0, 1, \cdots, n-2$. Then the obtained multigraph G has D as its degree sequence.

Now we show that G is super edge-connected. It is sufficient to check that $|[A,\overline{A}]_G| > k$, for any vertex set $A \subset V(G)$ with $|A| \geq 2$ and $|\overline{A}| = |V(G)\backslash A| \geq 2$.

Without loss of generality, we assume $A \subset V(H)$. If $q \geq 1$, then $|[A,\overline{A}]_G| \geq q|A| + k - q > k$. We assume that $q = 0$. If $|V(H)\backslash A| \geq 2$, then $|[A,\overline{A}]_G| \geq |\ [A, V(H)\backslash A]_H| > k - q = k$ by H is super edge-connected. If $|V(H)\backslash A| = 1$, then $|[A,\overline{A}]_G| \geq d_1 - 1 + k - q - 1 = d_1 - 2 + k > k$, since $d_1 \geq 3$.

Subcase 2.2.2. $k-q$ is odd and $n-1$ is odd.

Then r is odd and so $n-1-r$ is even. Obviously both k and $q+1$ are even or odd, and so $k-(q+1)$ is even.

In the following we will construct a super edge-connected multigraph G with D as its degree sequence by vertex set $V = \{v_1,\ v_2,\ \cdots,\ v_n\}$.

Let $V_1 = \{v_2,\ \cdots,\ v_{1+r}\}$ and $V_2 = \{v_{2+r},\ \cdots,\ v_n\}$.

We construct a multigraph G by the following steps.

Step 1: We add $\frac{k-(q+1)}{2}$ cycles with the vertex set $\{v_2,\ v_3,\ \cdots,\ v_n\}$ and the length of all the cycles is $n-1$, where one of the cycles(see Figure 4) is

$$C = v_2 v_3 \cdots v_r v_{n-1} v_{1+r} v_n v_2 \ (r = n-3)$$

or

$C' = v_2 v_3 \cdots v_{1+r} v_{2+r} v_{4+r} \cdots v_{2i+r} \cdots v_{n-1} v_{n-2} v_{n-4} \cdots v_{n-2(i+1)} \cdots v_{3+r} v_n v_2 \ (r < n-3)$.

Step 2: We add $q+1$ edges between v_1 and v_i $(i = 2, 3, \cdots, 1+r)$, q edges between v_1 and v_j $(j = 2+r, \cdots, n)$, and one edge between v_{l+r} and v_{l+1+r} $(l = 2, 3, \cdots, n-1-r)$.

Now we show that G is super edge-connected. It is sufficient to check that $|[B,\ \overline{B}]_G| > k$ for any vertex set $B \subset V(G)$ with $|B| \geq 2$ and $|\overline{B}| = |V \setminus B| \geq 2$.

Without loss of generality, we assume $v_1 \in B$, $|B| = p \geq 2$ and $|\overline{B}| = n - p \geq 2$.

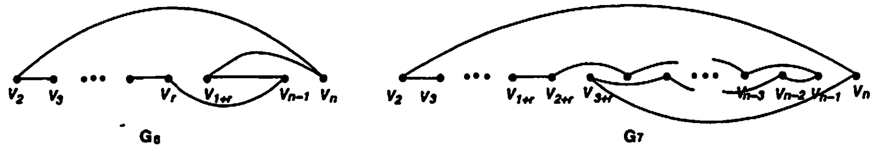

Figure 4.

Set $V' = V_1 \cup V_2 = \{v_2, v_3, \cdots, v_{1+r}\} \cup \{v_{2+r}, v_{3+r}, \cdots, v_n\} = \{v_2, v_3, \cdots, v_n\}$, $V_1' = B \cap V'$ and $V_2' = \overline{B} \cap V'$. As the number of the common edges of any cycle and any cut edge is even, we have $|[V_1', V_2']| \geq k-(q+1)$

If $\overline{B} \cap V_2 = \emptyset$ ($i.e. \overline{B} \cap V_1 = \overline{B}$), then

$$|[B, \overline{B}]_G| \geq |[\{v_1\}, \overline{B}]| + |[V_1', V_2']| \geq (n-p)(q+1) + k - (q+1) > k.$$

If $\overline{B} \cap V_2 \neq \emptyset$ and $\overline{B} \cap V_1 \neq \emptyset$. By the construction of the cycles with the vertex set $\{v_2, v_3, \cdots, v_n\}$, we obtain $|[V_1', V_2']| \geq k-(q+1)+1 = k-q$, and so

$$|[B, \overline{B}]_G| \geq |[\{v_1\}, \overline{B}]| + |[V_1', V_2']| \geq (q+1) + (n-p-1)q + k - q > k.$$

If $\overline{B} \cap V_2 = \overline{B}$ ($i.e. \overline{B} \cap V_1 = \emptyset$). By the construction of the cycles with the vertex set $\{v_2, v_3, \cdots, v_n\}$, we have $|[V_1', V_2']| \geq k-(q+1)+2 = k-q+1$ and so

$$|[B, \overline{B}]_G| \geq |[\{v_1\}, \overline{B}]| + |[V_1', V_2']| \geq (n-p)q + k - q + 1 > k.$$

□

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