Adjacent vertex distinguishing total colorings of graphs with four distinguishing constraints

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Abstract

Frequency assignment problem was produced in researching mobile communication networks. A proper total coloring of a graph G is a coloring of both edges and vertices of G such that no two adjacent or incident elements receive the same color. As known, the vertex distinguishing total coloring is one of suitable tools for investigating the frequency assignment problem. We introduce a new graph total coloring, called (4)-adjacent vertex distinguishing total coloring ((4)- AVDTC), in this paper. Our coloring contains the adjacent vertex distinguishing total coloring. The minimum number of colors required for every (4)-AVDTC of G is called the (4)-AVDTC chromatic number of G. We will show that using at most $\Delta(G) + 4$ colors can do at least 4 different adjacent vertex distinguishing actions to some communication networks G. The exact (4)-AVDTC chromatic numbers of several classes of graphs are determined here and a problem is presented.

Keywords: Total coloring; Adjacent vertex distinguishing total colorings; (4)-adjacent vertex distinguishing total colorings
MSC(2000): 05C15

1 Introduction

Graph coloring theory has a wide range of applications in many fields, such as physics, chemistry, computer science, network theory, social science, and more specific as, time tabling and scheduling, frequency assignment, register allocation, computer security, electronic banking, coding theory, communication network, logistics and so on. Since customers have increased dramatically, it will yield a confliction between the increasing customers and the limited expansion of communication network resources. In order to solve the frequency assignment problem, some scholars put forward the concept of vertex distinguishing colorings of graphs, which it had been concerned and studied extensively over one decade.

Burris and Schelp [3] introduced a non-traditional graph edge coloring: A proper edge coloring of a simple graph G is called a *vertex distinguishing edge coloring* (VDEC) if for any two distinct vertices u and v of G, the set of colors assigned to the edges incident to u differs from the set of colors assigned to the edges incident to v. The minimum number of colors required for all VDECs of G was denoted by $\chi'_s(G)$. Let $n_d = n_d(G)$ denote the number of vertices of degree d in G. It is clear that $\binom{\chi'_s(G)}{d} \geq n_d$ for all d ($\neq 0$) with respect to $\delta(G) \leq d \leq \Delta(G)$. Furthermore, Burris and Schelp presented a famous conjecture: *Let G be a simple graph having no isolated edge, at most one isolated vertex, maximum degree Δ and minimum degree δ, and let k be the smallest integer such that $\binom{k}{d} \geq n_d$ for all d ($\neq 0$) with respect to $\delta \leq d \leq \Delta$. Then $k \leq \chi'_s(G) \leq k+1$.* A weak version of the VDEC was introduced in [5], named as the *adjacent vertex distinguishing edge coloring* (AVDEC). Zhang et al. in [5] required that for each edge xy of G, the set of colors assigned to the edges incident to x differs from the set of colors assigned to the edges incident to y in an AVDEC, and used $\chi'_{as}(G)$ to denote the smallest number of k colors required for which G admits a k-AVDEC. They also proposed a conjecture: *Every simple graph G having no isolated edges and being not a cycle of five vertices holds $\chi'_{as}(G) \leq \Delta(G)+2$.* Vizing in [4] and Behzad in [1] presented independently the famous total coloring conjecture: *Let G be a simple graph with order $n \geq 2$, then G has its total chromatic number $\chi''(G) \leq \Delta(G) + 2$.* Based on the VDTC, Zhang et al. in [6] introduced a concept of *adjacent vertex distinguishing total coloring* (AVDTC), and showed a conjecture: *Let G be a simple graph with order $n \geq 2$; then G has its AVDTC chromatic number $\chi''_{as}(G) \leq \Delta(G)+3$.* It is very difficult to settle down the above conjectures, and no counterexamples to every one of the conjectures have been discovered up to now. As attacking the above problems and conjectures, we present several new colorings with many distinguishing constrains for considering graph coloring problems comprehensively than before. We will show that using at most $\Delta(G)+4$ colors can do at least 4 different distinguishing actions to some communication networks G.

Let G be a simple graph and $f: V(G)\cup E(G)\rightarrow\{1,2,\cdots,k\}$ be a proper total k-coloring of G. Setting $C(f,u)=\{f(e): e\in N_e(u)\}$, $C\langle f,u\rangle=\{f(x): x\in N(u)\}\cup\{f(u)\}$, $C[f,u]=C(f,u)\cup\{f(u)\}$ and $C_2[f,u]=C(f,u)\cup C\langle f,u\rangle$, where $N(u)$ stands for the set of neighbors of a vertex u, $N_e(u)$ is the set of edges incident to u. Let $C\{f;x\}=\{C(f,x),C\langle f,x\rangle,C[f,x],C_2[f,x]\}$. For each edge $xy\in E(G)$, the notation $C\{f;x\}\neq C\{f;y\}$ means that the four proper distinguishing constraints $C(f,x)\neq C(f,y)$, $C\langle f,x\rangle\neq C\langle f,y\rangle$, $C[f,x]\neq C[f,y]$ and $C_2[f,x]\neq C_2[f,y]$ hold true at the same time.

Definition 1.0 [6] Let $f: V(G)\cup E(G)\rightarrow\{1,2,\ldots,k\}$ be a proper total coloring of a simple graph G. We call f a *k-adjacent vertex distinguishing total coloring* (k-AVDTC) of G if $C[f,u]\neq C[f,v]$ for each edge $uv\in E(G)$. The minimum number of k colors required for which G admits a k-AVDTC is denoted as $\chi''_{as}(G)$ called the *AVDTC chromatic number* of G.

Definition 1.1 [7] Let $f: V(G)\cup E(G)\rightarrow\{1,2,\ldots,k\}$ be a proper total coloring of a simple graph G. We call f a *k-adjacent vertex strong distinguishing total coloring* (k-AVSDTC) of G if $C_2[f,u]\neq C_2[f,v]$ for each edge $uv\in E(G)$. The minimum number of k colors required for which G admits a k-AVSDTC is denoted as $\chi''_{ast}(G)$ called the *AVSDTC chromatic number* of G.

Motivated from the above definitions, we introduce a new graph total coloring, called (4)-adjacent vertex distinguishing total coloring ((4)-AVDTC) as follows.

Definition 1.2 Let $f: V(G)\cup E(G)\rightarrow\{1,2,\ldots,k\}$ be a proper total coloring of a simple graph G having $N(u)\neq N(v)$ for $uv\in E(G)$. We call f a *k-(4)-adjacent vertex distinguishing total coloring*(k-(4)-AVDTC) of G if $C\{f;x\}\neq C\{f;y\}$ for each edge $xy\in E(G)$. The minimum number of k colors required for which G admits a k-(4)-AVDTC is denoted by $\chi''_{(4)as}(G)$ and called the (4)-*AVDTC chromatic number* of G.

Clearly, complete graphs do not admit (4)-AVDTCs. For a simple graph G, it is not hard to see $\chi''_{as}(G) \leq \chi''_{ast}(G) \leq \chi''_{(4)as}(G)$. The following results will be used in the next section.

Lemma 1.3 [6] *Let G be a simple graph. If G contains adjacent vertices whose degrees are the maximum degree Δ, then $\chi''_{as}(G) \geq \Delta(G) + 2$.*

Theorem 1.4 [7] *Let P_n be a path with order n ($n \geq 3$). Then*

$$\chi''_{ast}(P_n) = \begin{cases} 4, & \text{if } n \equiv 1 \pmod 2; \\ 5, & \text{if } n \equiv 0 \pmod 2. \end{cases}$$

Theorem 1.5 [7] *Let C_n be a cycle of order n. Then*

$$\chi''_{ast}(C_n) = \begin{cases} 4, & \text{if } n \not\equiv 4, 10 \text{ and } 0 \pmod 2; \\ 5, & \text{otherwise.} \end{cases}$$

Theorem 1.6 [7] *Let $K_{m,n}$ be a complete bipartite graph with $m > n > 1$ and $m + n > 3$. Then*

$$\chi''_{ast}(K_{m,n}) = \begin{cases} m + 1, & \text{if } m - n \geq 2 \text{ or } m > 2 \text{ and } m > n = 1; \\ m + 2, & \text{if } m - n = 1; \\ m + 3, & \text{if } m = n. \end{cases}$$

2 Main results

Theorem 2.1 *Let P_n be a path with order $n \geq 4$. Then $\chi''_{(4)as}(P_n) = 5$.*

Proof We can write $P_n = v_1v_2 \cdots v_{n-1}v_n$ and define a total coloring f of P_n. We consider $P_4 = v_1v_2v_3v_4$, first of all. By the distinguishing rule, without loss of generality, we set $f(v_1) = 1, f(v_2) = 2$, then

$f(v_3) = 3$ (otherwise, $C\langle f, v_1\rangle = C\langle f, v_2\rangle$) and $f(v_4) = 4$ (otherwise, $C\langle f, v_i\rangle = C\langle f, v_{i+1}\rangle, i = 2, 3$). Set $f(v_1v_2) = 3$, then $f(v_2v_3) = 4$ (otherwise, $C_2[f, v_1] = C_2[f, v_2]$) and $f(v_3v_4) = 5$ (otherwise, $C_2[f, v_2] = C_2[f, v_3]$ or $C[f, v_2] = C[f, v_3]$ or $C_2[f, v_3] = C_2[f, v_4]$). Therefore, we obtain $\chi''_{(4)as}(P_4) = 5$. Notice that $\chi''_{(4)as}(P_n) \geq \chi''_{(4)as}(P_4) = 5$ for $n \geq 5$. We show $\chi''_{(4)as}(P_n) \leq 5$ in the following. In fact, we need only to prove that P_n has a 5-(4)-AVDTC.

By induction on orders of P_n for $n \geq 5$. Obviously, P_5 has a 5-(4)-AVDTC. Suppose that f is a 5-(4)-AVDTC of P_n for $n > 5$. Now we show that P_{n+1} has a 5-(4)-AVDTC f^* by $f^*(v_{n-1}) = \alpha_1$, $f^*(v_{n-1}v_n) = \alpha_2$, $f^*(v_n) = \alpha_3$, where $\{\alpha_1, \alpha_2, \alpha_3\} \subset \{1, 2, 3, 4, 5\}$, and defining $f^*(u) = f(u), u \in V(P_n)$; $f^*(v_nv_{n+1}) = \alpha_1$; $f^*(v_{n+1}) = \alpha_4 \in \{1, 2, 3, 4, 5\} \setminus C_2[f, v_n]$.

By the definition of the coloring f^*, we have a table

vertices	$C(f^*, v_i)$	$C\langle f^*, v_i\rangle$	$C[f^*, v_i]$	$C_2[f^*, v_i]$
u	$C(f, v_i)$	$C\langle f, v_i\rangle$	$C[f, v_i]$	$C_2[f, v_i]$
v_n	$\{\alpha_1, \alpha_2\}$	$\{\alpha_1, \alpha_3, \alpha_4\}$	$\{\alpha_1, \alpha_2, \alpha_3\}$	$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$
v_{n+1}	$\{\alpha_1\}$	$\{\alpha_3, \alpha_4\}$	$\{\alpha_1, \alpha_4\}$	$\{\alpha_1, \alpha_3, \alpha_4\}$

and claim that f^* is a 5-(4)-AVDTC of P_{n+2} by the induction hypothese. Therefore, $\chi''_{(4)as}(P_n) = 5$ for $n \geq 4$. □

Theorem 2.2 *Let C_n be a cycle with order $n \geq 4$. Then* $\chi''_{(4)as}(C_n) = 5$.

Proof We denote a cycle C_n of order $n \geq 4$ as $C_n = v_1v_2\cdots v_nv_1$ and define a total coloring f of C_n in the following. First, we consider the case $C_4 = v_1v_2v_3v_4$. By the distinguishing rule, without loss of generality, we let $f(v_1) = 1, f(v_2) = 2$, $f(v_3) = 1$, then $f(v_4) = 3$ (otherwise, $C\langle f, v_i\rangle = \{1, 2\}$ for $i = 1, 2, 3, 4$); next set $f(v_1v_2) = 3$, then $f(v_2v_3) = 4$, $f(v_3v_4) = 5$ (otherwise, $C_2[f, v_2] = C_2[f, v_3]$) and $f(v_4v_1) = 2$ (otherwise, $C_2[f, v_1] = C_2[f, v_2]$). Therefore, $\chi''_{(4)as}(C_4) = 5$. Notice that $\chi''_{(4)as}(C_n) \geq \chi''_{(4)as}(C_4) = 5$. We prove $\chi''_{(4)as}(C_n) \leq 5$ by showing that C_n has a 5-(4)-AVDTC.

Case 1. $n \equiv 0 \pmod 5$. For $1 \leq i \leq n$, we define a total coloring f of C_n as follows: $f(v_i) = 1, i \equiv 1 \pmod 5$; $f(v_i) = 2$ for $i \equiv 2 \pmod 5$; $f(v_i) = 3$ for $i \equiv 3 \pmod 5$; $f(v_i) = 4, i \equiv 4 \pmod 5$;

$f(v_i) = 5$ for $i \equiv 0 \pmod 5$. $f(v_iv_{i+1}) = 3$ for $i \equiv 1 \pmod 5$; $f(v_iv_{i+1}) = 4$ for $i \equiv 2 \pmod 5$; $f(v_iv_{i+1}) = 5$ for $i \equiv 3 \pmod 5$; $f(v_iv_{i+1}) = 1$ for $i \equiv 4 \pmod 5$; $f(v_iv_{i+1}) = 2$ for $i \equiv 0 \pmod 5$. This coloring f shows

vertices	$C(f, v_i)$	$C\langle f, v_i\rangle$	$C[f, v_i]$	$C_2[f, v_i]$
v_1	$\{2,3\}$	$\{1,2,5\}$	$\{1,2,3\}$	$\{1,2,3,5\}$
v_2	$\{3,4\}$	$\{1,2,3\}$	$\{2,3,4\}$	$\{1,2,3,4\}$
v_3	$\{4,5\}$	$\{2,3,4\}$	$\{3,4,5\}$	$\{2,3,4,5\}$
v_4	$\{1,5\}$	$\{3,4,5\}$	$\{1,4,5\}$	$\{1,3,4,5\}$
v_5	$\{1,2\}$	$\{1,4,5\}$	$\{1,2,5\}$	$\{1,2,4,5\}$

For other vertices of C_n, we have $C(f, v_i) = C(f, v_j)$, $C\langle f, v_i\rangle = C\langle f, v_j\rangle$, $C[f, v_i] = C[f, v_j]$, $C_2[f, v_i] = C_2[f, v_j]$ for $i \equiv j \pmod 5$. Therefore, $\chi''_{(4)as}(C_n) = 5$ since f is realy a 5-(4)-AVDTC of C_n.

Case 2. $n \equiv 1 \pmod 5$. We construct a total coloring f of C_n as follows: $f(v_1) = 3$, $f(v_n) = 4$, $f(v_1v_2) = 1$, $f(v_nv_1) = 5$. For $2 \leq i \leq n-1$, we set $f(v_i) = 1$ for $i \equiv 1 \pmod 5$; $f(v_i) = 2$ for $i \equiv 2 \pmod 5$; $f(v_i) = 3$ for $i \equiv 3 \pmod 5$; $f(v_i) = 4$ for $i \equiv 4 \pmod 5$; $f(v_i) = 5$ for $i \equiv 0 \pmod 5$. $f(v_iv_{i+1}) = 3$ for $i \equiv 1 \pmod 5$; $f(v_iv_{i+1}) = 4, i \equiv 2 \pmod 5$; $f(v_iv_{i+1}) = 5$ for $i \equiv 3 \pmod 5$; $f(v_iv_{i+1}) = 1$ for $i \equiv 4 \pmod 5$; $f(v_iv_{i+1}) = 2$ for $i \equiv 0 \pmod 5$. Based on the coloring f, we have the color sets $C\{f, v_i\}$ of the key vertices v_i in the following table

vertices	$C(f, v_i)$	$C\langle f, v_i\rangle$	$C[f, v_i]$	$C_2[f, v_i]$
v_1	$\{1,5\}$	$\{2,3,4\}$	$\{1,3,5\}$	$\{1,2,3,4,5\}$
v_2	$\{1,4\}$	$\{2,3\}$	$\{1,2,4\}$	$\{1,2,3,4\}$
v_3	$\{4,5\}$	$\{2,3,4\}$	$\{3,4,5\}$	$\{2,3,4,5\}$
v_4	$\{1,5\}$	$\{3,4,5\}$	$\{1,4,5\}$	$\{1,3,4,5\}$
v_5	$\{1,2\}$	$\{1,4,5\}$	$\{1,2,5\}$	$\{1,2,4,5\}$
v_6	$\{2,3\}$	$\{1,2,5\}$	$\{1,2,3\}$	$\{1,2,3,5\}$
v_7	$\{3,4\}$	$\{1,2,3\}$	$\{2,3,4\}$	$\{1,2,3,4\}$
v_8	$\{4,5\}$	$\{2,3,4\}$	$\{3,4,5\}$	$\{2,3,4,5\}$
v_9	$\{1,5\}$	$\{3,4,5\}$	$\{1,4,5\}$	$\{1,3,4,5\}$
v_{10}	$\{1,2\}$	$\{1,4,5\}$	$\{1,2,5\}$	$\{1,2,4,5\}$
v_{n-1}	$\{1,2\}$	$\{4,5\}$	$\{1,2,5\}$	$\{1,2,4,5\}$
v_n	$\{2,5\}$	$\{3,4,5\}$	$\{2,4,5\}$	$\{2,3,4,5\}$

For other vertices of C_n, we have $C(f,v_i) = C(f,v_j)$, $C\langle f,v_i\rangle = C\langle f,v_j\rangle$, $C[f,v_i] = C[f,v_j]$, $C_2[f,v_i] = C_2[f,v_j]$ when $i \equiv j \pmod 5$ and $6 \le i,j \le n-2$. We conclude that f is a 5-(4)-AVDTC of C_n, thus, $\chi''_{(4)as}(C_n) = 5$.

Case 3. $n \equiv 2 \pmod 5$. We make a total coloring f of C_n by defining $f(v_1) = 3$, $f(v_n) = 4$, $f(v_1v_2) = 1$, $f(v_nv_1) = 5$; for $2 \le i < n-1$, we let $f(v_i) = 1$ for $i \equiv 1 \pmod 5$; $f(v_i) = 2$ for $i \equiv 2 \pmod 5$; $f(v_i) = 3$ for $i \equiv 3 \pmod 5$; $f(v_i) = 4, i \equiv 4 \pmod 5$; $f(v_i) = 5$ for $i \equiv 0 \pmod 5$. $f(v_iv_{i+1}) = 3$ for $i \equiv 1 \pmod 5$; $f(v_iv_{i+1}) = 4$ for $i \equiv 2 \pmod 5$; $f(v_iv_{i+1}) = 5, i \equiv 3 \pmod 5$; $f(v_iv_{i+1}) = 1$ for $i \equiv 4 \pmod 5$; $f(v_iv_{i+1}) = 2$ for $i \equiv 0 \pmod 5$. The construction of the coloring f enables us to list the color sets $C\{f,v_i\}$ of the key vertices v_i in the following table

vertices	$C(f,v_i)$	$C\langle f,v_i\rangle$	$C[f,v_i]$	$C_2[f,v_i]$
v_1	$\{1,5\}$	$\{2,3,4\}$	$\{1,3,5\}$	$\{1,2,3,4,5\}$
v_2	$\{1,4\}$	$\{2,3\}$	$\{1,2,4\}$	$\{1,2,3,4\}$
v_3	$\{4,5\}$	$\{2,3,4\}$	$\{3,4,5\}$	$\{2,3,4,5\}$
v_4	$\{1,5\}$	$\{3,4,5\}$	$\{1,4,5\}$	$\{1,3,4,5\}$
v_5	$\{1,2\}$	$\{1,4,5\}$	$\{1,2,5\}$	$\{1,2,4,5\}$
v_6	$\{2,3\}$	$\{1,2,5\}$	$\{1,2,3\}$	$\{1,2,3,5\}$
v_7	$\{3,4\}$	$\{1,2,3\}$	$\{2,3,4\}$	$\{1,2,3,4\}$
v_8	$\{4,5\}$	$\{2,3,4\}$	$\{3,4,5\}$	$\{2,3,4,5\}$
v_9	$\{1,5\}$	$\{3,4,5\}$	$\{1,4,5\}$	$\{1,3,4,5\}$
v_{10}	$\{1,2\}$	$\{1,4,5\}$	$\{1,2,5\}$	$\{1,2,4,5\}$
v_{n-1}	$\{2,3\}$	$\{1,4,5\}$	$\{1,2,3\}$	$\{1,2,3,4,5\}$
v_n	$\{3,5\}$	$\{1,3,4\}$	$\{3,4,5\}$	$\{1,3,4,5\}$

For other vertices of C_n, we have $C(f,v_i) = C(f,v_j)$, $C\langle f,v_i\rangle = C\langle f,v_j\rangle$, $C[f,v_i] = C[f,v_j]$, $C_2[f,v_i] = C_2[f,v_j]$ when $i \equiv j \pmod 5$ and $6 \le i,j \le n-2$. The above deduction shows that f is really a 5-(4)-AVDTC of C_n, that is, $\chi''_{(4)as}(C_n) = 5$.

Case 4. $n \equiv 3 \pmod 5$. We construct a total coloring f of C_n in the way $f(v_1) = 5$, $f(v_2) = 3$, $f(v_3) = 2$, $f(v_{n-1}) = 2$, $f(v_n) = 1$, $f(v_1v_2) = 1$. $f(v_2v_3) = 4$, $f(v_3v_4) = 3$, $f(v_{n-1}v_n) = 4$, $f(v_nv_1) = 2$; for $4 < i \le n-2$, $f(v_i) = 1$ for $i \equiv 1 \pmod 5$; $f(v_i) = 2$ for $i \equiv 2 \pmod 5$; $f(v_i) = 3$ for $i \equiv 3 \pmod 5$; $f(v_i) = 4, i \equiv 4 \pmod 5$;

$f(v_i) = 5$ for $i \equiv 0 \pmod 5$. $f(v_iv_{i+1}) = 3$ for $i \equiv 1 \pmod 5$; $f(v_iv_{i+1}) = 4, i \equiv 2 \pmod 5$; $f(v_iv_{i+1}) = 5$ for $i \equiv 3 \pmod 5$; $f(v_iv_{i+1}) = 1$ for $i \equiv 4 \pmod 5$; $f(v_iv_{i+1}) = 2$ for $i \equiv 0 \pmod 5$. This coloring f gives the following table

vertices	$C(f, v_i)$	$C\langle f, v_i\rangle$	$C[f, v_i]$	$C_2[f, v_i]$
v_1	$\{1,2\}$	$\{1,3,5\}$	$\{1,2,5\}$	$\{1,2,3,5\}$
v_2	$\{1,4\}$	$\{2,3,5\}$	$\{1,3,4\}$	$\{1,2,3,4,5\}$
v_3	$\{3,4\}$	$\{2,3,4\}$	$\{2,3,4\}$	$\{2,3,4\}$
v_4	$\{1,3\}$	$\{2,4,5\}$	$\{1,3,4\}$	$\{1,2,3,4,5\}$
v_5	$\{1,2\}$	$\{1,4,5\}$	$\{1,2,5\}$	$\{1,2,4,5\}$
v_6	$\{2,3\}$	$\{1,2,5\}$	$\{1,2,3\}$	$\{1,2,3,5\}$
v_7	$\{3,4\}$	$\{1,2,3\}$	$\{2,3,4\}$	$\{1,2,3,4\}$
v_8	$\{4,5\}$	$\{2,3,4\}$	$\{3,4,5\}$	$\{2,3,4,5\}$
v_9	$\{1,5\}$	$\{3,4,5\}$	$\{1,4,5\}$	$\{1,3,4,5\}$
v_{10}	$\{1,2\}$	$\{1,4,5\}$	$\{1,2,5\}$	$\{1,2,4,5\}$
v_{n-2}	$\{2,3\}$	$\{1,2,5\}$	$\{1,2,3\}$	$\{1,2,3,5\}$
v_{n-1}	$\{3,4\}$	$\{1,2\}$	$\{2,3,4\}$	$\{1,2,3,4\}$
v_n	$\{2,4\}$	$\{1,2,5\}$	$\{1,2,4\}$	$\{1,2,4,5\}$

For other vertices of C_n, we have $C(f, v_i) = C(f, v_j)$, $C\langle f, v_i\rangle = C\langle f, v_j\rangle$, $C[f, v_i] = C[f, v_j]$, $C_2[f, v_i] = C_2[f, v_j]$ when $i \equiv j \pmod 5$ and $6 \leq i, j \leq n-3$. Therefore, f is a 5-(4)-AVDTC of C_n, and hence $\chi''_{(4)as}(C_n) = 5$.

Case 5. $n \equiv 4 \pmod 5$. We build up a total coloring f of C_n by setting $f(v_{n-1}) = 1$, $f(v_n) = 4$, $f(v_{n-1}v_n) = 2$, $f(v_nv_1) = \{5\}$; for $1 < i \leq n-2$, $f(v_i) = 1$ for $i \equiv 1 \pmod 5$; $f(v_i) = 2$ for $i \equiv 2 \pmod 5$; $f(v_i) = 3$ for $i \equiv 3 \pmod 5$; $f(v_i) = 4$ for $i \equiv 4 \pmod 5$; $f(v_i) = 5$ for $i \equiv 0 \pmod 5$. $f(v_iv_{i+1}) = 3$ for $i \equiv 1 \pmod 5$; $f(v_iv_{i+1}) = 4, i \equiv 2 \pmod 5$; $f(v_iv_{i+1}) = 5$ for $i \equiv 3 \pmod 5$; $f(v_iv_{i+1}) = 1$ for $i \equiv 4 \pmod 5$; $f(v_iv_{i+1}) = 2$ for

$i \equiv 0 \pmod 5$). Thereby, we have

vertices	$C(f,v_i)$	$C\langle f,v_i\rangle$	$C[f,v_i]$	$C_2[f,v_i]$
v_1	$\{3,5\}$	$\{1,2,4\}$	$\{1,3,5\}$	$\{1,2,3,4,5\}$
v_2	$\{3,4\}$	$\{1,2,3\}$	$\{2,3,4\}$	$\{1,2,3,4\}$
v_3	$\{4,5\}$	$\{2,3,4\}$	$\{3,4,5\}$	$\{2,3,4,5\}$
v_4	$\{1,5\}$	$\{3,4,5\}$	$\{1,4,5\}$	$\{1,3,4,5\}$
v_5	$\{1,2\}$	$\{1,4,5\}$	$\{1,2,5\}$	$\{1,2,4,5\}$
v_6	$\{2,3\}$	$\{1,2,5\}$	$\{1,2,3\}$	$\{1,2,3,5\}$
v_7	$\{3,4\}$	$\{1,2,3\}$	$\{2,3,4\}$	$\{1,2,3,4\}$
v_8	$\{4,5\}$	$\{2,3,4\}$	$\{3,4,5\}$	$\{2,3,4,5\}$
v_9	$\{1,5\}$	$\{3,4,5\}$	$\{1,4,5\}$	$\{1,3,4,5\}$
v_{10}	$\{1,2\}$	$\{1,4,5\}$	$\{1,2,5\}$	$\{1,2,4,5\}$
v_{n-3}	$\{2,3\}$	$\{1,2,5\}$	$\{1,2,3\}$	$\{1,2,3,5\}$
v_{n-2}	$\{3,4\}$	$\{1,2\}$	$\{2,3,4\}$	$\{1,2,3,4\}$
v_{n-1}	$\{2,4\}$	$\{1,2,4\}$	$\{1,2,4\}$	$\{1,2,4\}$
v_n	$\{2,5\}$	$\{1,4\}$	$\{2,4,5\}$	$\{1,2,4,5\}$

Clearly, $C(f,v_i) = C(f,v_j)$, $C\langle f,v_i\rangle = C\langle f,v_j\rangle$, $C[f,v_i] = C[f,v_j]$ and $C_2[f,v_i] = C_2[f,v_j]$ when $i \equiv j \pmod 5$ and $6 \le i,j \le n-4$. Therefore, f is a 5-(4)-AVDTC of C_n, thus, $\chi''_{(4)as}(C_n) = 5$.

The proof of the theorem is completed. □

Theorem 2.3 *Let $K_{m,n}$ be a complete bipartite graph with $m \ge n \ge 1$ and $m+n > 3$. Then*

$$\chi''_{(4)as}(K_{m,n}) = \begin{cases} m+1, & \text{if } m-n \ge 2 \text{ or } m > 2 \text{ and } m > n = 1; \\ m+2, & \text{if } m-n = 1; \\ m+4, & \text{if } m = n. \end{cases}$$

Proof We can describe a complete bipartite graph $K_{m,n}$ by its vertex set $V(K_{m,n}) = \{u_1,u_2,\ldots,u_m\} \cup \{v_1,v_2,\ldots,v_n\}$ and its edge set $E(K_{m,n}) = \{u_iv_j : i = 1,2,\ldots,m,\ j = 1,2,\ldots,n\}$. Without loss of generality, suppose that f is a total coloring of $K_{m,n}$ and S is a color set under f. For the purpose of clarity, we write $\overline{C}(f,u) = S \setminus C(f,u)$, $\overline{C}\langle f,u\rangle = S \setminus C\langle f,u\rangle$, $\overline{C}[f,u] = S \setminus C[f,u]$ and $\overline{C_2}[f,u] = S \setminus C_2[f,u]$ for $u \in V(K_{m,n})$.

Case 1. $m - n \geq 2$, or $m > 2$ and $m > n = 1$. It is obvious for case $m > 2$ and $m > n = 1$. Now we consider case $m - n \geq 2$. By Theorem 1.6, we have $\chi''_{(4)as}(K_{m,n}) \geq \chi''_{ast}(K_{m,n}) \geq m + 1$. Now we prove that $K_{m,n}$ has an $(m+1)$-(4)-AVDTC f by setting $f(u_i) = i + 1$ for $i = 1, 2, \ldots, m$; $f(v_j) = 1$ for $j = 1, 2, \ldots, n$; $f(u_iv_j) = i+j+1$ for $i+j+1 \leq m+1$ and $f(u_iv_j) = i+j+1 \pmod{m}$ for $i+j+1 > m+1$ and $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$. Then we have $C(f, u_i) \neq C(f, v_j)$ and $C[f, u_i] \neq C[f, v_j]$ since the degrees $d(u_i) = n$ and $d(v_j) = m$ with $1 \leq i \leq m$ and $1 \leq j \leq n$. According to the definition of f, we obtain $C\langle f, u_i\rangle \neq C\langle f, v_j\rangle$ because $C\langle f, u_i\rangle = \{1, i+1\}$ and $C\langle f, v_j\rangle = \{1, 2, \ldots, m+1\}$; $C_2[f, u_i] \neq C_2[f, v_j]$ for $\overline{C_2}[f, u_1] = S \setminus C_2[f, u_1] = \{m+1\}$, $\overline{C_2}[f, u_i] = S \setminus C_2[f, u_i] = \{i\}$ for $2 \leq i \leq m$, $\overline{C_2}[f, v_j] = S \setminus C_2[f, v_j] = \emptyset$ for $1 \leq j \leq n$, where $S = \{1, 2, \ldots, m+1\}$. Therefore, f is a $m+1$-(4)-AVDTC of $K_{m,n}$, which means $\chi''_{(4)as}(K_{m,n}) = m + 1$.

Case 2. $m - n = 1$. By Theorem 1.6, $\chi''_{(4)as}(K_{m,n}) \geq \chi''_{ast}(K_{m,n}) \geq m + 2$. Without loss of generality, we define a total coloring f of $K_{m,n}$ by defining $f(u_1) = m + 2$, $f(u_i) = 2$ for $i = 2, 3, \ldots, n+1$; $f(v_j) = 1$ for $j = 1, 2, \ldots, n$; $f(u_iv_j) = i + j + 1$ for $i + j + 1 \leq m + 2$ and $f(u_iv_j) = i + j + 1 \pmod{m}$ for $i + j + 1 > m + 2$, $i = 1, 2, \ldots, n+1$ and $j = 1, 2, \ldots, n$. Then we have $C(f, u_i) \neq C(f, v_j)$ and $C[f, u_i] \neq C[f, v_j]$ for $d(u_i) = n$ and $d(v_j) = m$ when $1 \leq i \leq n+1$ and $1 \leq j \leq n$. According to the definition of f, $C\langle f, u_i\rangle \neq C\langle f, v_j\rangle$ since $C\langle f, u_1\rangle = \{1, m+2\}$ and $C\langle f, u_i\rangle = \{1, 2\}$ for $i = 2, \ldots, n+1$ and $C\langle f, v_j\rangle = \{1, 2, m+2\}$ for $j = 1, 2, \ldots, n$. And, $C_2[f, u_i] \neq C_2[f, v_j]$ for $\overline{C_2}[f, u_i] = S \setminus C_2[f, u_i] = \{i+1\}$ for $i = 1, 2, \ldots, n+1$ and $\overline{C_2}[f, v_j] = S \setminus C_2[f, v_j] = \emptyset$ for $j = 1, 2, \ldots, n$, where $S = \{1, 2, \ldots, m+2\}$. Thereby, we claim that f is an $(m+2)$-(4)-AVDTC of $K_{m,n}$, namely, $\chi''_{(4)as}(K_{m,n}) = m + 2$.

Case 3. $m = n$. By Theorem 1.6, $\chi''_{(4)as}(K_{m,m}) \geq \chi''_{ast}(K_{m,m}) = m + 3$. If $\chi''_{(4)as}(K_{m,m}) = m + 3$, then there must exist $C_2[f, u_i] = C_2[f, v_j]$ for some u_i, v_j with $1 \leq i, j \leq m$ under the total coloring f: a contradiction. So $\chi''_{(4)as}(K_{m,m}) \geq m + 4$. We now show $\chi''_{(4)as}(K_{m,m}) \leq m + 4$ by defining a total coloring f as: $f(u_1v_j) = j$ for $1 \leq j \leq m$; $f(u_iv_j) = (i + j) \pmod{m+2}$ for $2 \leq i \leq m$ and $1 \leq j \leq m$; $f(u_1) = m + 1$, $f(u_i) = m + 3$ for $2 \leq i \leq m$;

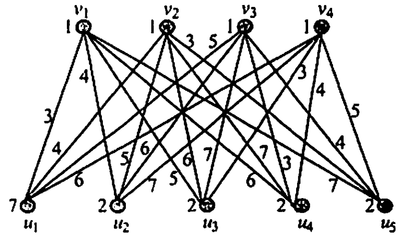

Figure 1: An example for illustrating Case 2 in the proof of Theorem 2.3.

$f(v_j) = m+4$ for $1 \le j \le m$. Thereby, we can compute

(i) $\overline{C}(f,u_1)\setminus\{m+3,m+4\} = \{m+1,m+2\}$, $\overline{C}(f,u_i)\setminus\{m+3,m+4\} = \{i-1,i\}$ for $2 \le i \le m$; $\overline{C}(f,v_1)\setminus\{m+3,m+4\} = \{2,m+2\}$, $\overline{C}(f,v_j)\setminus\{m+3,m+4\} = \{j-1,j+1\}$ for $2 \le j \le m$.

(ii) $C\langle f,u_1\rangle = \{m+1,m+4\}$, $C\langle f,u_i\rangle = \{m+3,m+4\}$ for $2 < i < m$, $C\langle f,v_j\rangle = \{m+1,m+3,m+4\}$ for $1 \le j \le m$.

(iii) $m+4 \notin C[f,u_i]$ for $1 \le i \le m$ and $m+4 \in C[f,v_j]$ for $1 \le j < m$.

(iv) $\overline{C_2}[f,u_1] = \{m+2,m+3\}$, $\overline{C_2}[f,u_i] = \{i-1,i\}$ for $2 \le i \le m$; $\overline{C_2}[f,v_1] = \{2,m+2\}$, $\overline{C_2}[f,v_j] = \{j-1,j+1\}$ for $2 \le j \le m-1$, $\overline{C_2}[f,v_m] = \{m-1\}$.

The above (i), (ii), (iii) and (iv) enable us to conclude that f is really an $(m+4)$-(4)-AVDTC of $K_{m,m}$, and hence $\chi''_{(4)as}(K_{m,m}) = m+4$.

The theorem is covered. □

By some empirical data, we present a problem for further researching vertex distinguishing colorings:

Problem *Let G be a simple, connected graph with $n \ge 3$ vertices. If $N(u) \ne N(v)$ for every edge $uv \in E(G)$, then $\chi''_{(4)as}(G) \le \Delta(G)+4$. Furthermore, whether there exists a proper subgraph H of G such that $\chi''_{(4)as}(H) > \chi''_{(4)as}(G)$.*

Acknowledgment This research was supported by the National Natural Science Foundation of China (No. 61163054, No. 61363060, No. 61163037 and No. 11171114). The authors are grateful to the anonymous referees for their valuable and helpful comments which lead to the improvement of this paper.

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