THE LINE GRAPH ASSOCIATED TO THE TOTAL GRAPH OF A COMMUTATIVE RING

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Abstract

Let R be a commutative ring with identity and $T(\Gamma(R))$ its total graph. The subject of this article is the investigation of the properties of the corresponding line graph $L(T(\Gamma(R)))$. The classification of all commutative rings whose line graphs are planar or toroidal is given. It is shown that for every integer $g \geq 0$ there are only finitely many commutative rings such that $\gamma(L(T(\Gamma(R)))) = g$.

Keywords: Total graph; line graph; genus of a graph; commutative ring.

1 Introduction

Let R be a commutative ring with identity, $Z(R)$ be the set of its zero divisors and $Z^*(R) = Z(R) \setminus \{0\}$. One may associate a graph to a given ring in various ways in order to investigate certain properties of that ring. One of the most common is the zero-divisor graph. This idea first appears in [3], where, for a ring R, the set of vertices is taken to be R and two vertices x and y are adjacent if and only if $xy = 0$. This paper primarily deals with the problem of graph coloring. Later, in the paper [2], the authors define the zero-divisor graph $\Gamma(R)$, where the set of vertices is taken to be $Z^*(R)$. They investigate various properties of this graph. This graph turned out to be quite interesting for many researchers. One of the most interesting problems is the question of the embedding of this graph into compact surfaces, where finite commutative rings play a special role.

In the paper [15], the list of all isomorphism classes of finite commutative rings such that the corresponding zero-divisor graph may be embedded into a torus is presented. In the paper [10] the list of all graphs with at most 14 vertices which are of the form $\Gamma(R)$ for some ring R is given, as well as the list of all isomorphism classes of the rings whose zero-divisor graphs are in that list.

In the paper [1], Anderson and Badawi introduce the total graph $T(\Gamma(R))$ whose set of vertices is R. Two vertices x and y are adjacent if and only if $x + y \in Z(R)$. The question of the embedding of this graph is discussed in [9]. In that paper all isomorphism classes of finite commutative rings whose total graphs are planar or toroidal are listed.

In graph theory, one associates a graph G with its line graph $L(G)$ so that the set of vertices of $L(G)$ is exactly the set of edges of G. Two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G have a common vertex. This allows one to investigate the properties of a graph G which depend only on the edges of that graph as the properties of the graph $L(G)$ which depend only on its vertices. This is very useful for various problems in graph theory. For example, to a matching in G there corresponds an independent set in $L(G)$. If G is connected and if its line graph $L(G)$ is known, one may, according to [14], completely determine G except in the case $L(G) = K^3$. In the paper [7], the authors investigate embeddings of the line graph $L(\Gamma(R))$ and present all isomorphism classes of finite commutative rings such that their line graphs are planar or toroidal.

Knowing the structure of the total graph $T(\Gamma(R))$, it is natural to investigate the structure of its line graph and to look into relations between them. In this paper we determine some properties of this graph. The main results are related to the embedding problem. We give the list of all isomorphism classes of finite commutative rings for which the associated line graph of the total graph is planar or toroidal. It is interesting to note that the list of all isomorphism classes of rings such that the total graph is planar (see [9]) is identical to the list of those for which the associated line graph is planar, while in the toroidal case this is not true. We also prove that for any integer $g \geq 0$ there are only finitely many isomorphism classes of finite rings whose line graph (of the total graph) has genus g.

In what follows, all rings R are commutative with identity; $Z(R)$ is the set of zero divisors of R, $Z^*(R) = Z(R) \setminus \{0\}$ and $Reg(R) = R \setminus Z(R)$. By a graph G, we mean the simple undirected graph without loops, with the set of vertices $V = V(G)$ and the set of edges $E = E(G)$. The *degree* of the vertex $v \in V$, denoted by $\deg(v)$ is the number of vertices adjacent to the vertex v and $\delta(G) = \min\{\deg(v) \mid v \in V(G)\}$ is the *minimal degree* of the graph G. A graph is *regular* of the degree r if every vertex has the degree r. The vertices x and y are adjacent if they are connected by an edge. If for every two vertices x and y there exists a path connecting them, then we say that this graph is *connected.* A graph G is *complete* if any two vertices are adjacent. If the vertices of the graph G may be separated into two disjoint sets of cardinalities m and n, such that vertices are adjacent if and only if they do not belong to the same set, then the graph G is a complete bipartite graph. For complete and complete bipartite graphs, we use the notation K^n and $K^{m,n}$. In particular, $K^{1,n}$ is a *star* graph. A graph G is of genus 0 or *planar* if it can be embedded into a plane. If it cannot be embedded into a plane, but it can be embedded into a torus, it is of genus 1 or toroidal. One may find all necessary results from commutative ring theory in [8] and from graph theory in [13].

2 Embeddings of the graph L(TΓ(R))

Let R be a commutative ring with identity and $T(\Gamma(R))$ its total graph. For simplicity of notation we use $\mathrm{T}\Gamma(R)$ for the total graph and $\mathrm{L}(\mathrm{T}\Gamma(R))$ for its line graph. If for elements $x, y \in R$ one has $x+y \in Z(R)$, then we have a vertex in the graph $\mathrm{L}(\mathrm{T}\Gamma(R))$ and we denote that vertex by $[x, y]$. From the definition of the graph $\mathrm{T}\Gamma(R)$, it follows that the degree of every vertex of this graph depends on the number of zero divisors, as well as on whether 2 is a zero divisor in R or not. It is easy to show (see [9]) that the following proposition holds.

Proposition 2.1 *Let R be a finite commutative ring with identity and let x be a vertex of the graph $\mathrm{T}\Gamma(R)$. Then*

$$\deg(x) = \begin{cases} |Z(R)|-1, & 2 \in Z(R) \text{ or } x \in Z(R) \\ |Z(R)|, & \text{otherwise.} \end{cases}$$

Theorem 2.2 *Let R be a finite commutative ring which is not a field. Then the following equality holds*

$$\delta(L(\mathrm{T}\Gamma(R))) = 2|Z(R)| - 4.$$

Proof. Let $[u, v]$ be a vertex of $\mathrm{L}(\mathrm{T}\Gamma(R))$. According to 2.1 $\deg([u, v])$ is either $2|Z(R)|-2$, $2|Z(R)|-3$ or $2|Z(R)|-4$. It is enough to show that at least one vertex of the graph $\mathrm{L}(\mathrm{T}\Gamma(R))$ has degree $2|Z(R)|-4$. For example, we can take as that vertex, the vertex $[0, x]$, where $x \in Z^*(R)$.□

Let n be a non-negative integer and S_n an orientable compact surface of genus n. The genus of the graph G, denoted by $\gamma(G)$ is the smallest n such that G may be embedded into S_n. Graphs of genus 0 are planar and graphs of genus 1 are toroidal. If H is a subgraph of G, then $\gamma(H) \leq \gamma(G)$. By the well-known theorem of Kuratowski, the graph G is planar if and only if it does not contain a subdivision of K^5 or $K^{3,3}$. Let us give a brief review of known results concerning the genus of complete and complete bipartite graphs.

Proposition 2.3 [11, 12]

$$\gamma(K^n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil, n \geq 3; \gamma(K^{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil, m, n \geq 2,$$

where $\lceil x \rceil$ *is the smallest integer greater than or equal to* x.

By the previous proposition, we have that K^n is planar for $n \leq 4$ and toroidal for $5 \leq n \leq 7$, and that graphs $K^{3,n}$ where $3 \leq n \leq 6$ as well as the graph $K^{4,4}$ are toroidal. It happens that these particular graphs are often subgraphs of $\mathrm{T}\Gamma(R)$; hence their line graphs are often subgraphs of $\mathrm{L}(\mathrm{T}\Gamma(R))$. Therefore it is important to know their genus for this work. In [7], the autors have investigated the genus of the graphs $L(K^n)$ and $L(K^{m,n})$. Although there is no general formula for the genus of these graphs (such as the formula from Proposition 2.3), one may find some useful estimates and the complete determination of this genus for some special cases.

Using the well-known Euler formula, one may prove the following useful proposition (or see the proof in [15]).

Proposition 2.4 *Let* G *be a simple graph with* n *vertices, where* $n \geq 3$. *Let* $\gamma(G) = g$ *and* $\delta(G)$ *be the minimal degree of the graph* G. *Then:*

$$\delta(G) \leq 6 + \frac{12g - 12}{n}.$$

Corollary 2.5 *(i) Let* G *be a simple graph with* $\gamma(G) = 1$. *Then* $\delta(G) \leq 6$, *i.e.,* G *must have a vertex* x *such that* $\deg(x) \leq 6$. *Moreover,* $\delta(G) = 6$ *if and only if* G *is actually the triangulation of the torus which is* 6*-regular.*

(ii) Let G *be a simple planar graph* ($\gamma(G) = 0$). *Then* $\delta(G) \leq 5$, *i.e.,* G *must have a vertex* x *such that* $\deg(x) \leq 5$.

Lemma 2.6 *Let* R *be a finite commutative ring such that* $|Z(R)| \geq 5$. *Then* $\mathrm{L}(\mathrm{T}\Gamma(R))$ *is not planar.*

Proof. According to Corollary 2.5 and Theorem 2.2, one has $\delta(L(\mathrm{T}\Gamma(R))) = 2|Z(R)| - 4 \geq 6$, so the result follows. □

According to Lemma 2.6, in investigation of planarity of $\mathrm{L}(\mathrm{T}\Gamma(R))$ we may restrict our attention only to rings with a small number of zero divisors. Let us first examine the non-local case.

Theorem 2.7 *Let R be a finite commutative ring with identity which is not local. Then $L(T\Gamma(R))$ is planar if and only if R is isomorphic to one of the rings $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_3$.*

Proof. We may assume that $R \cong R_1 \times R_2 \times \cdots \times R_n$, where $|R_1| \leq |R_2| \leq \cdots \leq |R_n|$ with $n \geq 2$ and each R_i is local.

1) $n \geq 3$: Then $|Z(R)| \geq 7$, so $L(T\Gamma(R))$ cannot be planar by Lemma 2.6.

2) $R = R_1 \times R_2$ and for at least one of them, for example for R_2 one has $|R_2| \geq 5$. Then $L(T\Gamma(R))$ is not planar by Lemma 2.6, since $(0, a_i) \in Z(R)$, $a_i \in R_2$, $i = 1, \dots, 5$.

3) $R = R_1 \times R_2$ and $|R_1|, |R_2| \leq 4$. Let us look at all possibilities.

3.1) $R = \mathbb{Z}_2 \times \mathbb{Z}_2$. Then $L(T\Gamma(R)) = C_4$ is obviously planar.

3.2) $R = \mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_6$. Then $L(T\Gamma(R))$ is also planar (see figures 1 and 2).

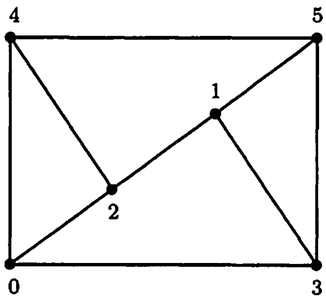

Figure 1. $T\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_3)$

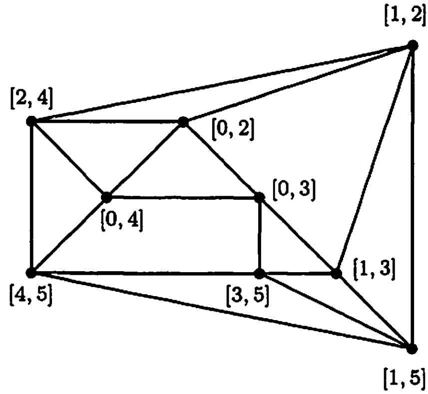

Figure 2. $L(T\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_3))$

3.3) $|R_1| = 2$, $|R_2| = 4$. The possibilities are: $\mathbb{Z}_2 \times \mathbb{F}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$. The case $\mathbb{Z}_2 \times \mathbb{F}_4$ is discussed in Theorem 2.10 where it is proved that $L(T\Gamma(\mathbb{Z}_2 \times \mathbb{F}_4))$ is not even embeddable into a torus. It is easy to check that the remaining two rings have 6 zero divisors, so their line graphs are not planar by Lemma 2.6.

3.4) $R = \mathbb{Z}_3 \times \mathbb{Z}_3$. In this case $|Z(R)| = 5$, so $L(T\Gamma(R))$ is not planar by Lemma 2.6.

3.5) $|Z(R)| \geq 7$ for the remaining cases, so $L(T\Gamma(R))$ is not planar by Lemma 2.6. $\square$

We now turn to the local case. Let R be a finite local ring with maximal ideal $M \neq \{0\}$. Then $Z(R) = M$, so, since zero divisors form an ideal, $T\Gamma(R)$ is disconnected and its structure is described by Theorem 2.2 from [1]:

1) If $2 \in Z(R)$, then $T\Gamma(R)$ is a disjoint union of $|R/M|$ copies of the complete graph $K^{|M|}$.

2) If $2 \notin Z(R)$, then $T\Gamma(R)$ is a disjoint union of the complete graph $K^{|M|}$ and $\frac{|R/M|-1}{2}$ complete bipartite graphs $K^{|M|,|M|}$.

In addition to these results, we use in the following proof the following known results. If R is a finite local ring with maximal ideal M, then $|R| = p^k$, where p is a prime, and $|M| \mid |R|$. We use the list of these rings up to the order p^5 ([4, 5]) and the result that $|R| \leq |Z(R)|^2$ ([6]).

Theorem 2.8 *Let R be a finite local commutative ring with identity which is not a field. Then $L(T\Gamma(R))$ is planar if and only if R is isomorphic to one of the following rings: $\mathbb{Z}_4$, $\mathbb{Z}_2[X]/(X^2)$, $\mathbb{Z}_8$, $\mathbb{Z}_2[X]/(X^3)$, $\mathbb{Z}_2[X,Y]/(X,Y)^2$, $\mathbb{Z}_4[X]/(2X,X^2)$, $\mathbb{Z}_4[X]/(2X,X^2-2)$, $\mathbb{F}_4[X]/(X^2)$ or $\mathbb{Z}_4[X]/(X^2+X+1)$.*

Proof. Let $|R| = p^k$. Note that it is enough to consider the cases $p \in \{2, 3\}$. Namely, if $p \geq 5$, since R is not a field and since $|M| \mid p^k$, one must have $|Z(R)| = |M| \geq 5$. According to Lemma 2.6, $L(T\Gamma(R))$ is not planar.

1) $k = 2$. The possibilities are: $\mathbb{F}_p[X]/(X^2)$ and $\mathbb{Z}_{p^2}$.

If R is one of the rings $\mathbb{F}_2[X]/(X^2)$ and $\mathbb{Z}_{2^2}$, then $L(T\Gamma(R))$ is obviously planar (it has 2 vertices). If $R = \mathbb{F}_3[X]/(X^2)$ or $R = \mathbb{Z}_{3^2}$, then $2 \notin Z(R)$, so $T\Gamma(R) = K^3 \cup K^{3,3}$. It may be shown (see [7]) that $\gamma(L(K^{3,3})) = 1$, so these are not planar.

2) $k = 3$. If $p = 3$ (i.e., $|R| = 27$), then $2 \notin Z(R)$; therefore $K^{3,3} \subseteq T\Gamma(R)$. So, $L(T\Gamma(R))$ is not planar. Therefore, it is enough to consider the case $p = 2$. Since R is not a field, the possibilities are: $\mathbb{Z}_{2^3}$, $\mathbb{Z}_2[X]/(X^3)$, $\mathbb{Z}_2[X,Y]/(X,Y)^2$, $\mathbb{Z}_4[X]/(2X,X^2)$ and $\mathbb{Z}_4[X]/(2X,X^2-2)$. It is not difficult to see that they all have isomorphic total graphs — disjoint union of two complete graphs K^4. Since $\gamma(L(K^4)) = 0$, it follows that the corresponding line graphs are planar

3) $k = 4$. Let us now look at local rings of order p^4. For the same reasons as in the case $k = 3$, we only need to consider the case $p = 2$, i.e., $|R| = 16$ with $|M| \mid 16$. The case $|M| = 1$ does not occur since R is not a field. The case $|M| = 2$ leads to the contradiction that $16 = |R| \leq |Z(R)|^2 = |M|^2 = 4$. If $|M| = |Z(R)| = 8$, then according to Lemma 2.6 the graph is not planar. Therefore, we only need to consider the case $|M| = 4$. There are exactly 21 non-isomorphic local commutative rings with identity of order 16 ([5]). From all of them, only two satisfy the condition $|M| = 4$: $\mathbb{F}_4[X]/(X^2)$ and $\mathbb{Z}_4[X]/(X^2+X+1)$. They have isomorphic total graphs — disjoint unions of 4 complete graphs K^4. Since $L(K^4)$ is planar, we conclude that for those two rings $L(T\Gamma(R))$ is planar.

4) $k \geq 5$. We can only consider the case $p = 2$, $|R| = 2^k$. The case $|M| = 4$ is not possible since $|R| \leq |M|^2$. In all other cases, the ring

contains too many zero divisors, so according to Lemma 2.6 these graphs are not planar. □

The list of all finite commutative rings with identity for which the line graph of the total graph is planar:

$$\mathbb{Z}_2\times\mathbb{Z}_2,\quad \mathbb{Z}_2\times\mathbb{Z}_3,\quad \mathbb{Z}_4,\quad \mathbb{Z}_2[X]/(X^2),\quad \mathbb{Z}_8,\quad \mathbb{Z}_2[X]/(X^3),$$
$$\mathbb{Z}_2[X,Y]/(X,Y)^2,\quad \mathbb{Z}_4[X]/(2X,X^2),\quad \mathbb{Z}_4[X]/(2X,X^2-2),$$
$$\mathbb{F}_4[X]/(X^2),\quad \mathbb{Z}_4[X]/(X^2+X+1).$$

Lemma 2.9 *Let R be a finite commutative ring with identity. Then:*

$$|Z(R)|>5 \implies \gamma(\mathrm{L}(\mathrm{T}\Gamma(R)))>1.$$

Proof. By Theorem 2.2 we have

$$\delta(\mathrm{L}(\mathrm{T}\Gamma(R)))=2|Z(R)|-4>6.$$

The claim follows by Proposition 2.4. □

Theorem 2.10 *Let R be a finite commutative ring with identity which is not local. Then* $\mathrm{L}(\mathrm{T}\Gamma(R))$ *does not have genus equal to* 1.

Proof. By Lemma 2.9 it is enough to discuss the decomposition $R=R_1\times R_2$ with $|R_1|\le|R_2|$. We give the proof by discussing the cardinality of the ring R_2.

1) $|R_2|\ge5$. Then $|Z(R)|\ge6$, so $\gamma(\mathrm{L}(\mathrm{T}\Gamma(R)))>1$ by Lemma 2.9.

2) $|R_2|=4$. We have two possibilities.

2.1) $|Z(R_2)|=2$. Then $|Z(R)|\ge6$, so $\gamma(\mathrm{L}(\mathrm{T}\Gamma(R)))>1$ by Lemma 2.9.

2.2) $|Z(R_2)|=1$. Then R_2 is isomorphic to the field $\mathbb{F}_4$. Let us discuss the minimal case $R=\mathbb{Z}_2\times\mathbb{F}_4$. We have that $\delta(\mathrm{T}\Gamma(R))=4$, so $\delta(\mathrm{L}(\mathrm{T}\Gamma(R)))=6$. From Corollary 2.5, we see that $\mathrm{L}(\mathrm{T}\Gamma(R))$ is of genus 1 if and only if this graph is the regular triangulation of a torus. Let us prove that this particular graph does not give this triangulation. Let $R=\mathbb{Z}_2\times\mathbb{F}_4$, $\mathbb{F}_4=\mathbb{Z}_2[X]/(X^2+X+1)$ and let x be the corresponding class. Then $Z(R)=\{v_1,v_2,v_3,v_4,v_5\}$ and $Reg(R)=\{v_6,v_7,v_8\}$, where $v_1=(0,0)$, $v_2=(0,x)$, $v_3=(0,x+1)$, $v_4=(0,1)$, $v_5=(1,0)$, $v_6=(1,x)$, $v_7=(1,x+1)$ and $v_8=(1,1)$.

The graph $\mathrm{T}\Gamma(R)$ is a regular graph of degree 4 with 8 vertices and 16 edges so $\mathrm{L}(\mathrm{T}\Gamma(R))$ is a regular graph of degree 6 with 16 vertices and 48 edges (it is obvious that the line graph of the regular graph of degree r with n vertices is also regular of degree $2r-2$ with $nr/2$ vertices and $nr(r-1)/2$ edges). To simplify the notation let us denote the vertex $[v_i,v_j]$

of $L(T\Gamma(R))$ by $w_{i,j}$. The graph $L(T\Gamma(\mathbb{Z}_2 \times \mathbb{F}_4))$ has 16 vertices: $w_{1,2}$, $w_{1,3}$, $w_{1,7}$, $w_{1,8}$, $w_{2,3}$, $w_{2,5}$, $w_{2,7}$, $w_{3,4}$, $w_{3,7}$, $w_{4,5}$, $w_{4,6}$, $w_{4,8}$, $w_{5,6}$, $w_{5,8}$, $w_{6,7}$ and $w_{6,8}$. Vertices $w_{i,j}$ and $w_{k,l}$ are adjacent if and only if $\{i,j\} \cap \{k,l\} \neq \emptyset$, so the number of edges is 48. According to the number of vertices, edges and faces (Euler's theorem), $L(T\Gamma(\mathbb{Z}_2 \times \mathbb{F}_4))$ gives a torus triangulation. Every vertex of the graph must be a center of some hexagon (the degree of every vertex is 6). We prove that this is not possible, e.g., for the vertex $w_{1,2}$. Suppose that $w_{1,2}$ is the center of some hexagon. This hexagon must have vertices $w_{1,3}$, $w_{1,7}$, $w_{1,8}$, $w_{2,3}$, $w_{2,5}$ and $w_{2,7}$ as its vertices. Consider vertices $w_{2,5}$ and $w_{2,7}$. The following cases may occur:

a) The vertices $w_{2,5}$ and $w_{2,7}$ are adjacent (Figure 3a). This triangulation may not be extended further since the only vertex from the set of all vertices which is adjacent to both of them is the vertex $w_{2,3}$ and this vertex must be on the hexagon.

b) If they are separated by only one vertex, this can only be the vertex $w_{2,3}$ (Figure 3b). Since $w_{1,3}$ is not adjacent neither to $w_{2,5}$ nor to $w_{2,7}$, it must be antipodal to the vertex $w_{2,3}$. Then the vertex $w_{1,8}$ cannot be on the hexagon.

c) If the vertices $w_{2,5}$ and $w_{2,7}$ are antipodal (Figure 3c), then the vertex $w_{1,3}$ cannot be on the hexagon.

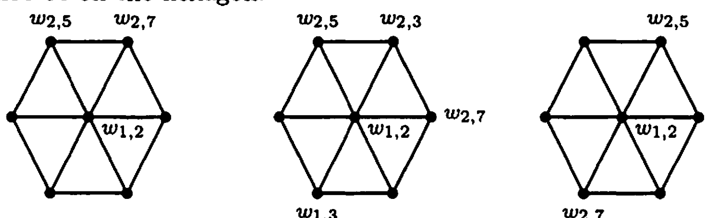

Figure 3. *a)* *b)* *c)*

3) $|R_2| = 3$. We have 2 possibilities.

3.1) $|R_1| = 2$. Then $R = \mathbb{Z}_2 \times \mathbb{Z}_3$, so $L(T\Gamma(R))$ is planar by Theorem 2.7.

3.2) $|R_1| = 3$. Then $R = \mathbb{Z}_3 \times \mathbb{Z}_3$ and $|Z(R)| = 5$. It follows that $\delta(T\Gamma(R)) = 4$ and $\delta(L(T\Gamma(R))) = 6$. If there is an an embedding, then all vertices must have degree 6 by Corollary 2.5. However. this is not true because $\deg([(0,1),(2,2)]) = 7$.

Since $L(T\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2))$ is planar, the proof is complete. □

So, only the local case remains.

Theorem 2.11 *Let R be a commutative local ring which is not a field. Then $L(T\Gamma(R))$ is toroidal if and only if R is isomorphic to one of the rings $\mathbb{Z}_9$ or $\mathbb{Z}_3[X]/(X^2)$.*

Proof. Let $M = Z(R)$ be the maximal ideal of R. By Lemma 2.9 it is enough to consider the case $|M| \leq 5$. First we prove that the case $|M| = 5$ may also be excluded.

Let $|M| = |Z(R)| = 5$ and $x \in Reg(R)$ be an arbitrary element. Then $\deg(x) = 5$ by Lemma 2.1. There exists $y \in Reg(R)$ such that x is adjacent to y in $T\Gamma(R)$, meaning that $x+y \in Z(R)$. Otherwise, x would be adjacent to 0 and that is not true. The edge x—y in $T\Gamma(R)$ connects two vertices of degree 5 in $T\Gamma(R)$, so the vertex $[x,y]$ in $L(T\Gamma(R))$ has degree 8. Since $\delta(T\Gamma(R)) = 4$, one has $\delta(L(T\Gamma(R))) = 6$, so by Corollary 2.5 in the case of toroidality all vertices of the graph $L(T\Gamma(R))$ must have degree 6 and that is not the case, since $\deg([x,y]) = 8$.

Let $|R| = p^k$. We only consider the cases $p = 2$ and $p = 3$.

1) $|R| = p^2$. The possibilities are $\mathbb{F}_{p^2}$, $\mathbb{Z}_p[X]/(X^2)$ and $\mathbb{Z}_{p^2}$. Since R is not a field and since the line graphs for $\mathbb{Z}_2[X]/(X^2)$ and $\mathbb{Z}_{2^2}$ are planar, we are left with rings $\mathbb{Z}_3[X]/(X^2)$ and $\mathbb{Z}_{3^2}$. Both of these rings have total graphs isomorphic to $K^3 \sqcup K^{3,3}$. Since $\gamma(L(K^{3,3})) = 1$, this graph is toroidal as it is the union of a toroidal and a planar graph.

2) $|R| = p^3$. According to [4] the possibilities are $\mathbb{F}_{p^3}$, $\mathbb{F}_p[X]/(X^3)$, $\mathbb{Z}_{p^2}[X]/(pX, X^2)$, $\mathbb{F}_p[X,Y]/(X,Y)^2$, $\mathbb{Z}_{p^3}$ and $\mathbb{Z}_{p^2}[X]/(pX, X^2 - \varepsilon p)$, where ε is not a square in $\mathbb{F}_p^*$. Since $\mathbb{F}_{p^3}$ is a field and since, by Theorem 2.8 the line graphs for $\mathbb{Z}_2[X]/(X^3)$, $\mathbb{Z}_4[X]/(2X, X^2)$, $\mathbb{F}_2[X,Y]/(X,Y)^2$, $\mathbb{Z}_{2^3}$ and $\mathbb{Z}_4[X]/(2X, X^2 - 2)$ are planar, we only have to discuss the rings $\mathbb{F}_3[X]/(X^3)$, $\mathbb{Z}_9[X]/(3X, X^2)$, $\mathbb{F}_3[X,Y]/(X,Y)^2$, $\mathbb{Z}_{27}$ and $\mathbb{Z}_9[X]/(3X, X^2 - 3)$. It is easy to see that for those rings one has $|M| = |Z(R)| = 9$, so according to Lemma 2.9 here we do not have line graphs of genus 1.

3) $|R| = p^4$.

3.1) $p = 3$ and $|R| = 81$, so $|M| \in \{1, 3, 9, 27\}$. The case $|M| = 1$ does not occur since R is not a field, while in the case $|M| = 3$ one has $|R| \leq 9$. In the remaining cases there are too many zero divisors.

3.2) $p = 2$ and $|R| = 16$, so $|M| \in \{1, 2, 4, 8\}$. By similar analysis only the case $|M| = 4$ (i.e., the rings $\mathbb{F}_4[X]/(X^2)$ and $\mathbb{Z}_4[X]/(X^2 + X + 1)$) remains. Their line graphs are planar by Theorem 2.8.

4) $|R| = p^k$, $k \geq 5$. It is clear that there are no toroidal line graphs in this case since $|M| = |Z(R)| \geq 8$. □

The list of finite commutative rings with identity such that the line graph of the total graph has genus 1 is as follows:

$$\mathbb{Z}_9, \quad \mathbb{Z}_3[X]/(X^2).$$

Theorem 2.12 *Let $g \geq 0$ be an integer. There are only finitely many commutative rings R such that $\gamma(L(T\Gamma(R))) = g$.*

Proof. We have proved that for $g = 0$ there are only 11 such rings, while for $g = 1$ there are only 2. Suppose now that $g \geq 2$. It follows from Theorem 2.2 that the inequality in Proposition 2.4 for the graph $\mathrm{L}(T\Gamma(R))$ reduces to

$$2|Z(R)| - 4 \leq 6 + \frac{12g - 12}{n}.$$

Since

$$n = |V(L(T\Gamma(R)))| \geq \frac{1}{2}|R|(|Z(R)| - 1),$$

substituting this in the previous inequality one gets

$$|R|(|Z(R)| - 1)(|Z(R)| - 5) \leq 12g - 12.$$

So, for $|Z(R)| \geq 6$ one has

$$|R| \leq 3g - 3.$$

On the other hand, since $|R| \leq |Z(R)|^2$, there are only finitely many cases such that $|Z(R)| < 6$. This completes the proof. □

Remark. *From the arguments in the previous proof, it is interesting to note here that for* $|Z(R)| \geq 6$, *there are no rings whose line graphs have genus* 2 *or* 3.

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