SOME RECURRENCE RELATIONS FOR q-BERNOULLI NUMBERS AND POLYNOMIALS

YAHUI YU AND YUAN HE

ABSTRACT. In this paper, we perform a further investigation for the q-analogues of the classical Bernoulli numbers and polynomials. By applying summation transform techniques, we establish some new recurrence relations for these type numbers and polynomials. We also present some illustrative special cases as well as immediate consequences of the main results.

1. INTRODUCTION

In his oft-cited papers [4, 5], Carlitz firstly brought out the concepts of the q-extensions of the classical Bernoulli numbers and polynomials. Since then, many authors have studied Carlitz's q-Bernoulli numbers and polynomials and discovered that these numbers and polynomial possess many surprising properties in many different areas of mathematics; see, for example, [8, 11, 15, 16, 17, 18, 19, 20, 21, 26, 27, 29]. This paper is primarily concerned with the q-analogues of the classical Bernoulli numbers and polynomials considered by Kupershmidt [23], Kim and Kim [22] different from Carlitz's q-Bernoulli numbers and polynomials. We establish some new recurrence relations for them following the recent work of Alzer and Kwong [3]. These results are the corresponding generalizations of some known formulae on the classical Bernoulli numbers and polynomials.

We adopt the common notation on q-series in the standard books [9]. Throughout this paper, the parameter q is a fixed nonzero complex number with $|q| < 1$. The q-shifted factorial $(a,q)_n$ is defined for positive integer n and complex number a by

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad (a;q)_{-n} = \frac{1}{(aq^{-n};q)_n}. \tag{1.1}$$

The q-number factorial $[n]_q!$ is defined for positive integer n by

$$[0]_q! = 1, \quad [n]_q! = [1]_q[2]_q\cdots[n-1]_q[n]_q, \tag{1.2}$$

with $[x]_q$ being the q-number given by $[x]_q = (1-q^x)/(1-q)$ for complex number x. The q-binomial coefficient (also called Gaussian binomial coefficient) $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is defined for non-negative integer n by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} = \frac{[n]_q!}{[k]_q!\cdot[n-k]_q!} \quad (k = 0, 1, \ldots, n). \tag{1.3}$$

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The q-analogue of exponential function $e_q(t)$ is defined by (see, e.g., [12, 13])

$$e_q(t) = \prod_{k=0}^{\infty} \frac{1}{(1-(1-q)q^k t)} = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} \quad (|t| < 1/|1-q|). \tag{1.4}$$

We now turn to the q-analogues of the classical Bernoulli numbers and polynomials. These type q-Bernoulli numbers $B_{n,q}$ and q-Bernoulli polynomials $B_{n,q}(x)$ are usually defined by the q-analogue of exponential function (see, e.g., [22, 23]):

$$\frac{t}{e_q(t)-1} = \sum_{n=0}^{\infty} B_{n,q} \frac{t^n}{[n]_q!} \quad \text{and} \quad \frac{t e_q(xt)}{e_q(t)-1} = \sum_{n=0}^{\infty} B_{n,q}(x) \frac{t^n}{[n]_q!}, \tag{1.5}$$

respectively. Obviously, the case $q \to 1$ in (1.5) yields the classical Bernoulli numbers B_n and Bernoulli polynomials $B_n(x)$ given by

$$\frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \quad \text{and} \quad \frac{t e^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \tag{1.6}$$

In fact, the above q-Bernoulli polynomials can also be defined recursively by the q-Bernoulli numbers, as follows, (see, e.g., [22])

$$B_{n,q}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^{n-k} B_{k,q} \quad (n \geq 0), \tag{1.7}$$

with the q-Bernoulli numbers obey the recurrence relation:

$$B_{0,q} = 1, \quad \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{1}{[k+1]_q} B_{n-k,q} = -B_{n,q} \quad (n \geq 1). \tag{1.8}$$

This paper is organized as follows. In the second section, we give some new recurrence relations for the q-Bernoulli numbers and polynomials described in (1.5), and deduce some known formulae including Alzer and Kwong [3] on the classical Bernoulli numbers and polynomials. The third section is contributed to the proof of the main result by applying summation transform techniques.

2. The restatement of results

Like the definition of the q-number shifted factorial stated in [9], we use the following notation: $[a]_{q;0} = 1$ and

$$[a]_{q;n} = [a]_q [a+1]_q \cdots [a+n-1]_q, \quad [a]_{q;-n} = \frac{1}{[a-1]_q [a-2]_q \cdots [a-n]_q}, \tag{2.1}$$

for positive integer n and complex number a. We present our results, as follows.

Theorem 2.1. *Let m, n be nonnegative integers and r be any integer. Then, for q-commuting variables x and y such that $xy = qyx$,*

$$\sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_q x^{m-k} q^{-nk-\binom{n+1}{2}} \frac{B_{n+k+r,q}(y)}{[n+k+1]_{q;r}}$$
$$= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-x)^{n-k} q^{-n(m+k+1)+\binom{k+1}{2}} \frac{B_{m+k+r,q}(x+y)}{[m+k+1]_{q;r}}$$
$$+ (-1)^{n+1} x^{m+n+1} \frac{[m]_q! \cdot [n]_q!}{[m+n+r]_q!} \sum_{i=0}^{r-1} \begin{bmatrix} m+n+r \\ i \end{bmatrix}_q \begin{bmatrix} n+r-1-i \\ n \end{bmatrix}_q$$
$$\times x^{r-1-i} B_{i,q}(y). \quad (2.2)$$

We next show some special cases of Theorem 2.1. In view of (1.3) and (2.1), we get that for non-negative integers n, k, r,

$$\frac{1}{[n+k+1]_{q;-r}} = [n+k]_q [n+k-1]_q \cdots [n+k+1-r]_q = [r]_q! \cdot \begin{bmatrix} n+k \\ r \end{bmatrix}_q. \quad (2.3)$$

Clearly, the case r being non-negative integer in Theorem 2.1 means the second summation of the right hand side of (2.2) vanishes. It follows from (2.3) and Theorem 2.1 that we state the following result.

Corollary 2.2. *Let m, n, r be nonnegative integers. Then, for q-commuting variables x and y such that $xy = qyx$,*

$$\sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_q \begin{bmatrix} n+k \\ r \end{bmatrix}_q x^{m-k} q^{-nk-\binom{n+1}{2}} B_{n+k-r,q}(y)$$
$$= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} m+k \\ r \end{bmatrix}_q (-x)^{n-k} q^{-n(m+k+1)+\binom{k+1}{2}} B_{m+k-r,q}(x+y). \quad (2.4)$$

Since the classical Bernoulli polynomials satisfy the different equation, as follows, (see, e.g., [1])

$$B_n(x+1) - B_n(x) = nx^{n-1} \quad (n \geq 0), \quad (2.5)$$

so by taking $q \to 1, m = n, x = 1$ and then substituting x for y in (2.4), we get that for nonnegative integers m, r,

$$\sum_{\substack{k=0 \\ k+m \text{ odd}}}^{m-1} \binom{m}{k} \binom{m+k}{r} B_{m+k-r}(x)$$
$$= \frac{1}{2} \sum_{k=0}^{m} \binom{m}{k} \binom{m+k-1}{r} (-1)^{m-k} (m+k) x^{m+k-r-1}, \quad (2.6)$$

which was discovered by Alzer and Kwong [3] and used to deduce some interesting identities for the classical Bernoulli numbers including the result:

$$\sum_{\substack{k=0\\ k+m\ odd}}^{m-1} \binom{m}{k}\binom{m+k}{r}\binom{m+k-r}{j} B_{m+k-r-j} = 0 \quad (0 \le j \le m-2-r), \tag{2.7}$$

which extends two formulae due to Kaneko [14], Chen and Sun [7], who proved (2.7) for the special cases $j = 1, r = 0$ and $j = 3, r = 0$, respectively.

On the other hand, since $B_{0,q}(x) = 1$, so by setting $r = 1$ in Theorem 2.1, we immediately obtain the following result.

Corollary 2.3. *Let m, n be nonnegative integers. Then, for q-commuting variables x and y such that $xy = qyx$,*

$$\sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_q x^{m-k} q^{-nk-\binom{n+1}{2}} \frac{B_{n+k+1,q}(y)}{[n+k+1]_q}$$
$$= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-x)^{n-k} q^{-n(m+k+1)+\binom{k+1}{2}} \frac{B_{m+k+1,q}(x+y)}{[m+k+1]_q}$$
$$+ (-1)^{n+1} x^{m+n+1} \frac{[m]_q! \cdot [n]_q!}{[m+n+1]_q!}. \tag{2.8}$$

It becomes obvious that by taking $q \to 1$ and $x + y + z = 1$ in Corollary 2.3, with the help of the symmetric distribution of the classical Bernoulli polynomials $B_n(1-x) = (-1)^n B_n(x)$ for non-negative integer n (see, e.g., [1]), we obtain the result of Sun [28] on the classical Bernoulli polynomials, namely

$$(-1)^m \sum_{k=0}^{m} \binom{m}{k} x^{m-k} \frac{B_{n+k+1}(y)}{n+k+1} + (-1)^n \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \frac{B_{m+k+1}(z)}{m+k+1}$$
$$= (-x)^{m+n+1} \frac{m! \cdot n!}{(m+n+1)!} \quad (m, n \ge 0). \tag{2.9}$$

See also [7, 10] for different proofs of (2.9). It is worth noticing that the formula (2.9) can be used to give the formula of Neuman and Schonbach [25] on the classical Bernoulli numbers. For example, substituting $i-a$ for x and making the summation operation $\sum_{i=1}^{a-1}$ in both sides of (2.5), we get that for non-negative integers a, n,

$$B_n(1-a) = B_n + (-1)^n n \sum_{i=1}^{a-1} (a-i)^{n-1}. \tag{2.10}$$

Hence, by setting $x = a, y = 0, z = 1 - a$ in (2.9), in light of (2.10) and the familiar binomial theorem, we obtain that for non-negative integers m, n, a,

$$(-1)^{n+1}\sum_{k=0}^{m}\binom{m}{k}\frac{B_{n+k+1}}{n+k+1}a^{m-k}+(-1)^{m+1}\sum_{k=0}^{n}\binom{n}{k}\frac{B_{m+k+1}}{m+k+1}a^{n-k}$$
$$=\frac{m!\cdot n!}{(m+n+1)!}a^{m+n+1}-\sum_{i=1}^{a-1}i^n(a-i)^m, \quad (2.11)$$

which was considered by Neuman and Schonbach [25] from the point of view of numerical analysis. For different proofs of (2.11), one may consult [2, 6].

3. The proof of main result

In order to complete the proof of Theorem 2.1, we need the famous q-Chu-Vandermonde summation formula described in [9], as follows.

Lemma 3.1. *Let r be a positive integer, and let ${}_{r+1}\phi_r$ be the basic hypergeometric series defined by*

$${}_{r+1}\phi_r\left[\begin{matrix}a_1, a_2, \ldots, a_{r+1}\\ b_1, b_2, \ldots, b_r\end{matrix}; q, t\right]=\sum_{n=0}^{\infty}\frac{(a_1, a_2, \ldots, a_{r+1}; q)_n}{(q, b_1, b_2, \ldots, b_r; q)_n}t^n \quad (3.1)$$

with $(a_1, \ldots, a_m; q)_n = (a_1; q)_n \cdots (a_m; q)_n$. Then, for any non-negative integer n,

$${}_2\phi_1\left[\begin{matrix}q^{-n}, a_1\\ b_1\end{matrix}; q, q\right]=\frac{a_1^n(b_1/a_1; q)_n}{(b_1; q)_n}. \quad (3.2)$$

We next give the detailed proof of Theorem 2.1.

The proof of Theorem 2.1. We firstly state the addition theorem for q-Bernoulli polynomials, i.e., for q-commuting variables x and y such that $xy = qyx$,

$$B_{n,q}(x+y)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q x^{n-k}B_{k,q}(y) \quad (n \geq 0), \quad (3.3)$$

which is easily derived by applying (1.4), (1.5) and the Cauchy product (see also [24, Theorem 2.2] for a similar proof). Let $k \in [0, n]$ and let λ be a variable associated with k. With the help of (3.3), we discover

$$\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q(-x)^{n-k}q^{\lambda}\frac{B_{m+k+r,q}(x+y)}{[m+k+1]_{q;r}}$$
$$=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{(-x)^{n-k}q^{\lambda}}{[m+k+1]_{q;r}}\sum_{i=0}^{m+k+r}\begin{bmatrix}m+k+r\\i\end{bmatrix}_q x^{m+k+r-i}B_{i,q}(y). \quad (3.4)$$

If we change the order of the summations in the right hand side of (3.4) then the above identity can be rewritten as

$$\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q(-x)^{n-k}q^{\lambda}\frac{B_{m+k+r,q}(x+y)}{[m+k+1]_{q;r}}$$
$$=\sum_{i=0}^{m+n+r}x^{m+n+r-i}B_{i,q}(y)\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}m+k+r\\i\end{bmatrix}_q\frac{(-1)^{n-k}q^{\lambda}}{[m+k+1]_{q;r}}. \quad (3.5)$$

For convenience, in the following we always denote M by

$$\mathrm{M}=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}m+k+r\\i\end{bmatrix}_q\frac{(-1)^{n-k}q^{\lambda}}{[m+k+1]_{q;r}}. \quad (3.6)$$

It is trivial to reduce M. Observe that for any integer r,

$$\frac{1}{[m+k+1]_{q;r}}=(1-q)^r\frac{(q;q)_{m+k}}{(q;q)_{m+k+r}}. \quad (3.7)$$

Hence, by applying $(q;q)_m=(q;q)_n(q^{n+1};q)_{m-n}$ for any integers m,n, we obtain

$$\begin{aligned}\mathrm{M} &= (-1)^n(1-q)^r\sum_{k=0}^{n}\frac{(-1)^k q^{\lambda}(q;q)_n(q;q)_{m+k}}{(q;q)_k(q;q)_{n-k}(q;q)_i(q;q)_{m+k+r-i}}\\ &= \frac{(-1)^n(1-q)^r(q;q)_m}{(q;q)_i(q;q)_{m+r-i}}\sum_{k=0}^{n}\frac{(-1)^k q^{\lambda}(q^{n-k+1};q)_k(q^{m+1};q)_k}{(q;q)_k(q^{m+r-i+1};q)_k}. \quad (3.8)\end{aligned}$$

If we take $a=q^k$ in the following identity (see, e.g., [9, pp. 24]):

$$(aq^{-k-n};q)_k=\frac{(q/a;q)_{n+k}}{(q/a;q)_n}\left(-\frac{a}{q}\right)^k q^{-kn-\binom{k}{2}}, \quad (3.9)$$

then we get

$$(q^{-n};q)_k=\frac{(q^{1-k};q)_{n+k}}{(q^{1-k};q)_n}(-1)^k q^{-kn+\binom{k}{2}}=(q^{n-k+1};q)_k(-1)^k q^{-kn+\binom{k}{2}}. \quad (3.10)$$

It follows from (3.8) and (3.10) that

$$\mathrm{M}=\frac{(-1)^n(1-q)^r(q;q)_m}{(q;q)_i(q;q)_{m+r-i}}\sum_{k=0}^{n}\frac{q^{\lambda+kn-\binom{k}{2}}(q^{-n};q)_k(q^{m+1};q)_k}{(q;q)_k(q^{m+r-i+1};q)_k}. \quad (3.11)$$

To determinate the value of the variable λ, with the help of Lemma 3.1, we choose $\lambda=-kn+\binom{k+1}{2}$ and get

$$\begin{aligned}\mathrm{M} &= \frac{(-1)^n(1-q)^r(q;q)_m}{(q;q)_i(q;q)_{m+r-i}}\cdot\frac{q^{n(m+1)}(q^{r-i};q)_n}{(q^{m+r-i+1};q)_n}\\ &= \frac{(-1)^n(1-q)^r q^{n(m+1)}(q^{r-i};q)_n}{(q;q)_i(q^{m+1};q)_{n+r-i}}. \quad (3.12)\end{aligned}$$

It is easy to see that $(q^{r-i};q)_n = (1-q^{r-i})\cdots(1-q^{n+r-i-1})$, which means $(q^{r-i};q)_n = 0$ when $r \le i \le n+r-1$ and for $n+r \le i \le m+n+r$,

$$\begin{aligned}(q^{r-i};q)_n &= (-1)^n q^{r-i}\cdots q^{r-i+n-1}(1-q^{i-r})\cdots(1-q^{i-r-n+1}) \\ &= (-1)^n q^{n(r-i)+\binom{n}{2}}\frac{(q;q)_{i-r}}{(q;q)_{i-r-n}}. \end{aligned} \tag{3.13}$$

So if we take $\lambda = -kn + \binom{k+1}{2}$, then we get that for integer $r \ge 1$,

$$M = \begin{cases} \frac{(-1)^n(1-q)^r q^{n(m+1)}(q^{r-i};q)_n}{(q;q)_i(q^{m+1};q)_{n+r-i}}, & 0 \le i \le r-1, \\ 0, & r \le i \le n+r-1, \\ (1-q)^r q^{n(m+1+r-i)+\binom{n}{2}}\frac{(q;q)_{i-r}}{(q;q)_i}\begin{bmatrix} m \\ i-r-n\end{bmatrix}_q, & n+r \le i \le m+n+r, \end{cases} \tag{3.14}$$

and for integer $r \le 0$,

$$M = \begin{cases} 0, & 0 \le i \le n+r-1, \\ (1-q)^r q^{n(m+1+r-i)+\binom{n}{2}}\frac{(q;q)_{i-r}}{(q;q)_i}\begin{bmatrix} m \\ i-r-n\end{bmatrix}_q, & n+r \le i \le m+n+r. \end{cases} \tag{3.15}$$

Hence, by applying (3.7), (3.14) and (3.15) to (3.5), we obtain

$$\begin{aligned}\sum_{k=0}^{n}\begin{bmatrix} n \\ k\end{bmatrix}_q (-x)^{n-k} q^{-n(m+k+1)+\binom{k+1}{2}}\frac{B_{m+k+r,q}(x+y)}{[m+k+1]_{q;r}} \\ = (-1)^n(1-q)^r\sum_{i=0}^{r-1} x^{m+n+r-i}B_{i,q}(y)\frac{(q^{r-i};q)_n}{(q;q)_i(q^{m+1};q)_{n+r-i}} \\ + \sum_{k=0}^{m}\begin{bmatrix} m \\ k\end{bmatrix}_q x^{m-k}q^{-nk-\binom{n+1}{2}}\frac{B_{n+k+r,q}(y)}{[n+k+1]_{q;r}}. \end{aligned} \tag{3.16}$$

Since $(q;q)_m = (q;q)_n(q^{n+1};q)_{m-n}$ for any integers m and n then

$$(q^{r-i};q)_n = (q^{r-i};q)_{n+r-1-i-(r-1-i)} = \frac{(q;q)_{n+r-1-i}}{(q;q)_{r-1-i}}, \tag{3.17}$$

and

$$(q^{m+1};q)_{n+r-i} = (q^{m+1};q)_{m+n+r-i-m} = \frac{(q;q)_{m+n+r-i}}{(q;q)_m}. \tag{3.18}$$

It follows from (3.17) and (3.18) that

$$\begin{aligned}\sum_{i=0}^{r-1} x^{m+n+r-i}B_{i,q}(y)\frac{(q^{r-i};q)_n}{(q;q)_i(q^{m+1};q)_{n+r-i}} \\ = x^{m+n+1}\frac{(q;q)_m(q;q)_n}{(q;q)_{m+n+r}}\sum_{i=0}^{r-1}\begin{bmatrix} m+n+r \\ i\end{bmatrix}_q\begin{bmatrix} n+r-1-i \\ n\end{bmatrix}_q \\ \times x^{r-1-i}B_{i,q}(y). \end{aligned} \tag{3.19}$$

Thus, combining (3.16) and (3.19) gives the desired result. This concludes the proof of Theorem 2.1.

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Department of Mathematics and Science, Luoyang Institute of Science and Technology, Luoyang, Henan 471023, People's Republic of China
E-mail address: henanyyh@163.com

Faculty of Science, Kunming University of Science and Technology, Kunming, Yunnan 650500, People's Republic of China
E-mail address: hyyhe@aliyun.com,hyyhe@outlook.com