Permutation Tableaux, Restricted Set Partitions and Labeled Dyck Paths

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Abstract. Permutation tableaux were introduced in the study of totally positive Grassmannian cells, and are connected with the steady state of asymmetric exclusion process which is an important model from statistical mechanics. In this paper, we firstly establish a shape preserving involution on the set of permutation tableaux of length n, which directly shows that the number of permutation tableaux of length n with k essential 1's equals the number of permutation tableaux of length n with $n-k$ unrestricted rows. In addition, we introduce three combinatorial structures, called free permutation tableaux, restricted set partitions and labeled Dyck paths. We discuss the properties about their internal structures and present the correspondence between the set of free permutation tableaux of length n and the set of restricted set partitions of $\{1,2,\ldots,n\}$, and we also give a bijection between the set of restricted set partitions of $\{1,2,\ldots,n\}$ and the set of labeled Dyck paths of length $2n$ and finally make a generalization of the latter bijection.
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1 Introduction

Permutation tableaux were introduced by Steingrímsson and Williams [19] in the study of totally positive Grassmannian cells [12, 16, 21]. They are closely related to the PASEP (partially asymmetric exclusion process) model in statistical physics [4, 7–9]. Permutation tableaux are also in one-to-one correspondence with alternative tableaux, see Viennot [20] and Nadeau [14].

A *permutation tableau* is defined by a Ferrers diagram possibly with empty rows such that the cells are filled with 0's and 1's, and

(1) each column contains at least one 1,

(2) there does not exist a 0 with a 1 above (in the same column) and a 1 to the left (in the same row).

The *length* of a permutation tableau is defined as the number of rows plus the number of columns. A 1 is called *topmost* 1 if it is the topmost 1 of some column. A 1 is said to be *essential* if it is the topmost 1 of some column or it is the leftmost 1 of some row. A 1 is called *superfluous* if it contains a 1 above itself in the same column. A 1 is called *superfluous-essential* 1 (*sup-ess* 1 for short), if it is the leftmost 1 of some row but it is not the topmost 1 of some column. Note that there is at most one sup-ess 1 in a row. A 0 is said to be *top-restricted* if there is a 1 above itself in the same column. Similarly, a 0 is defined as a *left-restricted* 0 if there is a 1 to its left in the same row. A row (resp., column) is *restricted* if it contains a top-restricted 0 (resp., left-restricted 0), otherwise it is called *unrestricted row* (resp., *unrestricted column*). A permutation tableau T of length n is labeled by the elements in the set $[n] = \{1, 2, \ldots, n\}$ in increasing order from the top right corner to the bottom left corner. The set $[n]$ is referred to as the *label set* of T. We use (i, j) to denote the cell with row label i and column label j. The *shape* of a permutation tableau T is defined as the shape of the underlying Ferrers diagram of T with empty rows allowed. In other words, the shape of T is a partition $(\lambda_1, \lambda_2, \ldots, \lambda_k)$, where λ_i is the number of cells in i-th row of the underlying Ferrers diagram of T, for $i = 1, 2, \ldots, k$. For example, Figure 1.1 exhibits a permutation tableau of shape $(5, 5, 4, 2, 1, 0)$ and length 11. There are six essential 1's in cells $(1, 5)$, $(1, 10)$, $(2, 3)$, $(2, 8)$, $(4, 6)$ and $(9, 10)$, but there is a unique sup-ess 1 in cell $(9, 10)$. There are five top-restricted 0's in cells $(2, 10)$, $(4, 8)$, $(4, 10)$, $(7, 8)$ and $(7, 10)$, and four left-restricted 0's in cells $(1, 3)$, $(1, 6)$, $(1, 8)$ and $(2, 6)$. Rows 2, 4 and 7 are restricted rows, columns 3, 6 and 8 are restricted columns, and the other rows or columns are unrestricted.

Corteel and Nadeau [6] found a bijection from permutation tableaux of length n with k columns and permutations of $[n]$ with k descents. Steingrímsson and Williams [19] established a one-to-one correspondence between permutation tableaux of length n with k rows and permutations of $[n]$ with k weak excedances. By using the conjugate operation on permutation tableaux, Corteel and Williams [8] introduced an involution on permutation tableaux (equivalently, on permutations) which generalizes the particle-hole symmetry of the PASEP, and reveals a symmetry in the PT chain, where PT chain is a Markov chain on the permutation tableaux. More details on permutation tableaux and permutations can also see [1,5]. On the other hand, Chen, Liu and Wang [2] gave a bijection between the set of permutation tableaux of length n with k rows and the set of linked partitions of $[n]$ with k blocks.

In this paper, we firstly establish a shape preserving involution on the

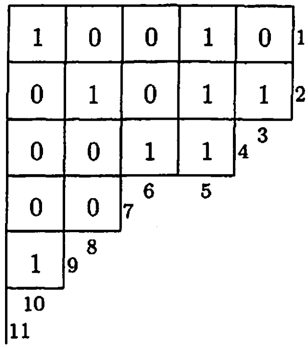

Figure 1.1: A permutation tableau T of length 11.

set of permutation tableaux of length n to directly determine the equidistribution property of the essential 1's and unrestricted rows. In addition, we define a kind of permutation tableaux as free permutation tableaux if they do not contain any top-restricted 0 or sup-ess 1, and we also introduce a new class of set partitions, called restricted set partitions. We discuss the internal structure of restricted set partitions from both the standard representation and the graphic representation. Furthermore, by utilizing the bijection of Chen, Liu and Wang [2], we show that the set of free permutation tableaux of length n is in one-to-one correspondence with the set of restricted set partitions of $[n]$. Finally, we present a labelling scheme on Dyck paths and obtain a bijection between the set of restricted set partitions of $[n]$ and the set of labeled Dyck paths of length $2n$ and make a generalization of the bijection.

2 An Involution on Permutation Tableaux

The objective of this section is to present a simple involution on the set of permutation tableaux of length n which directly show the equidistribution of the essential 1's and unrestricted rows. The involution also deduces some other interesting properties on the internal structure of permutation tableaux.

Let $T(n)$ denote the set of permutation tableaux of length n and let $T(n, k)$ denote the set of permutation tableaux of length n with k essential 1's. Let $T^*(n, k)$ denote the set of permutation tableaux of length n with k unrestricted rows. We have the following involution and the explicit construction is given in the proof.

Theorem 2.1. *For $n \geq 1$ and $0 \leq k \leq n-1$, there is a shape preserving involution φ on the set of permutation tableaux of length n such that*

(1) if $T \in T(n)$ is a permutation tableau of length n, then $T^* = \varphi(T)$ is also a permutation tableau of length n and $\varphi^2(T) = T$;

(2) if $T \in T(n,k)$ is a permutation tableau of length n with k essential 1's, then $T^* = \varphi(T) \in T^*(n, n-k)$ is a permutation tableau of length n with $n-k$ unrestricted rows.

Proof. Given a permutation tableau $T \in T(n)$, we shall first construct a permutation tableau $T^* = \varphi(T) \in T(n)$ of the same shape as T and then we show that (1) and (2) hold for all T and T^*.

If T consists of n empty rows, then we set $T^* = T$. Otherwise, we proceed to construct T^* according to the fillings of each row in T. There are two cases as follows.

Case 1: For any row i of T, if it contains at least one top-restricted 0 but does not contain any sup-ess 1, then we change all the top-restricted 0's in row i to 1's.

For example, row 2 in the permutation tableau shown in Figure 2.1 contains two top-restricted 0's in cells $(2,13)$ and $(2,14)$, but it does not contain any sup-ess 1. Then we change the top-restricted 0's to 1's. Also see row 4.

Case 2: For any row i of T, if it contains at least one sup-ess 1 but does not contain any top-restricted 0, then we change all superfluous 1's in row i to 0's, where the superfluous 1's are to the left of the leftmost topmost 1 that possibly exists in the row. Here a topmost 1 is called the leftmost topmost 1 in row i, if it is the leftmost one among all the topmost 1's in row i.

As an illustration, see row 5 of the permutation tableau $T \in T(15, 11)$ shown in Figure 2.1. There is a sup-ess 1 in cell $(5,14)$ but there does not exist any top-restricted 0 in row 5. Note that it contains two topmost 1's in cells $(5,7)$ and $(5,9)$, where the 1 in cell $(5,9)$ is the leftmost topmost 1 in the row. Then we change the superfluous 1's in cells $(5,11)$, $(5,12)$, $(5,13)$ and $(5,14)$ to 0's, but we do not change the 1 in cell $(5,8)$.

The other rows in T that do not satisfy the conditions in either Case 1 or Case 2 are fixed. Then we have the desired T^*. Note that we interchange top-restricted 0's with superfluous 1's in the two cases and there exist topmost 1's above the top-restricted 0's and superfluous 1's in the same column. Then it is easily known that the result $T^* = \varphi(T)$ is a permutation tableau of length n with the same shape as T and φ must be an involution on $T(n)$ such that $\varphi^2(T) = T$.

Finally we prove the correspondence between the number of essential 1's in T and the number of unrestricted rows in T^*. Note that the restricted rows in T^* are either obtained from the rows in T by changing the superfluous 1's to 0's mentioned in Case 2, or exactly the rows in T containing both sup-ess 1's and top-restricted 0's. That is to say the number of restricted rows in T^* equals the number of sup-ess 1's in T. Moreover, the number of rows of T^* is equal to the number of rows of T and the number of columns of T exactly equals the number of topmost 1's in T. Then the number of unrestricted rows in T^* is equal to the difference between n and the total number of topmost 1's and sup-ess 1's, i.e., the difference between n and the number of essential 1's in T. Hence if $T \in \mathcal{T}(n,k)$, then $T^* = \varphi(T) \in \mathcal{T}^*(n, n-k)$ is a permutation tableau of length n with $n-k$ unrestricted rows. The proof is completed. ∎

For example, Figure 2.1 illustrates the involution φ between a permutation tableau T of length 15 with 11 essential 1's and a permutation tableau T^* of length 15 with 4 unrestricted rows, where the 1's and 0's in bold type are interchanged with each other under the involution φ.

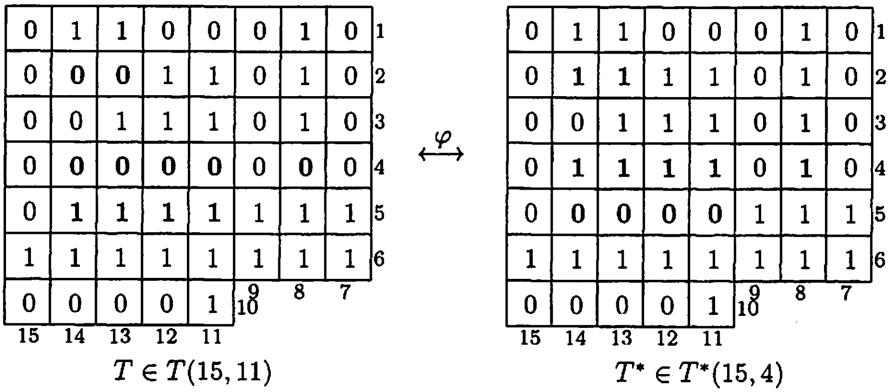

Figure 2.1: Involution φ.

Consequently, we conclude the equidistribution of the essential 1's and unrestricted rows.

Corollary 2.2. *The number of permutation tableaux of length n with k essential 1's is equal to the number of permutation tableaux of length n with $n-k$ unrestricted rows.*

In addition, we have the following properties of φ about the restricted columns.

Corollary 2.3. *Let T be a permutation tableau of length n with columns*

$j_1, j_2, \ldots, j_k$ being restricted. Then columns $j_1, j_2, \ldots, j_k$ are exactly the restricted columns of $T^* = \varphi(T)$.

Proof. First, we show that if columns $j_1, j_2, \ldots, j_k$ are restricted columns in a permutation tableau T, then columns $j_1, j_2, \ldots, j_k$ are also restricted in $T^* = \varphi(T)$.

For any $t = 1, 2, \ldots, k$, the restricted column j_t of T contains at least one left-restricted 0 in cell (i, j_t) for some row i, and there is a 1 to the left of the 0 in row i. Moreover, there does not exist any 1 above the left-restricted 0 and the left-restricted 0 in cell (i, j_t) will not be changed under the action of φ.

If there is a 1 in row i of T to the left of the left-restricted 0 in cell (i, j_t) such that it is the topmost 1 of some column j, where $j > j_t$, then the 1 is also to be the topmost 1 of column j in $T^* = \varphi(T)$, namely the 0 in cell (i, j_t) is also left-restricted in T^* and the column j_t is preserved to be restricted in T^*. Otherwise, if all the 1's in row i of T to the left of the 0 in cell (i, j_t) are superfluous 1's, then we consider the topmost 1's above these superfluous 1's in the same columns. Assume that one of these topmost 1's is in cell (m, j), where $m < i$ and $j > j_t$. Then the 1 in cell (m, j) remains unchanged in T^*, which implies that the 0 in cell (m, j_t) must be left-restricted, i.e., column j_t is also restricted in T^*.

Next, we prove that φ does not increase the number of restricted columns, i.e., if column j of T is unrestricted, then column j of $T^* = \varphi(T)$ is also unrestricted. Note that for any 0 in column j, the entries to the left of the 0 in the same row must be 0's. We choose a 0 in column j randomly and suppose that the 0 is in cell (i, j) for some row i. Then we discuss the entries to the left of it in the following two cases.

Case 1: If there is a top-restricted 0 in row i to the left of the left-restricted 0 in cell (i, j), then the 0 in cell (i, j) is also top-restricted (otherwise there must exist a left-restricted 0 in column j above the 0 in cell (i, j), a contradiction to column j being unrestricted). Thus all the top-restricted 0's in row i are either changed to 1's or unchanged simultaneously under the action of φ, which implies that there does not exist any left-restricted 0 in column j of T^*.

Case 2: If all the 0's in row i to the left of the 0 in cell (i, j) are not top-restricted, then the cells (m, ℓ) are filled with 0's, for any $m \leq i$ and $\ell > j$. Therefore the 0's in row i to the left of cell (i, j) are preserved to be 0's in T^*, i.e., the 0 in cell (i, j) is also not left-restricted in T^*. Since we choose the 0 in column j randomly, we have that column j is still unrestricted in T^*. Therefore there does not exist any left-restricted 0 in column j of $T^* = \varphi(T)$. ∎

From the construction of φ, it is not difficult to see that there exist a number of permutation tableaux T satisfying $\varphi(T) = T$, i.e., the fixed points of φ. Moreover, since $n!$ is even for $n \geq 2$ and φ is an involution on the set of permutation tableaux of length n, we deduce the following conclusion.

Corollary 2.4. *For $n \geq 2$, the number of permutation tableaux T of length n satisfying $\varphi(T) = T$ is even.*

In addition, the following corollary is easily obtained.

Corollary 2.5. *For $n \geq 1$ and $0 \leq k \leq n-1$, let T be a permutation tableau of length n with k essential 1's. If T satisfies $\varphi(T) = T$, then we conclude that*

(a) T also contains $n-k$ unrestricted rows;

(b) either T does not contain any top-restricted 0 or sup-ess 1, or each rows of T contains both of the two types of elements simultaneously.

3 Free Permutation Tableaux

We define a permutation tableau T as a free permutation tableau if T does not contain any top-restricted 0 or sup-ess 1. In this section, we shall consider the correspondence of the internal structure among the free permutation tableaux, restricted set partitions and labeled Dyck paths, where the latter two structures are defined in Section 3.2 and 3.3. To this end, we utilize linked partitions as an intermediate structure between free permutation tableaux and restricted set partitions, and present a bijection between restricted set partitions and labeled Dyck paths, which gives an indirect correspondence between free permutation tableaux and labeled lattice paths.

3.1 Permutation Tableaux and Linked Partitions

Linked partitions arise in the study of certain transforms in free probability theory, see Dykema [11]. A *linked partition* of $[n]$ is a collection of nonempty subsets $B_1, B_2, \ldots, B_k$ of $[n]$, called *blocks*, such that the union of $B_1, B_2, \ldots, B_k$ is $[n]$ and any two distinct blocks are nearly disjoint. Two blocks B_i and B_j are said to be *nearly disjoint* if for any $k \in B_i \cap B_j$, one of the following conditions holds:

(a) $k = \min(B_i), |B_i| > 1$ and $k \neq \min(B_j)$, or

(b) $k = \min(B_j), |B_j| > 1$ and $k \neq \min(B_i)$.

We adopt the *linear representation* of linked partitions, introduced by Chen, Wu and Yan [3]. For a linked partition τ of $[n]$, first we draw n

vertices $1, 2, \ldots, n$ on a horizontal line in increasing order. For each block $B = \{i_1, i_2, \ldots, i_k\}$, we write the elements $i_1, i_2, \ldots, i_k$ in increasing order, and use $\min(B)$ to denote the minimal element i_1 of B. If $k \geq 2$, then we draw an arc joining i_1 and any other vertex in B. We shall use a pair (i, j) to denote an arc between i and j, where we assume that $i < j$ and call i the *left-hand endpoint*, j the *right-hand endpoint*. Two arcs (i_1, j_1) and (i_2, j_2) form a *crossing* if $i_1 < i_2 < j_1 < j_2$, and they form a *nesting* if $i_1 < i_2 < j_2 < j_1$. For example, the linear representation of linked partition $\tau = \{1,3\}\{2,4,5\}\{4,6,7,9\}\{6,8\}\{10\}$ is illustrated in Figure 3.1, and there are a few of crossings formed by $(1,3)$ and $(2,4)$, $(1,3)$ and $(2,5)$, $(2,5)$ and $(4,9)$, etc., while there is only one nesting: $(4,9)$ and $(6,8)$.

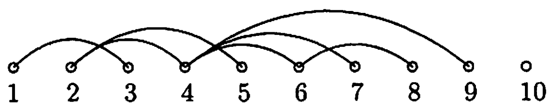

Figure 3.1: The linear representation of $\{1,3\}\{2,4,5\}\{4,6,7,9\}\{6,8\}\{10\}$.

Chen, Liu and Wang [2] gave a classification of vertices in the linear representation of a linked partition. Given a linked partition τ of $[n]$, if a vertex i is only a left-hand endpoint, then i is called an *origin*, or a vertex i is called a *transient* if it is both a left-hand point and a right-hand endpoint, or it is defined as a *singleton* if it is an isolated vertex, or a *destination* if it is only a right-hand endpoint. Figure 3.2 illustrates the four types of elements. We call a singleton k is *covered* by an arc (i, j) if $i < k < j$, and an arc (i, j) is *covered* by another arc (ℓ, m) if $\ell < i < j < m$. It is obvious that the arcs (i, j) and (ℓ, m) form a nesting.

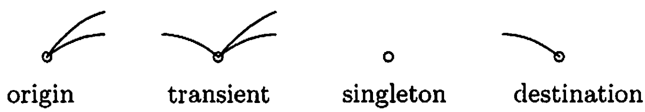

Figure 3.2: Four types of elements in linked partitions.

Chen, Liu and Wang [2, Theorem 3.1] gave a bijection between the set of permutation tableaux of length n and the set of linked partitions of $[n]$. They use a fact proved by Corteel and Nadeau [6] that a permutation tableau is determined by its topmost 1's and rightmost top-restricted 0's, where the rightmost top-restricted 0 is the rightmost top-restricted 0 in a row. We shall utilize the bijection to determine the correspondence between free permutation tableaux and restricted set partitions in Section 3.2.

Theorem 3.1 (Chen, Liu and Wang [2], Theorem 3.1). *For $n \geq 1$ and $1 \leq k \leq n$, there is a bijection ϕ between the set of permutation tableaux of length n with k rows and the set of linked partitions of $[n]$ with k blocks.*

Moreover, they also obtain a conclusion as follows.

Corollary 3.2 (Chen, Liu and Wang [2], Corollary 3.3). *For $0 \leq k \leq n-2$, the number of permutation tableaux of length n with k rightmost top-restricted 0's equals the number of linked partitions of $[n]$ with k transients.*

3.2 Free Permutation Tableaux and Restricted Set Partitions

In this subsection, we introduce a new kind of set partitions of $[n]$, called restricted set partitions, and discuss the correspondence between the set of restricted set partitions of $[n]$ and the set of free permutation tableaux of length n.

A *partition of* $[n]$ is a collection $\pi = \{B_1, B_2, \ldots, B_k\}$ such that

(1) for any $i = 1, 2, \ldots, k$, $B_i \neq \emptyset$;

(2) for any $i, j = 1, 2, \ldots, k$, if $i \neq j$, then $B_i \cap B_j = \emptyset$;

(3) $B_1 \cup B_2 \cup \cdots \cup B_k = [n]$.

It is easily known that an ordinary set partition of $[n]$ is exactly a linked partition of $[n]$ without any transient. Then we adopt the linear representation and the classification of vertices (except for the transients) defined in Section 3.1 to give a graphic representation for the ordinary set partitions. More information about set partitions, one could see Stanley [18].

A *restricted set partition* is an ordinary set partition of $[n]$ satisfying both of the following conditions:

(1) *Singleton-covered-avoiding condition* (*SCA condition* for short): if there is a singleton k in τ, then there does not exist any arc covering k.

(2) *Nesting-destroyed condition* (*ND condition* for short): if two arcs (i_1, j_1) and (i_2, j_2) form a nesting, where $i_1 < i_2 < j_2 < j_1$, then there must exist an arc (i_2, k) satisfying $k > j_1$. See Figure 3.3.

For example, Figure 3.4 illustrates a restricted set partition π of $\{1, 2, \ldots, 11\}$. Note that the unique singleton 9 of π is not covered by any arc, i.e., π satisfies the SCA condition, and there is a nesting in π formed by arcs $(3,7)$ and $(4,6)$, but there also exists the arc $(4,8)$ such that π satisfies the ND condition. For convenience, we call the structure described in ND condition shown in Figure 3.3 the *ND structure*.

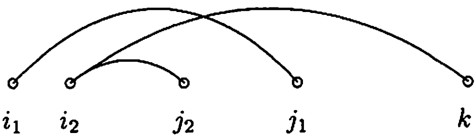

Figure 3.3: The ND structure.

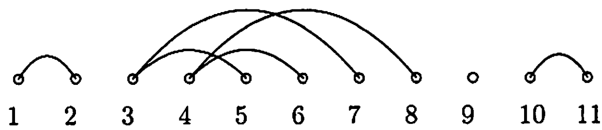

Figure 3.4: A restricted set partition $\pi = \{1,2\}\{3,5,7\}\{4,6,8\}\{9\}\{10,11\}$.

The SCA condition and ND condition present the restrictions on the linear representation of a restricted set partition. Here we have an equivalent and simpler definition for restricted set partitions. Given a set A, let $\max(A)$ and $\min(A)$ denote the maximum and minimum elements of A, respectively. Then we have the following result.

Theorem 3.3. *Let π be a partition of $[n]$ with m blocks $B_1, B_2, \ldots, B_m$ such that*

$$\max(B_1) < \max(B_2) < \cdots < \max(B_m). \tag{3.1}$$

Then π is a restricted set partition if and only if

$$\min(B_1) < \min(B_2) < \cdots < \min(B_m). \tag{3.2}$$

Proof. First, we prove that if a partition $\pi = \{B_1, B_2, \ldots, B_m\}$ of $[n]$ satisfies both (3.1) and (3.2), then π must be a restricted set partition, namely we prove that π satisfies both SCA condition and ND condition.

We prove this by contradiction. Suppose π contains a singleton k, where $1 < k < n$, and there is an arc (i,j) covering k with $1 \leq i < k < j \leq n$. Let $B_s = \{k\}$ and $\{i,j\} \in B_t$. Then $\min(B_s) = \max(B_s) = k$ and $i = \min(B_t)$, $j \leq \max(B_t)$. Since $i < k < j$ and all the blocks $B_1, B_2, \ldots, B_m$ of π satisfy (3.1), we have $s < t$ and $k = \max(B_s) < j \leq \max(B_t)$, but $k = \min(B_s) > i = \min(B_t)$, a contradiction to (3.2). Hence SCA condition holds. In addition, suppose that there are two arcs (i_1, j_1) and (i_2, j_2) forming a nesting with $i_1 < i_2 < j_2 < j_1$ such that there does not exist any arc (i_2, k) with $k > j_1$. Let $\{i_1, j_1\} \in B_t$ and $\{i_2, j_2\} \in B_s$. Then $s < t$, $j_2 \leq \max(B_s) < j_1 \leq \max(B_t)$, but $i_2 = \min(B_s) > i_1 = \min(B_t)$, a contradiction to (3.2). Therefore π also satisfies the ND condition. Above all, π is a restricted set partition of $[n]$.

On the other hand, we prove that if $\pi = \{B_1, B_2, \ldots, B_m\}$ is a restricted set partition and it satisfies (3.1), then π also satisfies (3.2). By contradiction, we assume that there exists a label $i \in \{1, 2, \ldots, m-1\}$ satisfying $\min(B_i) > \min(B_{i+1})$. Then since $\max(B_i) < \max(B_{i+1})$, B_i and B_{i+1} cannot be singleton blocks at the same time. Let $B_i = \{a_1, a_2, \ldots, a_k\}$, where $k \geq 1$ and $a_2 < a_3 < \cdots < a_k < a_1$ and let $B_{i+1} = \{b_1, b_2, \ldots, b_\ell\}$, where $\ell \geq 1$ and $b_2 < b_3 < \cdots < b_\ell < b_1$. From (3.1), we have $a_1 = \max(B_i) < b_1 = \max(B_{i+1})$. First, we assume $\ell \geq 2$. If $k = 1$, i.e., $B_i = \{a_1\}$ is a singleton block, then we have $a_1 < b_1$ and $a_1 = \min(B_i) > b_2 = \min(B_{i+1})$. That is to say that there exists an arc (b_2, b_1) covering the singleton a_1, a contradiction to the SCA condition. Thus $k \geq 2$, namely neither B_i nor B_{i+1} is a singleton block. Then it is easily known that $a_2 = \min(B_i) > b_2 = \min(B_{i+1})$ and $a_1 = \max(B_i) < b_1 = \max(B_{i+1})$, i.e., $b_2 < a_2 < a_1 < b_1$, which implies that in the linear representation of π, the arc (b_2, b_1) covers the arc (a_2, a_1) but there does not exist any arc (a_2, j) with $j > b_1$, a contradiction to the ND condition. Finally, there is only one case left: $k \geq 2$ and $\ell = 1$. Let $B_{i+1} = \{b_1\}$. Note that $a_1 = \max(B_i) < b_1 = \max(B_{i+1})$ and $a_2 = \min(B_i) > b_1 = \min(B_{i+1})$, i.e., $a_1 < b_1 < a_2$, a contradiction to $a_2 < a_1$. Therefore if π is a restricted set partition of $[n]$ satisfying (3.1), then π also satisfies (3.2).

By utilizing the bijection ϕ in Theorem 3.1 introduced by Chen, Liu and Wang [2], we directly obtain a one-to-one correspondence between free permutation tableaux and restricted set partitions.

Theorem 3.4. *For $n \geq 1$, the map ϕ is a bijection between the set of free permutation tableaux of length n and the set of restricted set partitions of $[n]$.*

Proof. Let T be a free permutation tableau of length n, namely T does not contain any top-restricted 0 or sup-ess 1, and let $\pi = \phi(T)$ is a linked partition of $[n]$. First, we prove that $\pi = \phi(T)$ is a restricted set partition of $[n]$, i.e., π satisfies both the SCA condition and the ND condition. We prove it by contradiction.

Suppose that there is a singleton k in $\pi = \phi(T)$ and there exists an arc covering k. Let (i, j) be the arc covering k with j being maximum. Then based on the construction of the map ϕ^{-1} in Chen, Liu and Wang [2, Theorem 3.1] from linked partitions to permutation tableaux, we surely obtain a permutation tableau $\phi^{-1}(\pi)$ containing a sup-ess 1 in cell (k, j), a contradiction. Therefore SCA condition holds.

Moreover, we assume that there are two arcs (i_1, j_1) and (i_2, j_2) forming a nesting in the linear representation of $\pi = \phi(T)$, where $i_1 < i_2 < j_2 < j_1$. Here we choose the arc (i_1, j_1) such that (i_1, j_1) covers the arc (i_2, j_2) and j_1 is maximum. It is easy to see that there must be a sup-ess 1 in cell (i_2, j_1) in the permutation tableau $\phi^{-1}(\pi)$, a contradiction. To avoid the existence of the sup-ess 1 in cell (i_2, j_1), we can only add a topmost 1 in row i_2 and to the left of cell (i_2, j_1), namely there must exist an arc (i_2, k) with $k > j_1$. Thus the ND condition holds.

On the other hand, from the proof mentioned above, it can be said with certainty that given any restricted set partition π of $[n]$, the permutation tableau $T = \phi^{-1}(\pi)$ must be a free permutation tableau of length n. The proof is completed. ∎

For example, Figure 3.5 illustrates the corresponding free permutation tableau of length 11 for the restricted set partition π in Figure 3.4.

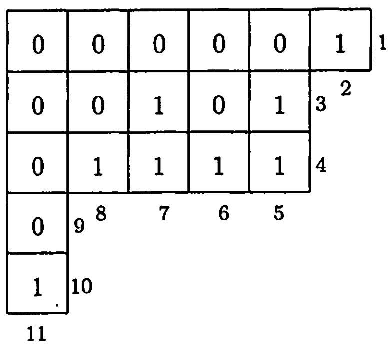

Figure 3.5: The corresponding free permutation tableau of π shown in Figure 3.4.

Furthermore, given a restricted set partition π of $[n]$, we find that the ND structure in the linear representation of π exactly corresponds to a "triangle" in the free permutation tableau $T = \phi^{-1}(\pi)$. Here a triangle in a free permutation tableau is defined as three cells $(i_1, j_1), (i_2, j_2)$ and (i_2, k), where $i_1 < i_2 < j_2 < j_1 < k$, each of which is filled with a topmost 1. For example, the ND structure in the restricted set partition shown in Figure 3.4 is formed by arcs $(3, 7)$, $(4, 6)$ and $(4, 8)$, whose corresponding triangle in the free permutation tableau shown in Figure 3.5 consists of three cells $(3, 7)$, $(4, 6)$ and $(4, 8)$ filled with topmost 1's. Thus we have a conclusion as follows.

Corollary 3.5. *The number of triangles in a free permutation tableau T equals the number of ND structures in the restricted set partition $\pi = \phi(T)$.*

3.3 Restricted Set Partitions and Labeled Dyck Paths

A *Dyck path of length* $2n$ is a lattice path on the plane from the origin $(0,0)$ to $(2n,0)$ consisting of up steps $u = (1,1)$ and down steps $d = (1,-1)$ such that the path does not go across the x-axis. It is well-known that the number of Dyck paths of length $2n$ equals the n-th Catalan number C_n, the sequence A000108 in OEIS [15]. In this part, we introduce a kind of labeled Dyck paths and present a bijection between the set of restricted set partitions of $[n]$ and the set of labeled Dyck paths of length $2n$ and we make a generalization of the bijection.

For $n \geq 1$, given a Dyck path P, we call a pair of two successive steps ud in P a *peak*. Suppose there are k peaks in P, $k \geq 1$, then we denote the *maximal successive segments of up steps* in P from left to right by $U_1, U_2, \ldots, U_k$, where U_1 starts at the origin $(0,0)$ and it is followed by the leftmost down step of P, but for any $i = 2, \ldots, k$, the step immediately followed by U_i and the step immediately following U_i are both down steps. Similarly, we denote the *maximal successive segments of down steps* in P from left to right by $D_1, D_2, \ldots, D_k$.

The *labeled Dyck paths* are defined as follows. Given a Dyck path of length $2n$, we use the alphabet $\{1, 2, \ldots, n\}$ to label all the up steps from left to right by increasing order. For all the down steps, if a down step is the down step of a peak ud, then we label the down step d with the same label as the up step u; otherwise we label the down steps from left to right by the labels in $\{1, 2, \ldots, n\}$ that has not been occupied by any down step such that the following conditions hold:

(C1) For any maximal successive segment D_i of down steps, let $D_i = d_1 d_2 \cdots d_j$ consist of j down steps from left to right with labels a_1, a_2, …, a_j, respectively. The labels $a_1, a_2, \ldots, a_j$ must satisfy

$$a_2 < a_3 < \cdots < a_j < a_1.$$

(C2) Let $D_i = d_1 d_2 \cdots d_j$ consisting of j down steps from left to right with labels $a_1, a_2, \ldots, a_j$, respectively, and let $D_{i+1} = d'_1 d'_2 \cdots d'_\ell$ consisting of ℓ down steps from left to right with labels $b_1, b_2, \ldots, b_\ell$, respectively. We demand the labels of D_i and D_{i+1} satisfy

$$a_1 < b_1 \text{ and } a_2 < b_2.$$

For example, Figure 3.6 illustrates a labeled Dyck path of length 26.

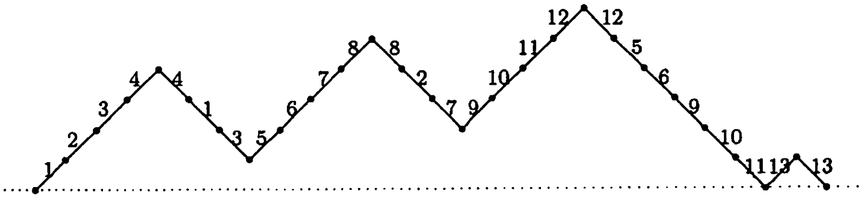

Figure 3.6: A labeled Dyck path of length 26.

Our labelling scheme of Dyck paths generalizes the one introduced by Du [10]. Moreover, from the conditions (C1) and (C2), we deduce the following interesting restriction on labeled Dyck paths.

Proposition 3.6. *Let P be a labeled Dyck paths. Then P avoids uudu, namely P does not contain any successive segment consisting of four steps u, u, d and u from left to right.*

Proof. We shall prove it by contradiction. Assume that the labeled Dyck path P contains a successive segment $uudu$. For the sake of convenience, we denote the segment $uudu$ by $u_1u_2d'u_3$. See Figure 3.7. According to the labeling scheme of Dyck paths, we suppose that the four steps u_1, u_2, d' and u_3 are labeled by $a, a+1, a+1$ and $a+2$, respectively, where $1 \leq a \leq n-2$. Denote the leftmost maximum successive segment of down steps to the right of $u_1u_2d'u_3$ by $D_j = d_1d_2 \cdots d_k$ with labels $b_1, b_2, \ldots, b_k$, respectively.

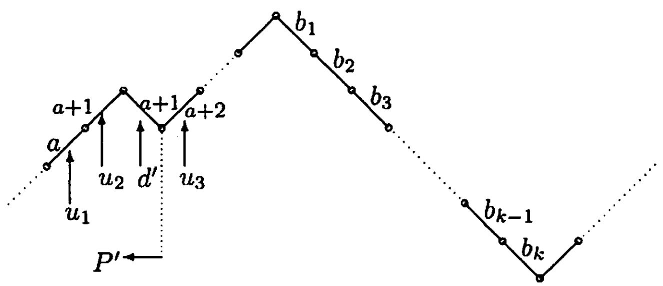

Figure 3.7: The labeled Dyck path P.

Based on (C1) and (C2), we have $b_2 < b_3 < \cdots < b_k < b_1$, $a+1 < b_1$ and $a+1 < b_2$. Let P' be the subpath of P starting at the origin $(0,0)$ and ending with d'. Since P' contains u_1u_2d' and P is a labeled Dyck path that does not go across the x-axis, we conclude that the number of up steps in

P' is at least one more than the number of down steps in P', which implies that the number of the labels occupied by up steps in P' is at least one more than the number of the labels of down steps in P'. Thus there exists a label $t \in \{1, 2, \ldots, a\}$ that has not been used by any down step of P'. Note that $t \leq a < a + 1 < b_2$ and b_2 is the minimum label among all the down steps to the right of u_3. That is to say that the label t does not been used in the whole labeled Dyck path P, a contradiction. ∎

More details about Dyck paths avoiding *uudu* are discussed in Mansour [13] and Sapounakis, Tansoulas and Tsikouras [17]. Now we present the main result of this part, a one-to-one correspondence between labeled Dyck paths and restricted set partitions.

Theorem 3.7. *For $n \geq 1$, there is a bijection ψ between the set of labeled Dyck paths of length $2n$ and the set of restricted set partitions of $[n]$.*

Proof. Let P be a labeled Dyck path of length $2n$. We construct a restricted set partition $\pi = \psi(P)$ such that if π satisfies (3.1), then π also satisfies (3.2).

Assume that P has m peaks and then it has m maximal successive segments of down steps, denoted by $D_1, D_2, \ldots, D_m$. For any $i = 1, 2, \ldots, m$, if $D_i = d_1 d_2 \cdots d_k$ consisting of k down steps from left to right with labels $a_1, a_2, \ldots, a_k$, respectively, then we set the i-th block of π as $B_i = \{a_2, a_3, \ldots, a_k, a_1\}$. Thus the set partition $\pi = \{B_1, B_2, \ldots, B_m\}$ of $[n]$ is obtained. Since the labels of down steps of P satisfy the conditions (C1) and (C2), π satisfies both (3.1) and (3.2), which implies that π is a restricted set partition of $[n]$. For example, the labeled Dyck path shown in Figure 3.6 corresponds to the restricted set partition

$$\pi = \{1, 3, 4\}\{2, 7, 8\}\{5, 6, 9, 10, 11, 12\}\{13\}.$$

See Figure 3.8.

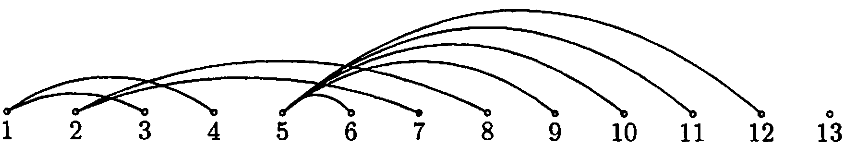

Figure 3.8: $\pi = \psi(P) = \{1, 3, 4\}\{2, 7, 8\}\{5, 6, 9, 10, 11, 12\}\{13\}$.

The inverse map of ψ can be described as follows. Let $\pi = \{B_1, \ldots, B_m\}$ be a restricted set partition of $[n]$ with m blocks satisfying both (3.1) and

(3.2). We construct a labeled Dyck path P of length $2n$ such that $P = \psi^{-1}(\pi)$. For any $i = 1, 2, \ldots, m$, we reorder the elements of B_i as $B_i = \{i_1, i_2, \ldots, i_k\}$ such that $i_2 < i_3 < \cdots < i_k < i_1$, where $k \geq 1$. First, we use B_i to construct the i-th maximal successive segment D_i of down steps of P. Let $D_i = d_1 d_2 \cdots d_k$ consisting of k down steps. We label d_j with i_j, for any $j = 1, 2, \ldots, k$. Let D_0 be an empty segment consisting of the origin $(0, 0)$ with label 0. For any $i = 0, 1, 2, \ldots, m-1$, suppose that the leftmost steps of D_i and D_{i+1} are labeled by a and b, respectively, where $1 \leq a < b \leq n$. Then we set $U_{i+1} = u_1 u_2 \cdots u_{b-a}$ consisting of $b - a$ up steps with labels $a + 1$, $a + 2$, ..., b, respectively. Finally, we set

$$P = U_1 D_1 U_2 D_2 \cdots U_m D_m$$

by joining the last vertex of U_i and the first vertex of D_i, for $i = 1, \ldots, m-1$.

Note that the up steps of P are labeled with $\{1, 2, \ldots, n\}$ from left to right by increasing order. Moreover, for any $i = 1, 2, \ldots, m$, we suppose that the up step followed by the maximal successive segment of down steps D_i is labeled by t. Then the leftmost down step of D_i is also labeled by t, and t is the maximal label among all the down steps in $D_1, D_2, \ldots, D_i$, which implies that the total number of down steps in $D_1, D_2, \ldots, D_i$ is less than or equal to t. That is to say that the number of up steps is always greater than or equal to the number of down steps in the subpath ending with D_i, for any $i = 1, 2, \ldots, m$, namely P forms a Dyck path of length $2n$ without going across the x-axis. In addition, since π is a partition of $[n]$ and its blocks $\{B_1, B_2, \ldots, B_m\}$ satisfy both (3.1) and (3.2), the labels of up steps and down steps of P satisfy the labelling scheme of Dyck paths. Especially, the labels of down steps satisfy both (C1) and (C2). Therefore P is the desired labeled Dyck path of length $2n$. The proof is completed.

Furthermore, we deduce the following corollaries.

Corollary 3.8. *For $n \geq 1$ and $1 \leq m \leq n$, the map ψ is a bijection between the set of labeled Dyck paths of length $2n$ with m peaks and the set of restricted set partitions of $[n]$ with m blocks.*

Corollary 3.9. *For $n \geq 1$ and $1 \leq m \leq n$, the composition of $\phi \circ \psi^{-1}$ is a bijection between the set of free permutation tableaux of length n with m rows and the set of labeled Dyck paths of length $2n$ with m peaks.*

From the construction of ψ, we generally obtain that ψ also gives the one-to-one correspondence between the ordinary set partitions and general labeled Dyck paths, where a general labeled Dyck path is a labeled Dyck path but without the restriction $a_2 < b_2$ in (C2). Specifically, we conclude it as follows.

Corollary 3.10. *The bijection ψ is also a bijection between the set of ordinary set partitions of $[n]$ and the set of general labeled Dyck path of length $2n$.*

For example, Figure 3.9 illustrates the corresponding general labeled Dyck path of length 18 for the ordinary set partition

$$\pi = \{3\}\{4,2\}\{5,1\}\{8,7\}\{9,6\},$$

where we order the five blocks of π by their maximal elements by increasing order from left to right, and its standard representation mentioned in Section 3.1 is $\pi = \{1,5\}\{2,4\}\{3\}\{6,9\}\{7,8\}$.

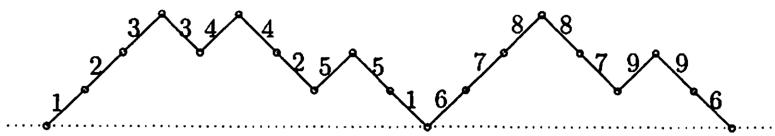

Figure 3.9: A general labeled Dyck path.

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