Regular sparse anti-magic squares with maximum density

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Abstract

Sparse anti-magic squares are useful in constructing vertex-magic labelings for bipartite graphs. An $n \times n$ array based on $\{0, 1, \cdots, nd\}$ is called *a sparse anti-magic square of order n with density d* ($d < n$), denoted by SAMS(n, d), if its row-sums, column-sums and two main diagonal sums constitute a set of $2n + 2$ consecutive integers. A SAMS(n, d) is called *regular* if there are d positive entries in each row, each column and each main diagonal. In this paper, some constructions of regular sparse anti-magic squares are provided and it is shown that there exists a regular SAMS$(n, n-1)$ if and only if $n \geq 4$.

Keywords: Magic square; Anti-magic square; Sparse; Regular; Vertex-magic labeling

1 Introduction

Magic squares and their various generalizations have been objects of interest for many centuries and in many cultures. A lot of work has been done on the constructions of magic squares, for more details, the interested reader may refer to [1–4] and the references therein.

An anti-magic square of order n is an $n \times n$ array with entries consisting of n^2 consecutive nonnegative integers such that the row-sums, column-sums and two main diagonal sums constitute a set of consecutive integers. Usually, the main diagonal from upper left to lower right is called *the left* *diagonal*, the other is called *the right diagonal*. The existence of an anti-magic square has been solved completely by Cormie et al (see [5, 6]). It was shown that there exists an anti-magic square of order n if and only if $n \geq 4$.

Sparse magic squares are a generalization of magic squares. For positive integers n, d $(d < n)$, an $n \times n$ array based on $0, 1, \cdots, nd$ is called *sparse magic square of order n with density d*, denoted by SMS(n, d), if its row-sums, column-sums and two main diagonal sums is the same. A SMS(n, d) is called *regular* if there exist d non-zero elements in each row, each column and each main diagonal. The existence of a regular SMS(n, d) has been solved completely by Li and Su [7]. It was shown that there exists a regular SMS(n, d) if and only if $d \geq 3$ when n is odd and d is even, $d \geq 4$ when n is even.

Sparse anti-magic squares are a generalization of anti-magic squares. For positive integers n, d $(d < n)$, let A be an $n \times n$ array with entries consisting of $0, 1, \cdots, nd$ and let S_A be the set of row-sums, column-sums and two main diagonal sums of A. Then A is called a *sparse anti-magic square of order n with density d*, denoted by SAMS(n, d), if S_A consists of $2n + 2$ consecutive integers. In [8], a SAMS(n, d) is also called a *sparse totally anti-magic square*. A SAMS(n, d) is called *regular* if all of its rows, columns and two main diagonal contain d positive entries. As an example, a regular SAMS$(4, 3)$ is listed below.

$$A = \begin{pmatrix} 0 & 1 & 3 & 12 \\ 4 & 11 & 0 & 2 \\ 9 & 6 & 8 & 0 \\ 7 & 0 & 10 & 5 \end{pmatrix}.$$

It is readily checked that all elements of A consists of $\{0, 1, 2, \cdots, 12\}$, $S_A = \{16, 17, \cdots, 24, 25\}$ and all of its rows, columns and two main diagonal contain 3 positive entries.

Sparse anti-magic squares are useful in graph theory. For example, they can be used to construct a vertex-magic total labeling for bipartite graphs, see [8] and the references therein.

In this paper, we investigate the existence of a regular sparse anti-magic square with maximum density (when $d = n - 1$). It is not difficult to see that there is no SAMS$(n, n - 1)$ for all $n = 1, 2, 3$. So to consider the existence of a regular SAMS$(n, n - 1)$, we need only to consider the case of $n \geq 4$. We shall prove the following.

Theorem 1.1. *There exists a regular SAMS$(n, n - 1)$ if and only if $n \geq 4$.*

Some constructions of sparse anti-magic squares are given in Section 2. The existence of a regular SAMS$(n, n-1)$ with n odd and even is considered in Section 3 and Section 4, respectively.

2 Constructions of sparse anti-magic squares

In this section, we shall provide two constructions of sparse anti-magic squares based on quasi or pseudo sparse anti-magic squares.

Let a, b be integers and let $[a, b]$ be the set of integers v such that $a \leq v \leq b$. Let A be an array based on Z and let $G(A)$ be the set of non-zero elements of A.

For positive integers n and d even ($d < n$), an $n \times n$ array A is called *uniform regular sparse array of order n with density d*, denoted by URSA(n, d), if $G(A) = [-nd/2, -1] \cup [1, nd/2]$, there are $d/2$ positive entries and $d/2$ negative entries in each row, each column and each main diagonals.

A URSA(n, d), A, is called a *quasi sparse anti-magic square*, denoted by QSAMS(n, d), if $S_A = [-n, n+1]$, where the sum of all elements in the left diagonal and the right diagonal is $n+1$ and 0, respectively.

A quasi sparse anti-magic square can be used to construct a regular sparse anti-magic square. We have the following.

Construction 2.1. *Let n, d be positive integer and d even. If there exists a QSAMS(n, d), then there exists a regular SAMS(n, d).*

Proof. Let $A = (a_{i,j})$ be a QSAMS(n, d) and $p = nd/2$. Denote $B = (b_{i,j})$, where

$$b_{i,j} = \begin{cases} a_{i,j} + p, & \text{if } a_{i,j} > 0, \\ 0, & \text{if } a_{i,j} = 0, \\ a_{i,j} + p + 1, & \text{if } a_{i,j} < 0. \end{cases}$$

Since $G(A) = [-nd/2, -1] \cup [1, nd/2]$, we have $G(B) = [1, nd]$. On the other hand, there are $d/2$ positive entries and $d/2$ negative entries in each row, column and main diagonal of A. So there are d positive integers in each row, column and main diagonal of B. Note that $S_A = [-n, n+1]$, from which it follows $S_B = [dp - n + d/2, dp + n + d/2 + 1]$. Thus, B is the desired regular SAMS(n, d). □

Let n, d be both even and $B = (b_{i,j})$, $0 \leq i, j \leq n-1$, be a URSA(n, d), then B is called a *pseudo sparse anti-magic square*, denoted by PSAMS(n, d), if the following properties hold:

(i) Row-sums are all 0, column-sums are $n/2$ or $-n/2$, two main diagonal sums are both $n/2 + 2$.

(ii) There exists a n-set $H = \{j_0, j_1, \cdots, j_{n-1}\}$, such that $b_{i,j_i} = 0$, $0 \leq i \leq n-1$; There exists exactly one i such that $i = j_i$, moreover, $\sum_{0 \leq s \leq n-1} b_{s,j_i} > 0$; There exists exactly one i' such that $i' + j_{i'} = n-1$, moreover, $\sum_{0 \leq s \leq n-1} b_{s,j_{i'}} > 0$.

A pseudo sparse anti-magic square can be used to construct a regular sparse anti-magic square of even order. We have the following.

Construction 2.2. *Let n, d be both even. If there exists a PSAMS(n,d), then there exists a regular SAMS$(n, d+1)$.*

Proof. Suppose that $B = (b_{i,j})$, $0 \le i, j \le n-1$, is a PSAMS(n,d) with the properties (i) and (ii) mentioned above. We can write $H = H_0 \cup H_1 \cup H_2$, where

$$H_0 = \{j_i | i = j_i \text{ or } i = n-1-j_i\},$$
$$H_1 = \{j_i | \sum_{0 \le s \le n-1} b_{s,j_i} > 0\} \backslash H_0 = \{s_k | k = 0, 1, \cdots, n/2-3\},$$
$$H_2 = \{j_i | \sum_{0 \le s \le n-1} b_{s,j_i} < 0\} = \{t_k | k = 0, 1, \cdots, n/2-1\}.$$

Let $p = n(d+1)/2$, $u = p + n/2$, $v = p - n/2 + 1$. Denote $B' = (b'_{i,j})$, where

$$b'_{i,j_i} = \begin{cases} u, & \text{if } j_i = i, \\ u-1, & \text{if } j_i = n-1-i, \\ u-2-k, & \text{if } j_i = s_k \in H_1, \\ v+k, & \text{if } j_i = t_k \in H_2, \end{cases}$$

when $j \ne j_i$,

$$b'_{i,j} = \begin{cases} b_{i,j} + u, & \text{if } b_{i,j} > 0, \\ 0, & \text{if } b_{i,j} = 0, \\ b_{i,j} + v, & \text{if } b_{i,j} < 0. \end{cases}$$

It can be shown that B' is a regular SAMS$(n, d+1)$. In fact, noting that B is a PSAMS(n,d), so there are $d+1$ non-zero entries in each row, column and two main diagonals of B'. The set of non-zero elements of B' is that

$$\begin{aligned} G(B') &= [u+1, u+nd/2] \cup [v-nd/2, v-1] \cup [u-n/2+1, u] \cup [v, v+n/2-1] \\ &= [u-n/2+1, u+nd/2] \cup [v-nd/2, v+n/2-1] \\ &= [p+1, 2p] \cup [1, p] \\ &= [1, n(d+1)]. \end{aligned}$$

Now we consider the sum set of B'. Let s_{h_i}, s_{l_i}, s_{d_1} and s_{d_2} be the i-th row sum, the i-th column sum, the left diagonal sum and the right diagonal sum. Then we have

$$\bigcup_{i=0}^{n-1}\{s_{h_i}\} = [(u+v)d/2+v, (u+v)d/2+u],$$

$$\bigcup_{i=0}^{n-1}\{s_{l_i}\} = [(u+v)d/2-n/2+v, (u+v)d/2+v-1] \cup [(u+v)d/2+u+1, (u+v)d/2+n/2+u],$$

$$\bigcup_{i=1}^{2}\{s_{d_i}\} = \{(u+v)d/2+n/2+u+1, (u+v)d/2+n/2+u+2\}.$$

Thus,

$$\begin{aligned} S_{B'} &= \bigcup_{i=0}^{n-1}\{s_{h_i}\} \cup \bigcup_{i=0}^{n-1}\{s_{l_i}\} \cup \bigcup_{i=1}^{2}\{s_{d_i}\} \\ &= [(u+v)d/2-n/2+v, (u+v)d/2+n/2+u+2] \\ &= [(d+1)p+d/2-n+1, (d+1)p+d/2+n+2]. \end{aligned}$$

So, B' is the desired regular SAMS$(n, d+1)$. □

To illustrate Construction 2.2, we give an example in the following.

Example 1. There exists a regular SAMS$(6,5)$.

Proof. Let

$$B = \begin{pmatrix} 0 & -3 & -10 & 8 & 5 & 0 \\ -9 & -6 & 0 & 0 & 12 & 3 \\ 9 & 0 & 6 & -7 & -8 & 0 \\ 0 & 2 & -4 & 7 & 0 & -5 \\ -1 & 0 & 11 & -11 & 0 & 1 \\ 4 & 10 & 0 & 0 & -12 & -2 \end{pmatrix}.$$

It is readily checked that B is a PSAMS$(6,4)$ having the properties (i) and (ii) mentioned above, here, $H = \{0,2,5,4,1,3\}$. We write

$$H = H_0 \cup H_1 \cup H_2,$$

where

$$H_0 = \{0,1\}, \quad H_1 = \{2\}, \quad H_2 = \{5,4,3\} = \{t_k | k = 0,1,2\}.$$

For $n = 6$ and $d = 4$, let $p = n(d+1)/2 = 15$, $u = p + n/2 = 18$, $v = p - n/2 + 1 = 13$.

Denote $B' = (b'_{i,j})$, where

$$b'_{i,j_i} = \begin{cases} 18, & \text{if } j_i = i, \\ 17, & \text{if } j_i = 5 - i, \\ 16, & \text{if } j_i \in H_1, \\ 13 + k, & \text{if } j_i = t_k \in H_2, \end{cases}$$

when $j \neq j_i$,

$$b'_{i,j} = \begin{cases} b_{i,j} + 18, & \text{if } b_{i,j} > 0, \\ 0, & \text{if } b_{i,j} = 0, \\ b_{i,j} + 13, & \text{if } b_{i,j} < 0. \end{cases}$$

Then we get

$$B' = \begin{pmatrix} 18 & 10 & 3 & 26 & 23 & 0 \\ 4 & 7 & 16 & 0 & 30 & 21 \\ 27 & 0 & 24 & 6 & 5 & 13 \\ 0 & 20 & 9 & 25 & 14 & 8 \\ 12 & 17 & 29 & 2 & 0 & 19 \\ 22 & 28 & 0 & 15 & 1 & 11 \end{pmatrix}.$$

It is readily checked that all elements of B' consists of $[0, 30]$, $S_{B'} = [72, 85]$ and all of its rows, columns and two main diagonal contain 5 positive entries. So, B' is a regular SAMS$(6, 5)$. □

The following is straight-forward but is useful in our recursive construction for regular sparse anti-magic squares.

Lemma 2.3. *If there exist arrays $A_k = (a_{i,j}^{(k)})$, $k = 1, 2, \cdots, m$, with the following properties:*
(i) $G(A_k) = [-x_k, -1] \cup [1, x_k]$, $k = 1, 2, \cdots, m$.
(ii) *For each k, $1 \le k \le m$, the number of positive integers is the same as that of negative integers in each row, each column and each main diagonal of A_k.*
Then there exist arrays $B_1, B_2, \cdots, B_m$ such that
(a) $\bigcup_{k=1}^{m} G(B_k) = [-\sum_{k=1}^{m} x_k, -1] \cup [1, \sum_{k=1}^{m} x_k]$.
(b) *For each k, $1 \le k \le m$, the number of positive integers is the same as that of negative integers in each row, each column and each main diagonal of B_k, the sums of all elements in the corresponding rows (columns or main diagonals) of B_k and A_k are the same.*

Proof. Let $c_1 = 0$, $c_2 = x_1$, $c_k = \sum_{s=1}^{k-1} x_s$, $k = 3, \cdots, m$. For each $k \in$

$\{1,2,\cdots,m\}$, denote $B_k = (b_{i,j}^{(k)})$, where

$$b_{i,j}^{(k)} = \begin{cases} a_{i,j}^{(k)} + c_k, & \text{if } a_{i,j}^{(k)} > 0, \\ 0, & \text{if } a_{i,j}^{(k)} = 0, \\ a_{i,j}^{(k)} - c_k, & \text{if } a_{i,j}^{(k)} < 0. \end{cases}$$

Then it is readily checked that $B_1, B_2, \cdots, B_m$ are the desired arrays. □

3 Regular SAMS$(n, n-1)$ with n odd

In this section, we shall prove that there exists a regular SAMS$(n, n-1)$ for all odd $n \geq 5$. By Construction 2.1, it suffices to show that there exists a QSAMS$(n, n-1)$ for all odd $n \geq 5$. We start with some direct constructions of several small value n.

Lemma 3.1. *There exists a QSAMS$(n, n-1)$ for all $n \in \{5, 7, 9, 11\}$.*

Proof. For $n = 5$, let

$$B = \begin{pmatrix} 6 & -10 & 0 & -2 & 5 \\ -1 & 10 & -9 & 3 & 0 \\ -8 & 0 & -3 & 9 & 4 \\ 8 & -5 & 2 & 0 & -4 \\ 0 & 1 & 7 & -6 & -7 \end{pmatrix}.$$

It is readily checked that B is the desired QSAMS$(5, 4)$.

For $n = 7$, let

$$B = \begin{pmatrix} 4 & -20 & -9 & 8 & 14 & 0 & -4 \\ 16 & -2 & 2 & -17 & -12 & 9 & 0 \\ 0 & 18 & 17 & -5 & 5 & -15 & -14 \\ -21 & -8 & 0 & 20 & 12 & -7 & 7 \\ -1 & 1 & -19 & -10 & 0 & 21 & 10 \\ 11 & 0 & -3 & 3 & -16 & -13 & 13 \\ -11 & 15 & 19 & 0 & -6 & 6 & -18 \end{pmatrix}.$$

It is readily checked that B is the desired QSAMS$(7, 6)$.

For $n = 9$, let

$$B = \begin{pmatrix} 14 & -33 & -25 & -11 & 19 & 0 & 27 & -9 & 9 \\ -1 & 2 & 13 & -34 & -23 & -12 & 29 & 35 & 0 \\ 36 & 0 & -2 & 4 & 12 & -35 & -21 & -13 & 22 \\ -14 & 30 & 0 & 32 & -3 & 5 & 11 & -36 & -18 \\ -28 & -26 & -15 & 31 & 0 & 21 & -4 & 8 & 10 \\ 1 & 18 & -29 & -24 & -16 & 33 & 0 & 26 & -5 \\ 24 & -6 & 3 & 17 & -30 & -22 & -17 & 0 & 25 \\ 0 & 20 & 28 & -7 & 6 & 16 & -31 & -20 & -19 \\ -27 & -10 & 23 & 0 & 34 & -8 & 7 & 15 & -32 \end{pmatrix}.$$

It is readily checked that B is the desired QSAMS(9, 8).

For $n = 11$, let B be

$$\begin{pmatrix} 28 & -54 & -35 & -26 & -19 & 34 & 54 & 0 & -2 & 2 & 17 \\ 6 & 21 & 24 & -52 & -37 & -29 & -16 & 55 & 0 & 41 & -6 \\ 42 & -10 & 10 & 19 & 26 & -49 & -40 & -31 & -14 & 0 & 39 \\ 44 & 36 & 0 & -8 & 8 & 16 & 29 & -47 & -42 & -33 & -12 \\ -23 & -22 & 38 & 0 & 53 & -5 & 5 & 14 & 31 & -45 & -44 \\ -55 & -34 & -25 & -20 & 0 & 48 & 45 & -3 & 3 & 12 & 33 \\ 22 & 23 & -53 & -36 & -27 & -18 & 0 & 40 & 52 & -1 & 1 \\ -11 & 11 & 20 & 25 & -51 & -38 & -30 & -15 & 46 & 37 & 0 \\ 0 & 43 & -9 & 9 & 18 & 27 & -48 & -41 & -32 & -13 & 51 \\ -17 & 0 & 49 & 35 & -7 & 7 & 15 & 30 & -46 & -43 & -28 \\ -39 & -24 & -21 & 50 & 47 & 0 & -4 & 4 & 13 & 32 & -50 \end{pmatrix}.$$

It is readily checked that B is the desired QSAMS(11, 10). □

We shall take advantage of the quasi sparse anti-magic squares given in Lemma 3.1 to construct a QSAMS$(n, n-1)$ for all odd $n \geq 13$. To do this, some arrays with special properties are needed.

Lemma 3.2. *There exists a URSA(9, 8), $A = (a_{ij})$, $0 \leq i, j \leq 8$, having the property that $a_{4,4} = 0$ and $\sum_{0 \leq i \leq 8} a_{i,4} = \sum_{0 \leq j \leq 8} a_{4,j} = 0$, the set of remaining row-sums, column-sums is $[-9, -2] \cup [2, 9]$ and there are four positive integers in the set of row-sums and the set of column-sums, respectively, the left diagonal sum is 8, the right diagonal sum is 0.*

Proof. Let

$$A = \begin{pmatrix} -1 & -8 & 0 & 6 & 10 & 17 & 12 & -18 & -11 \\ 0 & 5 & 3 & -2 & -10 & -17 & -12 & 14 & 11 \\ 1 & 4 & -9 & 0 & -7 & -13 & -16 & 18 & 15 \\ -3 & 0 & -5 & 9 & 7 & 13 & 16 & -14 & -15 \\ 8 & -6 & 2 & -4 & 0 & 36 & -35 & -36 & 35 \\ -24 & 23 & -19 & 20 & 34 & -31 & 0 & 31 & -32 \\ -26 & 25 & -21 & 22 & 33 & -30 & 32 & -29 & 0 \\ 26 & -23 & 19 & -20 & -33 & 0 & -28 & 30 & 27 \\ 24 & -25 & 21 & -22 & -34 & 29 & 28 & 0 & -27 \end{pmatrix}.$$

It is readily checked that A is the desired array. □

For even m and even n, an $m \times n$ array T is called *near-uniform* if $G(T) = [-mn/2, -1] \cup [1, mn/2]$, there are $n/2$ positive entries and $n/2$ negative entries in each row, there are $m/2$ positive entries and $m/2$ negative entries in each column.

Lemma 3.3. *For all positive integer $t \geq 2$, there exists an $8 \times 2t$ near-uniform array with the property that column-sums are all 0 and row-sums are $2t$ or $-2t$.*

Proof. (i) For $t = 2s$, $s \geq 1$, let

$$C_i = \begin{pmatrix} -1-16i & -2-16i & 3+16i & 4+16i \\ 1+16i & 2+16i & -3-16i & -4-16i \\ 5+16i & 6+16i & -7-16i & -8-16i \\ -5-16i & -6-16i & 7+16i & 8+16i \\ -9-16i & -10-16i & 11+16i & 12+16i \\ -13-16i & -14-16i & 15+16i & 16+16i \\ 9+16i & 10+16i & -11-16i & -12-16i \\ 13+16i & 14+16i & -15-16i & -16-16i \end{pmatrix},$$

$0 \leq i \leq s-1$. Then $C = (C_0, C_1, \cdots, C_{s-1})$ is the required array.

(ii) For $t = 2s+1$, let

$$C'_0 = \begin{pmatrix} -1 & 2 & -3 & -4 & 5 & 7 \\ 1 & -2 & 3 & 4 & -5 & -7 \\ 6 & 8 & -9 & -10 & 11 & -12 \\ -6 & -8 & 9 & 10 & -11 & 12 \\ -13 & 14 & -15 & -16 & 17 & 19 \\ -18 & -20 & 21 & 22 & -23 & 24 \\ 13 & -14 & 15 & 16 & -17 & -19 \\ 18 & 20 & -21 & -22 & 23 & -24 \end{pmatrix},$$

$$C'_j = \begin{pmatrix} -9-16j & -10-16j & 11+16j & 12+16j \\ 9+16j & 10+16j & -11-16j & -12-16j \\ 13+16j & 14+16j & -15-16j & -16-16j \\ -13-16j & -14-16j & 15+16j & 16+16j \\ -17-16j & -18-16j & 19+16j & 20+16j \\ -21-16j & -22-16j & 23+16j & 24+16j \\ 17+16j & 18+16j & -19-16j & -20-16j \\ 21+16j & 22+16j & -23-16j & -24-16j \end{pmatrix},$$

$1 \leq j \leq s-1$. When $s = 1$, C'_0 is the required 8×6 array. When $s \geq 2$, $C = (C'_0 \ C'_1 \ \cdots \ C_{s-1})$ is the required $8 \times 2t$ array. □

Lemma 3.4. *There exists a QSAMS$(n, n-1)$ for all odd $n \geq 5$.*

Proof. For each odd $n \geq 5$, we can write $n = 8k+w$, where $w \in \{5, 7, 9, 11\}$, $k \geq 0$.

When $k = 0$, $n = w$, the desired QSAMS$(n, n-1)$ is given by Lemma 3.1.

Suppose that $k \geq 0$ and B is a QSAMS$(n, n-1)$, where $n = 8k + w$. We shall show that there exists a QSAMS$(n+8, n+7)$.

Let A be a URSA$(9, 8)$ having the property mentioned in Lemma 3.2. Let C be an $8 \times (n-1)$ near-uniform array having the property mentioned in Lemma 3.3. Let D be the transpose of C. If necessary, we can perform row permutations to C and independently perform column permutations to D so that the signs of corresponding row (column) sums match, i.e.,

$$\sum_{0 \leq j \leq 8} a_{i,j} \sum_{0 \leq j \leq n-2} c_{i,j} > 0, \ 0 \leq i \leq 3,$$

$$\sum_{0\le j\le 8} a_{i+1,j} \sum_{0\le j\le n-2} c_{i,j} > 0,\ 4 \le i \le 7,$$

$$\sum_{0\le i\le 8} a_{i,j} \sum_{0\le i\le n-2} d_{i,j} > 0,\ 0 \le j \le 3,$$

and

$$\sum_{0\le i\le 8} a_{i,j+1} \sum_{0\le i\le n-2} d_{i,j} > 0,\ 4 \le j \le 7.$$

Clearly,

$$G(A) = [-36, -1] \cup [1, 36],$$
$$G(B) = [-n(n-1)/2, -1] \cup [1, n(n-1)/2],$$
$$G(C) = G(D) = [-4(n-1), -1] \cup [1, 4(n-1)]$$

and the number of positive integers is the same as that of negative integers in each row, column of A, B, C and D, respectively. By Lemma 2.3, there exist four arrays $A' = (a'_{i,j})$, $B' = (b'_{i,j})$, $C' = (c'_{i,j})$ and $D' = (d'_{i,j})$ such that

$$\begin{aligned} &G(A') \cup G(B') \cup G(C') \cup G(D') \\ &= [-n(n-1)/2 - 8(n-1) - 36, -1] \cup [1, n(n-1)/2 + 8(n-1) + 36] \\ &= [-(n+8)(n+7)/2, -1] \cup [1, (n+8)(n+7)/2]. \end{aligned}$$

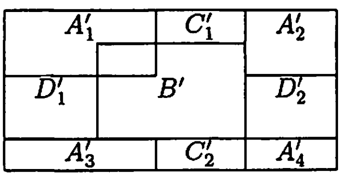
and the number of positive integers is the same as that of negative integers in each row, column of A', B', C' and D', respectively. Meanwhile, the number of positive integers is the same as that of negative integers in each main diagonal of A' and B', and the central entry $a'_{4,4} = 0$.

We write A', C' and D' in the following forms:

$$A' = \begin{pmatrix} A'_1 & A'_2 \\ A'_3 & A'_4 \end{pmatrix}, \quad C' = \begin{pmatrix} C'_1 \\ C'_2 \end{pmatrix}, \quad D' = \begin{pmatrix} D'_1 & D'_2 \end{pmatrix},$$

where A'_1, A'_2, A'_3, A'_4 are $5\times 5, 5\times 4, 4\times 5, 4\times 4$ subarrays of A', C'_1 and C'_2 are $4\times(n-1)$ subarrays of C', D'_1 and D'_2 are $(n-1)\times 4$ subarrays of D'.

Construct an $(n+8)\times(n+8)$ array $E = (e_{i,j})$ as follows.

$E =$ 

where $e_{4,4} = b'_{0,0}$. It is not difficult to check that E is the desired QSAMS$(n+8, n+7)$.

In fact, $G(E) = G(A') \cup G(B') \cup G(C') \cup G(D') = [-(n+8)(n+7)/2, (n+8)(n+7)/2]$.

Let s_{h_i}, s_{l_i}, s_{d_1} and s_{d_2} be the i-th row sum, the i-th column sum, the left diagonal sum and the right diagonal sum of E, respectively. We have

$$\begin{aligned}\bigcup_{i=0}^{3}\{s_{h_i}, s_{l_i}\} \cup \bigcup_{j=n+4}^{n+7}\{s_{h_j}, s_{l_j}\} &= [-9-n+1, -2-n+1] \cup [2+n-1, 9 \\ &\quad +n-1] \\ &= [-n-8, -n-1] \cup [n+1, n+8], \\ \bigcup_{i=4}^{n+3}\{s_{h_i}, s_{l_i}\} &= [-n, -1] \cup [1, n].\end{aligned}$$

Noting that the left diagonal sums of A and B are 8 and $n+1$, respectively, the right diagonal sums of A and B are both 0, we have $s_{d_1} = (n+1) + 8 = n+9$, $s_{d_2} = 0 + 0 = 0$. Then

$$S_E = \bigcup_{i=0}^{n+7}\{s_{h_i}, s_{l_i}\} \cup \{s_{d_1}, s_{d_2}\} = [-n-8, n+9]$$

and there are $8n+7$ integers in each row, each column and each main diagonal, where the left diagonal sum is $n+9$, the right diagonal sum is 0. Thus, E is the desired QSAMS$(n+8, n+7)$. □

Theorem 3.5. *There exists a regular SAMS$(n, n-1)$ for all odd $n \geq 5$.*

Proof. Combining Lemma 3.4 and Construction 2.1 gives the proof. □

4 Regular SAMS$(n, n-1)$ with n even

In this section, we shall prove that for all even $n \geq 4$, there exists a regular SAMS$(n, n-1)$. For $n = 4$, the desired regular SAMS$(4, 3)$ is given in Section 1. For even $n \geq 6$, to construct a regular SAMS$(n, n-1)$, by Construction 2.2, it suffices to show that there exists a PSAMS$(n, n-2)$. We start with some direct constructions of several small value n.

Lemma 4.1. *There exists a PSAMS$(n, n-2)$ for all $n \in \{6, 8, 10, 12\}$.*

Proof. For $n = 6$, the desired PSAMS$(6, 4)$ is given in Example 1.

For $n = 8$, let

$$B=\begin{pmatrix} 12 & 0 & 20 & 23 & -23 & -20 & 0 & -12 \\ 0 & -11 & -24 & -18 & 18 & 24 & 11 & 0 \\ 0 & 15 & -10 & -13 & 13 & 10 & -15 & 0 \\ -16 & 0 & 14 & 9 & -9 & -14 & 0 & 16 \\ 3 & -4 & -21 & 0 & 0 & 21 & -3 & 4 \\ 6 & -5 & 0 & -19 & 19 & 0 & -6 & 5 \\ -7 & 8 & 17 & 0 & 0 & -17 & 7 & -8 \\ -2 & 1 & 0 & 22 & -22 & 0 & 2 & -1 \end{pmatrix}.$$

It is readily checked that B is a PSAMS$(8,6)$. Here, $H=\{1,0,7,6,3,5,4,2\}$.

For $n=10$, let

$$B=\begin{pmatrix} 0 & 10 & 3 & -26 & 20 & -20 & 26 & -12 & -1 & 0 \\ 6 & 11 & 0 & 18 & 21 & -21 & -19 & 0 & -11 & -5 \\ -7 & 0 & -6 & -18 & 27 & -28 & 19 & 12 & 1 & 0 \\ -29 & -33 & 37 & 13 & 0 & 0 & -13 & -37 & 33 & 29 \\ 30 & 34 & -38 & 0 & -14 & 14 & 0 & 38 & -34 & -30 \\ -31 & -35 & 39 & 0 & 15 & -15 & 0 & -39 & 35 & 31 \\ 32 & 36 & -40 & -16 & 0 & 0 & 16 & 40 & -36 & -32 \\ 0 & -8 & 9 & 17 & -22 & 28 & -24 & -2 & 0 & 2 \\ 7 & 0 & -9 & 24 & -25 & 25 & -23 & 5 & 0 & -4 \\ -3 & -10 & 0 & -17 & -27 & 22 & 23 & 0 & 8 & 4 \end{pmatrix}.$$

It is readily checked that B is a PSAMS$(10,8)$. Here, $H=\{0,2,9,4,3,6,5,8,1,7\}$.

For $n=12$, let B be

$$\begin{pmatrix} 18 & 0 & 57 & -60 & -46 & 36 & -36 & 46 & 60 & -57 & 0 & -18 \\ 0 & -17 & -51 & 54 & 40 & -30 & 30 & -40 & -54 & 51 & 17 & 0 \\ 0 & 23 & -16 & 59 & 45 & -35 & 35 & -45 & -59 & 16 & -23 & 0 \\ -24 & 0 & -56 & 15 & -39 & 29 & -29 & 39 & -15 & 56 & 0 & 24 \\ 25 & 48 & 0 & -53 & 14 & 34 & -34 & -14 & 53 & 0 & -48 & -25 \\ -31 & -42 & 50 & 0 & 44 & -13 & 13 & -44 & 0 & -50 & 42 & 31 \\ -26 & -47 & -55 & -58 & -38 & 0 & 0 & 38 & 58 & 55 & 47 & 26 \\ 32 & 41 & 49 & 52 & 0 & 19 & -19 & 0 & -52 & -49 & -41 & -32 \\ 3 & -4 & 22 & 12 & -20 & 0 & 0 & 20 & -12 & -22 & -3 & 4 \\ 8 & -7 & -11 & -21 & 0 & 27 & -27 & 0 & 21 & 11 & -8 & 7 \\ -9 & 10 & 5 & 0 & -43 & -33 & 33 & 43 & 0 & -5 & 9 & -10 \\ -2 & 1 & 0 & -6 & 37 & -28 & 28 & -37 & 6 & 0 & 2 & -1 \end{pmatrix}.$$

It is readily checked that B is a PSAMS$(12,10)$. Here, $H=\{1,0,11,10,2,3,5,7,6,4,8,9\}$.

□

We shall show that there exists a PSAMS$(n,n-2)$ for all even $n\geq 14$ by means of some special arrays.

Lemma 4.2. *There exists an* 8×8 *array* $A=(a_{i,j})$ *having the following properties:*
(i) $G(A)=[-24,-1]\cup[1,24]$.
(ii) *There are* 3 *positive entries and* 3 *negative entries in each row, column and there are* 4 *positive entries and* 4 *negative entries in each main diagonal.*

(iii) *Row-sums are all* 0, *column-sums are* 4 *or* -4, *two main diagonal sums are both* 4.

(iv) *There exists a set* $H = \{j_0, j_1, \cdots, j_7\}$, *such that* $a_{i,j_i} = 0$, $0 \le i \le 7$.

Proof. Let

$$A = \begin{pmatrix} -1 & 0 & 1 & 13 & -13 & -9 & 0 & 9 \\ 0 & 2 & -10 & -14 & 14 & 10 & -2 & 0 \\ 17 & 21 & -3 & 0 & 0 & 3 & -21 & -17 \\ -18 & -22 & 0 & 4 & -4 & 0 & 22 & 18 \\ -19 & -23 & 0 & -8 & 8 & 0 & 23 & 19 \\ 20 & 24 & 7 & 0 & 0 & -7 & -24 & -20 \\ 0 & -6 & -11 & -15 & 15 & 11 & 6 & 0 \\ 5 & 0 & 12 & 16 & -16 & -12 & 0 & -5 \end{pmatrix}.$$

It is readily checked that A is the desired array. Here, $H = \{1, 0, 3, 2, 5, 4, 7, 6\}$. □

Lemma 4.3. *For all positive integer* $t \ge 2$, *there exists an* $8 \times 2t$ *near-uniform array having the property that row-sums are all* 0 *and column-sums are* 4 *or* -4.

Proof. Let

$$C = \begin{pmatrix} -1 & 1 & -1-8 & 1+8 & \cdots & -1-8y & 1+8y \\ 2 & -2 & 2+8 & -2-8 & \cdots & 2+8y & -2-8y \\ -3 & 3 & -3-8 & 3+8 & \cdots & -3-8y & 3+8y \\ 4 & -4 & 4+8 & -4-8 & \cdots & 4+8y & -4-8y \\ -5 & 5 & -5-8 & 5+8 & \cdots & -5-8y & 5+8y \\ 6 & -6 & 6+8 & -6-8 & \cdots & 6+8y & -6-8y \\ -7 & 7 & -7-8 & 7+8 & \cdots & -7-8y & 7+8y \\ 8 & -8 & 8+8 & -8-8 & \cdots & 8+8y & -8-8y \end{pmatrix},$$

where $y = t - 1$. Then C is the desired array. □

Lemma 4.4. *For all positive integer* $t \ge 2$, *there exists a* $2t \times 8$ *near-uniform array having the property that row-sums are all* 0 *and column-sums are* t *or* $-t$.

Proof. Let D be

$$\begin{pmatrix} -1 & 1 & -1-2t & 1+2t & 1+4t & -1-4t & 1+6t & -1-6t \\ 2 & -2 & 2+2t & -2-2t & -2-4t & 2+4t & -2-6t & 2+6t \\ -3 & 3 & -3-2t & 3+2t & 3+4t & -3-4t & 3+6t & -3-6t \\ 4 & -4 & 4+2t & -4-2t & -4-4t & 4+4t & -4-6t & 4+6t \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -2t+1 & 2t-1 & -4t+1 & 4t-1 & 6t-1 & -6t+1 & 8t-1 & -8t+1 \\ 2t & -2t & 4t & -4t & -6t & 6t & -8t & 8t \end{pmatrix}.$$

It is readily checked that D is the desired array. □

Lemma 4.5. *There exists a PSAMS$(n, n-2)$ for all even $n \geq 6$.*

Proof. For each even $n \geq 6$, we can write $n = 8k + w$, where $w \in \{6, 8, 10, 12\}$, $k \geq 0$.

When $k = 0$, $n = w$, the desired PSAMS$(n, n-2)$ is given by Lemma 4.1.

When $k \geq 1$, let $A_m = A = (a_{i,j})$, $1 \leq m \leq k$, where A is the same as in Lemma 4.2. Let $B = (b_{i,j})$ be a PSAMS$(w, w-2)$ coming from Lemma 4.1. For $1 \leq m \leq k$, let $t_m = w/2 + 4(m-1)$ and $C_m = (c_{i,j}^{(m)})$ be an $8 \times 2t_m$ near-uniform array with the property mentioned in Lemma 4.3. If necessary, we can do some column permutations to C_m so that
when $m = 1$,

$$\sum_{0 \leq i \leq w-1} b_{i,j} \sum_{0 \leq i \leq 7} c_{i,j}^{(1)} > 0, \ 0 \leq j \leq w-1,$$

when $m \geq 2$,

$$\sum_{0 \leq i \leq 7} a_{i,j} \sum_{0 \leq i \leq 7} c_{i,j+4u}^{(m)} > 0, \ 0 \leq j \leq 3 \ \textit{and} \ 0 \leq u \leq m-2.$$

$$\sum_{0 \leq i \leq w-1} b_{i,j} \sum_{0 \leq i \leq 7} c_{i,j+4(m-1)}^{(m)} > 0, \ 0 \leq j \leq w-1,$$

$$\sum_{0 \leq i \leq 7} a_{i,j} \sum_{0 \leq i \leq 7} c_{i,j-4+w+4u}^{(m)} > 0, \ 4 \leq j \leq 7 \ \textit{and} \ m-1 \leq u \leq 2m-3,$$

Let $D_m = (d_{i,j}^{(m)})$ be a $2t_m \times 8$ near-uniform array with the property mentioned in Lemma 4.4. If necessary, we can do some column permutations to D_m so that

$$\sum_{0 \leq i \leq 7} a_{i,j} \sum_{0 \leq i \leq 2t_m - 1} d_{i,j}^{(m)} > 0, \ 0 \leq j \leq 7.$$

Clearly,

$G(A_m) = [-24, -1] \cup [1, 24]$, $G(B) = [-w(w-2)/2, -1] \cup [1, w(w-2)/2]$,

$G(C_m) = G(D_m) = [-8t_m, -1] \cup [1, 8t_m] = [-(4w + 32(m-1)), -1] \cup [1, 4w + 32(m-1)]$.

By Lemma 2.3, there exist arrays A'_m, B', C'_m and D'_m, $1 \leq m \leq k$, such that

$$
\begin{aligned}
&(\bigcup_{m=1}^{k} G(A'_m)) \cup G(B') \cup (\bigcup_{m=1}^{k} G(C'_m)) \cup (\bigcup_{m=1}^{k} G(D'_m)) \\
&= [-24k - w(w-2)/2 - 2(16k^2 - 16k + 4kw), -1] \cup [1, 24k + w(w-2)/2 \\
&\quad + 2(16k^2 - 16k + 4kw)] \\
&= [-(8k+w)(8k+w-2)/2, -1] \cup [1, (8k+w)(8k+w-2)/2] \\
&= [-n(n-2)/2, -1] \cup [1, n(n-2)/2],
\end{aligned}
$$

the number of positive integers is same as that of negative integers in each row, each column and each main diagonal of A'_m and B', respectively, the number of positive integers is same as that of negative integers in each row, each column of C'_m and D'_m, respectively. The sums of all elements in the same row (column) of A_m and A'_m, B and B', C_m and C'_m, D_m and D'_m are the same, respectively.

Let $A'_m = (a'^{(m)}_{i,j})$ and $B' = (b'_{i,j})$, by the property of A and B, there exists a set $\{j_0, j_1, \cdots, j_7\}$, such that $a'^{(m)}_{i,j_i} = 0$, $0 \le i \le 7$, and there exists a set $\{j'_0, j'_1, \cdots, j'_{w-1}\}$, such that $b'_{i,j'_i} = 0$, $0 \le i \le w-1$, there exists exactly one i such that $i = j'_i$ and $\sum\limits_{0 \le s \le n-1} b_{s,j_i} > 0$, there exists exactly one i' such that $i' + j'_{i'} = w-1$ and $\sum\limits_{0 \le t \le w-1} b_{t,j_{i'}} > 0$.

We write A'_m, C'_m and D'_m in the following forms:

$$
A'_m = \begin{pmatrix} A_{m,1} & A_{m,2} \\ A_{m,3} & A_{m,4} \end{pmatrix}, \quad C'_m = \begin{pmatrix} C_{m,1} \\ C_{m,2} \end{pmatrix}, \quad D'_m = \begin{pmatrix} D_{m,1} & D_{m,2} \end{pmatrix},
$$

where $1 \le m \le k$, $A_{m,1}$, $A_{m,2}$, $A_{m,3}$ and $A_{m,4}$ are 4×4 sub-arrays of A'_m, $C_{m,1}$ and $C_{m,2}$ are $4 \times 2t_m$ sub-arrays of C'_m, $D_{m,1}$ and $D_{m,2}$ are $2t_m \times 4$ sub-arrays of D'_m.

Construct an $n \times n$ array $E = (e_{r,t})$ below.

<table>
<tr><td>$A_{k,1}$</td><td colspan="9">$C_{k,1}$</td><td>$A_{k,2}$</td></tr>
<tr><td rowspan="9">$D_{k,1}$</td><td>$\ddots$</td><td colspan="7">$\vdots$</td><td></td><td rowspan="9">$D_{k,2}$</td></tr>
<tr><td rowspan="7">$\cdots$</td><td>$A_{3,1}$</td><td colspan="5">$C_{3,1}$</td><td>$A_{3,2}$</td><td rowspan="7">$\cdots$</td></tr>
<tr><td rowspan="5">$D_{3,1}$</td><td>$A_{2,1}$</td><td colspan="3">$C_{2,1}$</td><td>$A_{2,2}$</td><td rowspan="5">$D_{3,2}$</td></tr>
<tr><td rowspan="3">$D_{2,1}$</td><td>$A_{1,1}$</td><td>$C_{1,1}$</td><td>$A_{1,2}$</td><td rowspan="3">$D_{2,2}$</td></tr>
<tr><td>$D_{1,1}$</td><td>B'</td><td>$D_{1,2}$</td></tr>
<tr><td>$A_{1,3}$</td><td>$C_{1,2}$</td><td>$A_{1,4}$</td></tr>
<tr><td>$A_{2,3}$</td><td colspan="3">$C_{2,2}$</td><td>$A_{2,4}$</td></tr>
<tr><td>$A_{3,3}$</td><td colspan="5">$C_{3,2}$</td><td>$A_{3,4}$</td></tr>
<tr><td></td><td colspan="7">$\vdots$</td><td>$\ddots$</td></tr>
<tr><td>$A_{k,3}$</td><td colspan="9">$C_{k,1}$</td><td>$A_{k,4}$</td></tr>
</table>

It is not difficult to check that E is a PSAMS$(n, n-2)$. In fact, by the properties of A'_m, B', C'_m and D'_m, it is easy to see that there are $(n-2)/2$ positive entries and $(n-2)/2$ negative entries in each row, column and each main diagonal of E, respectively. The set of non-zero elements of E is

$$\begin{aligned} G(E) &= (\bigcup_{m=1}^{k} G(A'_m)) \cup G(B') \cup (\bigcup_{m=1}^{k} G(C'_m)) \cup (\bigcup_{m=1}^{k} G(D'_m)) \\ &= [-n(n-2)/2, -1] \cup [1, n(n-2)/2] \end{aligned}$$

Row-sums of E are all 0, column-sums of E are $w/2+4k = n/2$ or $-w/2-4k = -n/2$. Note that two main diagonal sums of A are both 4 and two main diagonal sums of B are both $w/2+2$, so two main diagonal sums of E are both $4k + w/2 + 2 = n/2 + 2$.

By the property of A'_m, B' and the construction of E, there exists a set $H = \{t_0, t_1, \cdots, t_{n-1}\}$, such that $s_{r,t_r} = 0$, $0 \le r \le n-1$, there exists exactly one r such that $r = t_r$ and $\sum_{0 \le l \le n-1} e_{l,t_r} > 0$, there exists exactly one r' such that $r' + t_{r'} = n-1$ and $\sum_{0 \le v \le n-1} e_{v,t_{r'}} > 0$.

Thus, E is the desired PSAMS$(n, n-2)$. □

Theorem 4.6. *There exists a regular SAMS$(n, n-1)$ for all even $n \ge 4$.*

Proof. A regular SAMS$(4,3)$ is given in Section 1. For even $n \ge 6$, the corresponding regular SAMS$(n, n-1)$ is obtained by Lemma 4.5 and Construction 2.2. □

The proof of Theorem 1.1 Just combining Theorem 3.5 and Theorem 4.6, the proof is obtained.

□

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