ON ALGEBRAIC IDENTITIES ON A NEW INTEGER SEQUENCE WITH FOUR PARAMETERS

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ABSTRACT. In this work, we define a new integer sequence related to Fibonacci and Pell sequences with four parameters and then derived some algebraic identities on it including, the sum of first non-zero terms, recurrence relations, rank of its terms, powers of companion matrix and the limit of cross-ratio of four consecutive terms of it.

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1. PRELIMINARIES.

Fibonacci, Lucas and Pell numbers and their generalizations arise in the examination of various areas of science and art. In fact, these numbers are special cases of a sequence which is defined as a linear combination as follows:

$$a_{n+k} = c_1 a_{n+k-1} + c_2 a_{n+k-2} + \cdots + c_k a_n, \tag{1.1}$$

where $c_1, c_2, \cdots, c_k$ are real constants. The applications and identities related with them can be seen in [1, 3, 4, 6, 7, 8, 12]. Fibonacci numbers form a sequence defined by the following recurrence relation: $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. The first Fibonacci numbers are $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \cdots$ (sequence A000045 in OEIS). The characteristic equation of F_n is $x^2 - x - 1 = 0$ and hence the roots of it are $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Johannes Kepler pointed out that the ratio of consecutive Fibonacci numbers converges to the golden ratio as the limit, that is, $\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \alpha$. Like every sequence defined by linear recurrence, the Fibonacci numbers F_n have a closed-form solution. It has become known as Binet's formula $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ for $n \geq 0$. Lucas numbers L_n [3, 4, 7, 11] are defined by $L_0 = 2, L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. The first Lucas numbers are $2, 1, 3, 4, 7, 11, 18, 29, 47, 76, \cdots$ (sequence A000032 in OEIS). There are a lot of algebraic identities between Fibonacci and Lucas numbers. Some of them can be given as follows: $L_n = F_{n-1} + F_{n+1}$, $F_{m+n} = \frac{F_m L_n + L_m F_n}{2}$, $L_n^2 - 5F_n^2 = 4(-1)^n$, $F_{m-n} = \frac{(-1)^n (F_m L_n - L_m F_n)}{2}$ and $F_{2n} = F_n L_n$. The Pell numbers are defined by the recurrence relation $P_0 = 0$, $P_1 = 1$ and $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$. The first few terms of the sequence are $0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378 \cdots$ (sequence A000129 in OEIS). Some identities for Pell numbers can be found in [2, 5, 10, 13].

2. Results.

In [11, 12, 13, 14], the first and second authors derived some nice results on Lucas, Fibonacci, Pell sequences and balancing and oblong numbers involving the Pell equation, respectively. In this section, we aim to define a new integer sequence related to Fibonacci and Pell sequences with four parameters and then derive some algebraic identities on it. For this reason we set $T_0 = 0, T_1 = 0$, $T_2 = -3, T_3 = 12$ and

$$T_n = -5T_{n-1} - 5T_{n-2} + 2T_{n-3} + 2T_{n-4} \tag{2.1}$$

for $n \geq 4$. The characteristic equation of (2.1) is $x^4 + 5x^3 + 5x^2 - 2x - 2 = 0$ and hence the roots of it are

$$\alpha = \frac{-1+\sqrt{5}}{2},\ \beta = \frac{-1-\sqrt{5}}{2},\ \gamma = -2+\sqrt{2} \text{ and } \delta = -2-\sqrt{2}. \tag{2.2}$$

Then we can give the following theorems.

Theorem 2.1. *The generating function for T_n is*

$$T(x) = \frac{-3x^3 - 3x^2}{-2x^4 - 2x^3 + 5x^2 + 5x + 1}.$$

Proof. The generating function $T(x)$ should be the function whose formal power series expansion at $x = 0$ has the form

$$T(x) = \sum_{n=0}^{\infty} T_n x^n = T_0 + T_1 x + T_2 x^2 + \cdots + T_n x^n + \cdots.$$

Hence we get

$$\begin{aligned}(1 + 5x + 5x^2 - 2x^3 - 2x^4)T(x) &= T_0 + (T_1 + 5T_0)x + (T_2 + 5T_1 + 5T_0)x^2 \\ &\quad + (T_3 + 5T_2 + 5T_1 - 2T_0)x^3 \\ &\quad + (T_4 + 5T_3 + 5T_2 - 2T_1 - 2T_0)x^4 \\ &\quad + (T_5 + 5T_4 + 5T_3 - 2T_2 - 2T_1)x^5 + \cdots \\ &\quad + (T_n + 5T_{n-1} + 5T_{n-2} - 2T_{n-3} - 2T_{n-4})x^n \\ &\quad + \cdots.\end{aligned}$$

Since $T_0 = 0, T_1 = 0, T_2 = -3, T_3 = 12$ and $T_n = -5T_{n-1} - 5T_{n-2} + 2T_{n-3} + 2T_{n-4}$, we deduce that $(-2x^4 - 2x^3 + 5x^2 + 5x + 1)T(x) = -3x^3 - 3x^2$ and hence the result is obvious. □

Theorem 2.2. *Let T_n denote the $n-$th number. Then the Binet-like formula for T_n is*

$$T_n = \left(\frac{\gamma^n - \delta^n}{\gamma - \delta}\right) - \left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)$$

for $n \geq 0$, where α, β, γ and δ are defined in (2.2).

Proof. In above theorem, we find that the generating function for T_n is

$$T(x) = \frac{-3x^3 - 3x^2}{-2x^4 - 2x^3 + 5x^2 + 5x + 1}.$$

Since $-2x^4 - 2x^3 + 5x^2 + 5x + 1 = (2x^2 + 4x + 1)(-x^2 + x + 1)$, we can rewrite $T(x)$ as

$$T(x) = \frac{x}{2x^2 + 4x + 1} - \frac{x}{-x^2 + x + 1}. \tag{2.3}$$

It is known in [9] that the generating function for the sequence $U_n = PU_{n-1} - QU_{n-2}$ with parameters P and Q, defined as $U_0 = 0, U_1 = 1$ is

$$U(x)(P, Q) = \frac{x}{1 - Px + Qx^2}.$$

Hence for $P = -4, Q = 2$ and for $P = -1, Q = -1$, we get

$$U(x)(-4, 2) = \frac{x}{2x^2 + 4x + 1} \quad \text{and} \quad U(x)(-1, -1) = \frac{x}{-x^2 + x + 1}. \tag{2.4}$$

(2.3) and (2.4) yield that $T(x) = U(x)(-4, 2) - U(x)(-1, -1)$. So

$$T_n = \left(\frac{\gamma^n - \delta^n}{\gamma - \delta}\right) - \left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)$$

as we wanted. □

Theorem 2.3. *Let T_n denote the $n-$th number. Then the sum of first non-zero terms of T_n is*

$$\sum_{i=1}^{n} T_i = \frac{6T_n + T_{n-1} - 4T_{n-2} - 2T_{n-3} - 6}{7}. \tag{2.5}$$

Proof. Notice that $T_n = -5T_{n-1} - 5T_{n-2} + 2T_{n-3} + 2T_{n-4}$. So

$$5T_{n-1} + 5T_{n-2} = 2T_{n-3} + 2T_{n-4} - T_n. \tag{2.6}$$

Applying (2.6), we deduce that

$$\begin{aligned} 5T_3 + 5T_2 &= 2T_1 + 2T_0 - T_4 \\ 5T_4 + 5T_3 &= 2T_2 + 2T_1 - T_5 \end{aligned}$$

$$
\begin{array}{rcl}
5T_5+5T_4 & = & 2T_3+2T_2-T_6 \\
(2.7) \qquad & & \cdots \\
5T_{n-2}+5T_{n-3} & = & 2T_{n-4}+2T_{n-5}-T_{n-1} \\
5T_{n-1}+5T_{n-2} & = & 2T_{n-3}+2T_{n-4}-T_n.
\end{array}
$$

If we sum of both sides of (2.7), then we obtain $5T_2+5T_{n-1}+10(T_3+T_4+\cdots+T_{n-2}) = -(T_4+T_5+\cdots+T_{n-1}+T_n)+4(T_1+T_2+\cdots+T_{n-5}+T_{n-4})+2T_{n-3}+2T_0$ and hence

$$
\begin{aligned}
T_1+T_2+T_3+T_4+\cdots+T_{n-1}+T_n = \; & 2T_{n-3}+2T_0-5T_2-5T_{n-1} \\
& +T_1+T_2+T_3+T_1+T_2+T_3 \\
& +T_4+\cdots+T_{n-5}+T_{n-4} \\
& -10(T_3+T_4+\cdots+T_{n-4}) \\
& -10(T_{n-3}+T_{n-2}).
\end{aligned}
$$

Adding $-T_{n-1}+T_2-6T_n$ to both sides of above equation, we conclude that

$$T_1+T_2+\cdots+T_n = \frac{6T_n+T_{n-1}-4T_{n-2}-2T_{n-3}-6}{7}.$$

This completes the proof. □

From the above theorem, we can give the following two theorems which can be proved similarly.

Theorem 2.4. *Let T_n denote the $n-th$ number.*

(1) *If n is even, then*

$$\sum_{i=0}^{\frac{n-2}{2}} T_{2i} = \frac{-4T_n-17T_{n-1}-2T_{n-2}+6T_{n-3}-3}{7}$$

$$\sum_{i=0}^{\frac{n}{2}} T_{2i+1} = \frac{-32T_n-17T_{n-1}+12T_{n-2}+6T_{n-3}-3}{7}.$$

(2) *If n is odd, then*

$$\sum_{i=0}^{\frac{n+1}{2}} T_{2i} = \frac{-32T_n-17T_{n-1}+12T_{n-2}+6T_{n-3}-3}{7}$$

$$\sum_{i=0}^{\frac{n-3}{2}} T_{2i+1} = \frac{-4T_n-17T_{n-1}-2T_{n-2}+6T_{n-3}-3}{7}$$

for $n \geq 3$.

Theorem 2.5. *Let T_n denote the $n-$th number.*

(1) *If n is even, then*

$$\sum_{i=0}^{\frac{n}{2}} T_{2i+1} - \sum_{i=0}^{\frac{n-2}{2}} T_{2i} = -4T_n + 2T_{n-2}$$

(2) *If n is odd, then*

$$\sum_{i=0}^{\frac{n-3}{2}} T_{2i+1} - \sum_{i=0}^{\frac{n+1}{2}} T_{2i} = 4T_n - 2T_{n-2}$$

for $n \geq 3$.

Theorem 2.6. *Let T_n denote the $n-$th number. Then the recurrence relations on the terms of T_n are*

$$T_{2n} = 15T_{2n-2} - 41T_{2n-4} + 24T_{2n-6} - 4T_{2n-8}$$

and

$$T_{2n+1} = 15T_{2n-1} - 41T_{2n-3} + 24T_{2n-5} - 4T_{2n-7}$$

for $n \geq 4$.

Proof. Notice that $T_{2n} = -5T_{2n-1} - 5T_{2n-2} + 2T_{2n-3} + 2T_{2n-4}$. So

$$\begin{aligned}
T_{2n} &= -5T_{2n-1} - 5T_{2n-2} + 2T_{2n-3} + 2T_{2n-4} \\
&= -5(-5T_{2n-2} - 5T_{2n-3} + 2T_{2n-4} + 2T_{2n-5}) \\
&\quad -5(-5T_{2n-3} - 5T_{2n-4} + 2T_{2n-5} + 2T_{2n-6}) \\
&\quad +2(-5T_{2n-4} - 5T_{2n-5} + 2T_{2n-6} + 2T_{2n-7}) + 2T_{2n-4} \\
&= 25T_{2n-2} + 25T_{2n-3} - 10T_{2n-4} - 10T_{2n-5} + 25T_{2n-3} + 25T_{2n-4} \\
&\quad -10T_{2n-5} - 10T_{2n-6} - 8T_{2n-4} - 10T_{2n-5} + 4T_{2n-6} + 4T_{2n-7} \\
&= 25T_{2n-2} - 10(-5T_{2n-3} - 5T_{2n-4} + 2T_{2n-5} + 2T_{2n-6}) - 10T_{2n-5} \\
&\quad -43T_{2n-4} + 14T_{2n-6} + 4T_{2n-7} - 10T_{2n-6} + 10T_{2n-6} - 4T_{2n-8} \\
&= 15T_{2n-2} - 41T_{2n-4} + 24T_{2n-6} - 4T_{2n-8}.
\end{aligned}$$

The other assertion can be proved similarly. □

Theorem 2.7. *Let T_n denote the $n-$th number. Then*

(1) $\alpha^n + \beta^n = (-1)^n(2F_n + F_{n-3})$ *for $n \geq 3$.*
(2) $\alpha^n - \beta^n = (-1)^{n+1}F_n\sqrt{5}$ *for $n \geq 0$.*

(3)

$$\gamma^n + \delta^n = \begin{cases} 2^{\frac{n+2}{2}}(P_n + P_{n-1}) & \textit{if } n \textit{ is even} \\ -2^{\frac{n+3}{2}} P_n & \textit{if } n \textit{ is odd} \end{cases}$$

and

$$\gamma^n - \delta^n = \begin{cases} -2^{\frac{n+2}{2}} P_n & \textit{if } n \textit{ is even} \\ -2^{\frac{n+1}{2}}(P_n + P_{n-1}) & \textit{if } n \textit{ is odd} \end{cases}$$

for $n \geq 1$.

Proof. (1) Recall that the roots of Fibonacci sequence are $a = \frac{1+\sqrt{5}}{2}$ and $b = \frac{1-\sqrt{5}}{2}$. So the roots of T_n are

$$-a = \frac{-1-\sqrt{5}}{2} = \beta \text{ and } -b = \frac{-1+\sqrt{5}}{2} = \alpha.$$

Since $F_n = \frac{a^n - b^n}{a-b}$, we get

$$(2.8) \qquad 2F_n = \frac{2(a^n - b^n)}{\sqrt{5}} = \frac{2}{\sqrt{5}}\left((-\beta)^n - (-\alpha)^n\right) = (-1)^n \frac{2}{\sqrt{5}}\left(\beta^n - \alpha^n\right)$$

and similarly

$$(2.9) \qquad F_{n-3} = \frac{a^{n-3} - b^{n-3}}{\sqrt{5}} = (-1)^n \left(\frac{(-2+\sqrt{5})}{\sqrt{5}}\beta^n + \frac{(2+\sqrt{5})}{\sqrt{5}}\alpha^n \right).$$

Applying (2.8) and (2.9), we get

$$2F_n + F_{n-3} = (-1)^n \frac{2}{\sqrt{5}}\left(\beta^n - \alpha^n\right) + (-1)^n \left(\frac{(-2+\sqrt{5})}{\sqrt{5}}\beta^n + \frac{(2+\sqrt{5})}{\sqrt{5}}\alpha^n \right)$$

and hence $(-1)^n(2F_n + F_{n-3}) = \alpha^n + \beta^n$.

(2) Since $-a = \beta$ and $-b = \alpha$, we get

$$F_n = \frac{a^n - b^n}{\sqrt{5}} = \frac{1}{\sqrt{5}}\left((-\beta)^n - (-\alpha)^n\right) = \frac{(-1)^n}{\sqrt{5}}\left(\beta^n - \alpha^n\right)$$

and hence $(-1)^{n+1}F_n\sqrt{5} = (-1)^{2n+1}\left(\beta^n - \alpha^n\right) = \alpha^n - \beta^n$.

(3) For the Pell numbers P_n, Tekcan proved in [13] that

$$P_n + P_{n+1} = \frac{a^{n+1} + b^{n+1}}{2},$$

where $a = 1+\sqrt{2}$, $b = 1-\sqrt{2}$. So $P_n + P_{n-1} = \frac{a^n + b^n}{2}$. Let n be even. Then by binomial series expansion, we get

$$\frac{a^n + b^n}{2} = \sum_{i=0}^{\frac{n}{2}} \binom{n}{2i} 2^i.$$

Therefore

$$(2.10) \qquad 2^{\frac{n+2}{2}}(P_n + P_{n-1}) = \sum_{i=0}^{\frac{n}{2}} \binom{n}{2i} 2^{n+1-i}.$$

Notice that $\gamma = -2 + \sqrt{2}$ and $\delta = -2 - \sqrt{2}$. So we deduce that

$$(2.11) \qquad \gamma^n + \delta^n = \sum_{i=0}^{\frac{n}{2}} \binom{n}{2i} 2^{n+1-i}.$$

Applying (2.10) and (2.11), we conclude that $\gamma^n + \delta^n = 2^{\frac{n+2}{2}}(P_n + P_{n-1})$. Similarly it can be shown that if n is odd, then $\gamma^n + \delta^n = -2^{\frac{n+3}{2}} P_n$. The last assertion can be proved similarly. □

From Theorems 2.5 and 2.7, we can give the following result concerning the relationship among the sequences F_n, P_n and T_n.

Theorem 2.8. *Let T_n denote the $n-th$ number.*

(1) *If n is even, then*

$$2^{\frac{n+2}{2}}(P_n + P_{n-1}) - 2F_n - F_{n-3} = -3T_n + 2T_{n-2}$$

(2) *If n is odd, then*

$$-2^{\frac{n+3}{2}} P_n + 2F_n + F_{n-3} = -3T_n + 2T_{n-2}$$

for $n \geq 3$.

Theorem 2.9. *Let T_n denote the $n-th$ number. Then the sum of two consecutive terms of T_n is*

$$T_n + T_{n-1} = -3T_{n-1} + 7T_{n-3} - 2T_{n-5}$$

for $n \geq 5$.

Proof. Notice that $T_n = -5T_{n-1} - 5T_{n-2} + 2T_{n-3} + 2T_{n-4}$. Hence

$$\begin{aligned} T_n &= -5T_{n-1} - 5T_{n-2} + 2T_{n-3} + 2T_{n-4} \\ &= -4T_{n-1} - T_{n-1} - 5T_{n-2} + 2T_{n-3} + 2T_{n-4} \\ &= -4T_{n-1} - (-5T_{n-2} - 5T_{n-3} + 2T_{n-4} + 2T_{n-5}) \\ &\quad -5T_{n-2} + 2T_{n-3} + 2T_{n-4} \\ &= -4T_{n-1} + 7T_{n-3} - 2T_{n-5} \\ &= -T_{n-1} - 3T_{n-1} + 7T_{n-3} - 2T_{n-5}. \end{aligned}$$

So $T_n + T_{n-1} = -3T_{n-1} + 7T_{n-3} - 2T_{n-5}$. □

From the above theorem, we can give the following theorem which we will use it in later theorem can be proved by induction on n.

Theorem 2.10. *Let T_n denote the $n-th$ number. Then the sum of two consecutive T_n numbers is always divisible by 3, that is, $\frac{T_n+T_{n-1}}{3} \in \mathbb{Z}$.*

Now we consider the rank of T_n numbers. The rank of T_n is defined to be

$$\rho(T_n) = \begin{cases} p & \text{if } p \text{ is the smallest prime with } p|T_n \\ \infty & \text{if } T_n \text{ is prime.} \end{cases}$$

Theorem 2.11. *Let T_n denote the $n-th$ number. Then the rank of T_n is*

$$\rho(T_n) = \begin{cases} 2 & \text{if } n \equiv 0 (\text{mod } 3) \\ 3 & \text{if } n \equiv 1, 2 (\text{mod } 3) \end{cases}$$

for $n \geq 2$.

Proof. Let $n \equiv 0(\text{mod } 3)$, say $n = 3k$ for some integer $k \geq 1$. Let $k = 1$. Then $T_3 = 12 = 2^2 \cdot 3$. So $\rho(T_3) = 2$. Let us assume that the rank of T_{3k-3} is 2, that is, $\rho(T_{3k-3}) = 2$. In this case $T_{3k-3} = 2^a \cdot A$ for some positive integers $a \geq 1$ and $A > 0$. Then $T_{3k} = -5T_{3k-1} - 5T_{3k-2} + 2T_{3k-3} + 2T_{3k-4}$. Taking $T_{3k-1} \to -5T_{3k-2} - 5T_{3k-3} + 2T_{3k-4} + 2T_{3k-5}$, we deduce that

$$\begin{aligned} T_{3k} &= -5T_{3k-1} - 5T_{3k-2} + 2T_{3k-3} + 2T_{3k-4} \\ &= 20T_{3k-2} + 27T_{3k-3} - 8T_{3k-4} - 10T_{3k-5} \\ &= 2[10T_{3k-2} + 27 \cdot 2^{a-1} \cdot A - 4T_{3k-4} - 5T_{3k-5}]. \end{aligned}$$

So $\rho(T_{3k}) = 2$.

Now let $n \equiv 2(\text{mod } 3)$, say $n = 2 + 3k$ for some integer $k \geq 0$. Let $k = 0$, then $n = 2$ and hence $T_2 = -3 = 3 \cdot (-1)$. So $\rho(T_2) = 3$. Let us assume that the rank of T_n is 3 for $n - 1$, that is, $\rho(T_{n-1}) = 3$. So $T_{n-1} = 3^b \cdot B$ for some integers $b \geq 1$ and $B > 0$. Then we get

$$\begin{aligned} T_n &= -5T_{n-1} - 5T_{n-2} + 2T_{n-3} + 2T_{n-4} \\ &= -5T_{n-1} + 27T_{n-3} + 27T_{n-4} - 10(T_{n-5} + T_{n-6}). \end{aligned} \tag{2.12}$$

By virtue of Theorem 2.10 that $T_n + T_{n-1} = 3K$ for some $K \neq 0$ and hence $T_{n-5} + T_{n-6} = 3K$. Therefore (2.12) becomes

$$\begin{aligned} T_n &= -5T_{n-1} + 27T_{n-3} + 27T_{n-4} - 10(T_{n-5} + T_{n-6}) \\ &= -5T_{n-1} + 27T_{n-3} + 27T_{n-4} - 10(3K) \\ &= 3[-5B \cdot 3^{b-1} + 9T_{n-3} + 9T_{n-4} - 10K]. \end{aligned} \tag{2.13}$$

So the rank of T_n is hence 3, that is, $\rho(T_n) = 3$. □

From the above theorem one can easily realize that if $n \equiv 0 \pmod 3$, say $n = 3k$ for $k \geq 1$. Then T_n is positive if k is odd, or negative if k is even. If $n \equiv 1 \pmod 3$, say $n = 1 + 3k$ for $k \geq 1$. Then T_n is positive if k is even, or negative if k is odd and if $n \equiv 2 \pmod 3$, say $n = 2 + 3k$ for $k \geq 0$. Then T_n is positive if k is odd, or negative if k is even. Also T_n is always divisible by 3, that is, $\frac{T_n}{3} \in \mathbb{Z}$. So we set the sequence $Q_0 = 0$, $Q_1 = 0$ and

$$Q_n = \frac{T_n}{3} \tag{2.14}$$

for every $n \geq 2$. Now we set

$$M = M(T_n) = \begin{bmatrix} -5 & -5 & 2 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{(companion matrix)}$$

$$A = A(T_n) = \begin{bmatrix} -45 & 12 & -3 & 0 \end{bmatrix} \text{ and } B = B(T_n) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Theorem 2.12. *Let T_n denote the $n-$th number.*

(1) *If n is odd, then*

$$M^n = \begin{bmatrix} \sum\limits_{i=2}^{n+2}(-1)^iQ_i & -5\left(\sum\limits_{i=2}^{n+1}(-1)^{i+1}Q_i\right) - 2Q_n & -2Q_{n+1} & -2\sum\limits_{i=0}^{n+1}(-1)^iQ_i \\ \sum\limits_{i=2}^{n+1}(-1)^{i+1}Q_i & -5\left(\sum\limits_{i=2}^{n}(-1)^iQ_i\right) - 2Q_{n-1} & -2Q_n & -2\sum\limits_{i=0}^{n}(-1)^{i+1}Q_i \\ \sum\limits_{i=2}^{n}(-1)^iQ_i & -5\left(\sum\limits_{i=2}^{n-1}(-1)^{i+1}Q_i\right) - 2Q_{n-2} & -2Q_{n-1} & -2\sum\limits_{i=0}^{n-1}(-1)^iQ_i \\ \sum\limits_{i=2}^{n-1}(-1)^{i+1}Q_i & -5\left(\sum\limits_{i=2}^{n-2}(-1)^iQ_i\right) - 2Q_{n-3} & -2Q_{n-2} & -2\sum\limits_{i=0}^{n-2}(-1)^{i+1}Q_i \end{bmatrix}$$

for $n \geq 5$.

(2) *If n is even, then*

$$M^n = \begin{bmatrix} \sum\limits_{i=2}^{n+2}(-1)^{i+1}Q_i & -5\left(\sum\limits_{i=2}^{n+1}(-1)^iQ_i\right) - 2Q_n & -2Q_{n+1} & -2\sum\limits_{i=0}^{n+1}(-1)^{i+1}Q_i \\ \sum\limits_{i=2}^{n+1}(-1)^iQ_i & -5\left(\sum\limits_{i=2}^{n}(-1)^{i+1}Q_i\right) - 2Q_{n-1} & -2Q_n & -2\sum\limits_{i=0}^{n}(-1)^iQ_i \\ \sum\limits_{i=2}^{n}(-1)^{i+1}Q_i & -5\left(\sum\limits_{i=2}^{n-1}(-1)^iQ_i\right) - 2Q_{n-2} & -2Q_{n-1} & -2\sum\limits_{i=0}^{n-1}(-1)^{i+1}Q_i \\ \sum\limits_{i=2}^{n-1}(-1)^iQ_i & -5\left(\sum\limits_{i=2}^{n-2}(-1)^{i-1}Q_i\right) - 2Q_{n-3} & -2Q_{n-2} & -2\sum\limits_{i=0}^{n-2}(-1)^iQ_i \end{bmatrix}$$

for $n \geq 4$.

Proof. Let n be odd. We prove it by induction: Let $n = 5$. Then

$$\begin{aligned}
&\begin{bmatrix}
\sum_{i=2}^{7}(-1)^iQ_i & -5\left(\sum_{i=2}^{6}(-1)^{i+1}Q_i\right)-2Q_5 & -2Q_6 & -2\sum_{i=0}^{6}(-1)^iQ_i \\
\sum_{i=2}^{6}(-1)^{i+1}Q_i & -5\left(\sum_{i=2}^{5}(-1)^{i}Q_i\right)-2Q_4 & -2Q_5 & -2\sum_{i=0}^{5}(-1)^{i+1}Q_i \\
\sum_{i=2}^{5}(-1)^iQ_i & -5\left(\sum_{i=2}^{4}(-1)^{i+1}Q_i\right)-2Q_3 & -2Q_4 & -2\sum_{i=0}^{4}(-1)^iQ_i \\
\sum_{i=2}^{5}(-1)^{i+1}Q_i & -5\left(\sum_{i=2}^{3}(-1)^{i}Q_i\right)-2Q_2 & -2Q_3 & -2\sum_{i=0}^{3}(-1)^{i+1}Q_i
\end{bmatrix} \\
= &\begin{bmatrix}
-890 & -1391 & -1317 & -387 \\
257 & 395 & -106 & -146 \\
-73 & -108 & 30 & 40 \\
20 & 27 & -8 & -10
\end{bmatrix} \\
= &\ M^5.
\end{aligned}$$

So it is true for $n = 5$. Let us assume that this relation is satisfied for $n-2$. We want to show that it is also satisfied for n. Since $M^n = M^2 \cdot M^{n-2}$, we get

$$\begin{aligned}
= &\begin{bmatrix}
20 & 27 & -8 & -10 \\
-5 & -5 & 2 & 2 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix} \times \\
&\begin{bmatrix}
\sum_{i=2}^{n}(-1)^iQ_i & -5\left(\sum_{i=2}^{n-1}(-1)^{i+1}Q_i\right)-2Q_{n-2} & -2Q_{n-1} & -2\sum_{i=0}^{n-1}(-1)^iQ_i \\
\sum_{i=2}^{n-1}(-1)^{i+1}Q_i & -5\left(\sum_{i=2}^{n-2}(-1)^{i}Q_i\right)-2Q_{n-3} & -2Q_{n-2} & -2\sum_{i=0}^{n-2}(-1)^{i+1}Q_i \\
\sum_{i=2}^{n-2}(-1)^iQ_i & -5\left(\sum_{i=2}^{n-3}(-1)^{i+1}Q_i\right)-2Q_{n-4} & -2Q_{n-3} & -2\sum_{i=0}^{n-3}(-1)^iQ_i \\
\sum_{i=2}^{n-3}(-1)^{i+1}Q_i & -5\left(\sum_{i=2}^{n-4}(-1)^{i}Q_i\right)-2Q_{n-5} & -2Q_{n-4} & -2\sum_{i=0}^{n-4}(-1)^{i+1}Q_i
\end{bmatrix} \\
= &\begin{bmatrix}
M_{11} & M_{12} & M_{13} & M_{14} \\
M_{21} & M_{22} & M_{23} & M_{24} \\
M_{31} & M_{32} & M_{33} & M_{34} \\
M_{41} & M_{42} & M_{43} & M_{44}
\end{bmatrix}.
\end{aligned}$$

Hence

$$\begin{aligned}
M_{21} &= -5\left(\sum_{i=2}^{n}(-1)^iQ_i\right) - 5\left(\sum_{i=2}^{n-1}(-1)^{i+1}Q_i\right) + 2\left(\sum_{i=2}^{n-2}(-1)^iQ_i\right) + \\
&\quad 2\left(\sum_{i=2}^{n-3}(-1)^{i+1}Q_i\right) \qquad (2.15)\\
&= -5(Q_2 - Q_3 - \cdots + Q_{n-1} - Q_n) - 5(Q_2 + Q_3 - \cdots + Q_{n-2} - Q_{n-1}) \\
&\quad +2(Q_2 - Q_3 - \cdots + Q_{n-3} - Q_{n-2}) + 2(-Q_2 + Q_3 + \cdots + Q_{n-4} - Q_{n-3}) \\
&= -2Q_{n-2} + 5Q_n - Q_2 + Q_3 + 5Q_2 - 2Q_0 - 2Q_1 + 2Q_1 + 2Q_2 - 2Q_2 \\
&\quad +5Q_{n-2} - 5Q_{n-2} - 5Q_{n-1} + 5Q_{n-1}.
\end{aligned}$$

Since $T_n = -5T_{n-1} - 5T_{n-2} + 2T_{n-3} + 2T_{n-4}$, we get

$$
(2.16) \quad \begin{aligned}
-Q_2 &= -Q_2 \\
Q_3 &= Q_3 \\
-Q_4 &= 5Q_3 + 5Q_2 - 2Q_1 - 2Q_0 \\
Q_5 &= -5Q_4 - 5Q_3 + 2Q_2 + 2Q_1 \\
-Q_6 &= 5Q_5 + 5Q_4 - 2Q_3 - 2Q_2 \\
&\cdots \\
-Q_{n-1} &= 5Q_{n-2} + 5Q_{n-3} - 2Q_{n-4} - 2Q_{n-5} \\
Q_n &= -5Q_{n-1} - 5Q_{n-2} + 2Q_{n-3} + 2Q_{n-4} \\
-Q_{n+1} &= 5Q_n + 5Q_{n-1} - 2Q_{n-2} - 2Q_{n-3}.
\end{aligned}
$$

If we sum the left and right side of (2.16), then we obtain

$$
\begin{aligned}
-Q_2 + Q_3 - \cdots + Q_n - Q_{n+1} &= -2Q_{n-2} + 5Q_n - Q_2 + Q_3 + 5Q_2 \\
&\quad -2Q_0 - 2Q_1 + 2Q_1 + 2Q_2 - 2Q_2 \\
&\quad +5Q_{n-2} - 5Q_{n-2} - 5Q_{n-1} + 5Q_{n-1} \\
&= \sum_{i=2}^{n+1} (-1)^{i+1} Q_i.
\end{aligned}
$$

So applying (2.15), the result is clear. The other cases for M_{ij} can be proved similarly. □

Theorem 2.13. *Let T_n denote the $n-th$ number. Then*

$$T_n = A(M^{n-4})^t B$$

for $n \geq 4$.

Proof. It can be proved by induction on n as in Theorem 2.12. □

Example 2.1. *Let $n = 12$. Then $T_{12} = -886896$. Since*

$$
M^8 = \begin{bmatrix}
35824 & 56723 & -14836 & -20954 \\
-10477 & -16561 & 4338 & 6118 \\
3059 & 4818 & -1266 & -1780 \\
-890 & -1391 & 368 & 514
\end{bmatrix}
$$

we get $A(M^8)^t B = -886896 = T_{12}$.

3. Cross-ratio of Four Consecutive T_n Numbers.

The cross-ratio is also an important quantity in complex analysis and also in the theory of discrete groups. Given any four different complex numbers z_1, z_2, z_3 and z_4, the cross-ratio defined as

$$[z_1, z_2; z_3, z_4] = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)} \tag{3.1}$$

is invariant under arbitrary Mobius (i.e., linear fractional) transformations. This definition can be extended to the entire Riemann sphere (i.e. $\mathbb{C} \cup \{\infty\}$) by continuity. More generally, the cross-ratio can be defined on any projective line (The Riemann Sphere is just the complex projective line). It is given by the above expression in any affine coordinate chart. Cross-ratios are invariant of projective geometry in the sense that they are preserved by projective transformations. The cross-ratio of four complex numbers is real if and only if the four numbers are either collinear or noncyclic.

Now we want the determine the limit of cross-ratio of four consecutive T_n numbers T_n, T_{n+1}, T_{n+2} and T_{n+3}. Let $[T_n, T_{n+1}; T_{n+2}, T_{n+3}]$ denote the cross-ratio of four consecutive T_n numbers. Then we can give the following theorem.

Theorem 3.1. *Let T_n, T_{n+1}, T_{n+2} and T_{n+3} be four consecutive T_n numbers. Then*

$$\lim_{n\to\infty} [T_n, T_{n+1}; T_{n+2}, T_{n+3}] = \frac{6+\delta}{7}.$$

Proof. Let T_n, T_{n+1}, T_{n+2} and T_{n+3} be four consecutive T_n numbers. Then by (3.1), we get

$$[T_n, T_{n+1}; T_{n+2}, T_{n+3}] = \frac{(T_n - T_{n+1})(T_{n+2} - T_{n+3})}{(T_{n+1} - T_{n+2})(T_{n+3} - T_n)}. \tag{3.2}$$

Note that $T_n = -5T_{n-1} - 5T_{n-2} + 2T_{n-3} + 2T_{n-4}$. So

$$T_n - T_{n+1} = \left[\frac{\gamma^n - \delta^n}{\gamma - \delta} - \frac{\alpha^n - \beta^n}{\alpha - \beta}\right] - \left[\frac{\gamma^{n+1} - \delta^{n+1}}{\gamma - \delta} - \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right].$$

Therefore

$$\begin{aligned}
T_{n+2} - T_{n+3} &= \left[\frac{\gamma^{n+2} - \delta^{n+2}}{\gamma - \delta} - \frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta}\right] \left[\frac{\gamma^{n+3} - \delta^{n+3}}{\gamma - \delta} - \frac{\alpha^{n+3} - \beta^{n+3}}{\alpha - \beta}\right] . \\
T_{n+1} - T_{n+2} &= \left[\frac{\gamma^{n+1} - \delta^{n+1}}{\gamma - \delta} - \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right] - \left[\frac{\gamma^{n+2} - \delta^{n+2}}{\gamma - \delta} - \frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta}\right] \\
T_{n+3} - T_n &= \left[\frac{\gamma^{n+3} - \delta^{n+3}}{\gamma - \delta} - \frac{\alpha^{n+3} - \beta^{n+3}}{\alpha - \beta}\right] - \left[\frac{\gamma^n - \delta^n}{\gamma - \delta} - \frac{\alpha^n - \beta^n}{\alpha - \beta}\right].
\end{aligned}$$

Hence (3.2) becomes

$$[T_n, T_{n+1}; T_{n+2}, T_{n+3}] = \frac{\left[\left(\frac{\gamma^n-\delta^n}{\gamma-\delta} - \frac{\alpha^n-\beta^n}{\alpha-\beta}\right) - \left(\frac{\gamma^{n+1}-\delta^{n+1}}{\gamma-\delta} - \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right)\right] \times \left[\left(\frac{\gamma^{n+2}-\delta^{n+2}}{\gamma-\delta} - \frac{\alpha^{n+2}-\beta^{n+2}}{\alpha-\beta}\right) - \left(\frac{\gamma^{n+3}-\delta^{n+3}}{\gamma-\delta} - \frac{\alpha^{n+3}-\beta^{n+3}}{\alpha-\beta}\right)\right]}{\left[\left(\frac{\gamma^{n+1}-\delta^{n+1}}{\gamma-\delta} - \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right) - \left(\frac{\gamma^{n+2}-\delta^{n+2}}{\gamma-\delta} - \frac{\alpha^{n+2}-\beta^{n+2}}{\alpha-\beta}\right)\right] \times \left[\left(\frac{\gamma^{n+3}-\delta^{n+3}}{\gamma-\delta} - \frac{\alpha^{n+3}-\beta^{n+3}}{\alpha-\beta}\right) - \left(\frac{\gamma^n-\delta^n}{\gamma-\delta} - \frac{\alpha^n-\beta^n}{\alpha-\beta}\right)\right]}.$$

Notice that $\alpha = \frac{-1+\sqrt{5}}{2}, \beta = \frac{-1-\sqrt{5}}{2}, \gamma = -2-\sqrt{2}$ and $\delta = -2-\sqrt{2}$. Combining these and taking the limit of both side of above equation, we conclude that

$$\lim_{n\to\infty}[T_n, T_{n+1}; T_{n+2}, T_{n+3}] = \frac{2+\sqrt{2}}{5+3\sqrt{2}} = \frac{6+\delta}{7}$$

as we wanted. □

From the above theorem we can give the following result.

Corollary 3.2. *Let T_n, T_{n+1}, T_{n+2} and T_{n+3} be four consecutive numbers. Then*

$$\begin{aligned}
\lim_{n\to\infty}[T_n, T_{n+1}; T_{n+3}, T_{n+2}] &= \frac{7}{6+\delta} \\
\lim_{n\to\infty}[T_n, T_{n+2}; T_{n+3}, T_{n+1}] &= -\delta - 1 \\
\lim_{n\to\infty}[T_n, T_{n+2}; T_{n+1}, T_{n+3}] &= -\frac{1}{\delta+1} \\
\lim_{n\to\infty}[T_n, T_{n+3}; T_{n+2}, T_{n+1}] &= \frac{2-\delta}{2} \\
\lim_{n\to\infty}[T_n, T_{n+3}; T_{n+1}, T_{n+2}] &= \frac{2}{2-\delta}.
\end{aligned}$$

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