On Row-Cyclic Array Codes Detecting and Correcting CT-Bursts Array Errors

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Abstract. In this paper, we study CT-burst array error [6] detection and correction in row-cyclic array codes [8].

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1. Introduction

Row-cyclic array codes equipped with m-metric [12] have already been introduced by the author in [8]. These codes are suitable for parallel channel communication systems. The author also gave the decoding methods for the correction/detection of random array errors [8] and usual burst array errors [10] in row-cyclic array codes. There are yet another kind of burst array errors that occur during parallel channel communication known as the CT-burst array errors [6]. In fact, the CT-burst array errors are the generalized version of usual burst array errors. In this paper, we study the CT-burst array error detection and correction in row-cyclic array codes.

2. Definitions and Notations

Let F_q be a finite field of q elements. Let $\text{Mat}_{m\times s}(F_q)$ denote the linear space of all $m \times s$ matrices with entries from F_q. An m-metric array code is a subset of $\text{Mat}_{m\times s}(F_q)$ and a linear m-metric array code is an F_q-linear subspace of $\text{Mat}_{m\times s}(F_q)$. Note that the space $\text{Mat}_{m\times s}(F_q)$ is identifiable with the space F_q^{ms}. Every matrix in $\text{Mat}_{m\times s}(F_q)$ can be represented as a $1 \times ms$ vector by writing the first row of matrix followed by second row and so on. Similarly, every vector in F_q^{ms} can be represented as an $m \times s$

matrix in $\text{Mat}_{m\times s}(F_q)$ by separating the co-ordinates of the vector into m groups of s-coordinates. The m-metric on $\text{Mat}_{m\times s}(F_q)$ is defined as follows [12]:

Definition 2.1. Let $Y \in \text{Mat}_{1\times s}(F_q)$ with $Y = (y_1, y_2, \cdots, y_s)$. Define row weight (or ρ-weight) of Y as

$$wt_\rho(Y) = \begin{cases} \max\{\ i \mid y_i \neq 0\} & \text{if } Y \neq 0 \\ 0 & \text{if } Y = 0. \end{cases}$$

Extending the definitions of wt_ρ to the class of $m \times s$ matrices as

$$wt_\rho(A) = \sum_{i=1}^{m} wt_\rho(R_i)$$

where $A = \begin{bmatrix} R_1 \\ R_2 \\ \cdots \\ R_m \end{bmatrix} \in \text{Mat}_{m\times s}(F_q)$ and R_i denotes the i^{th} row of A. Then wt_ρ satisfies $0 \leq wt_\rho(A) \leq n(= ms)\ \forall\ A \in \text{Mat}_{m\times s}(F_q)$ and determines a metric on $\text{Mat}_{m\times s}(F_q)$ known as m-metric (or ρ-metric).

Now we define CT-burst errors in linear array codes [6]:

Definition 2.2. A CT burst of order pr(or $p \times r$)$(1 \leq p \leq m, 1 \leq r \leq s)$ in the space $\text{Mat}_{m\times s}(F_q)$ is an $m \times s$ matrix in which all the nonzero entries are confined to some $p \times r$ submatrix which has non-zero first row and first column.

Note. For $p = 1$, Drfinition 2.1 reduces to the Definition of classical CT-bursts [3].

The following theorem gives a bound on the number of parity check digits for the correction of CT-burst array errors in linear m-metric array codes [6].

Definition 2.3. A CT-burst of order pr or less $(1 \leq p \leq m, 1 \leq r \leq s)$ in the space $\text{Mat}_{mtimess}(F_q)$ is a CT-burst of order cd(or $c \times d$) where $1 \leq c \leq p \leq m$ and $1 \leq d \leq r \leq s$.

Theorem 2.1. *An (n, k) linear m-metric array code $V \subseteq Mat_{m\times s}(F_q)$ where $n = ms$ that corrects all CT bursts of order $pr(1 \leq p \leq m, 1 \leq r \leq s)$*

must satisfy

$$q^{n-k} \geq 1 + T^{p\times r}_{m\times s}(F_q),$$

where $T^{p\times r}_{m\times s}(F_q)$ is the number of CT bursts of order $pr(1 \leq p \leq m, 1 \leq r \leq s)$ in $Mat_{m\times s}(F_q)$ and is given by

$$T^{p\times r}_{m\times s}(F_q) = \begin{cases} ms(q-1) & \text{if } p=1,\ r=1, \\ m(s-r+1)(q-1)q^{r-1} & \text{if } p=1, r\geq 2, \\ (m-p+1)s(q-1)q^{p-1} & \text{if } p\geq 2, r=1 \\ (m-p+1)(s-r+1)q^{r(p-1)}\times & \\ \times\left[(q^r-1)-(q^{r-1}-1)q^{1-p}\right] & \text{if } p\geq 2,\ r\geq 2. \end{cases} \tag{1}$$

Now, we define row-cyclic array codes [8].

Definition 2.4. An $[m \times s, k]$ linear array codes $C \subseteq \text{Mat}_{m\times s}(F_q)$ is said to be row-cyclic if

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{ms} \end{pmatrix} \in \mathbf{C}$$

$$\Longrightarrow \begin{pmatrix} a_{1s} & a_{11} & a_{12} & \cdots a_{1,s-1} \\ a_{2s} & a_{21} & a_{22} & \cdots a_{2,s-1} \\ \vdots & \vdots & \vdots & \vdots \\ a_{ms} & a_{m1} & a_{m2} & \cdots a_{m,s-1} \end{pmatrix} \in \mathbf{C}$$

i.e. the array obtained by shifting the columns of a code array cyclically by one position of the right and the last column occupying the first place is also a code array. In fact, a row-cyclic array code C of order $m \times s$ turns out to be $C = \bigoplus_{i=1}^{m} C_i$ where each C_i is a classical cyclic code of length s. Also, every matrix/array in $\text{Mat}_{m\times s}(F_q)$ can be identified with an m-tuple in $A_s^{(m)}$ where $A_s^{(m)}$ is the direct product of algebra A_s taken m times and A_s is the algebra of all polynomials over F_q modulo the polynomial $x^s - 1$

and this identification is given by

$$\theta : \mathrm{Mat}_{m\times s}(F_q) \to A_s^{(m)}$$

$$\theta(A) = \theta\begin{pmatrix} R_1 \\ \vdots \\ R_m \end{pmatrix} = \begin{pmatrix} \theta' R_1 \\ \theta' R_2 \\ \vdots \\ \theta' R_m \end{pmatrix} = (\theta' R_1, \theta' R_2, \cdots, \theta' R_m) \tag{2}$$

where $R_i (i = 1$ to $m)$ denotes the i^{th} row of A and $\theta' : F_q^s \longrightarrow A_s$ is given by

$$\theta'(a_0, a_1, \cdots, a_{s-1}) = a_0 + a_1 x + \cdots + a_{s-1} x^{s-1}.$$

An equivalent definition of row-cyclic array code is given by [8]:

Definition 2.5. An $m \times s$ linear array codes $C \subseteq \mathrm{Mat}_{m\times s}(F_q)$ is said to be row-cyclic if

$$C = \bigoplus_{i=1}^{m} C_i$$

where each C_i is an $[s, k_i, d_i]$ classical cyclic code equipped with m-metric. The parameters of row-cyclic array code C are given by $[m\times s, \sum_{i=1}^{m} k_i, \min_{i=1}^{m} d_i]$. If $g_i(x)$ is the generator polynomial of classical cyclic code C_i, then the m-tuple $(g_1(x) \cdots, g_m(x))$ is called the generator m-tuple of row cyclic code C.

3. Detection of CT-Burst Errors in Row-Cyclic Array Codes

In this section, we first obtain an upper bound on the order of CT-bursts that can be detected by a row-cyclic array code and then obtain the ratio of CT-bursts (of order exceeding the upper bound) to the total number of CT-bursts.

Theorem 3.1 *Let $C = \bigoplus_{i=1}^{m} C_i$ be an $[m \times s, \sum_{i=1}^{m} k_i, \min_{i=1}^{m} d_i]$ row-cyclic array code. Then no code array is a CT-burst of order $m \times r$ or less where*

$r = \min_{i=1}^{m}\{s - k_i\}$. *Therefore, every* $[m \times s, \sum_{i=1}^{m} k_i, \min_{i=1}^{m} d_i]$ *row-cyclic array code detects every CT-burst of order* $m \times \min_{i=1}^{m}\{s - k_i\}$ *or less.*

Proof. Let

$$A = (0 \quad B \quad 0) = \begin{pmatrix} & b_1 & \\ 0 & b_2 & 0 \\ & \vdots & \\ & b_m & \end{pmatrix}$$

$$= \begin{pmatrix} & b_1(x) & \\ 0 & b_2(x) & 0 \\ & \vdots & \\ & b_m(x) & \end{pmatrix} \in A_s^{(m)}$$

denote a CT-burst of order $m \times r$ or less ($r = \min_{i=1}^{m}\{s - k_i\}$) where B is a $m \times r$ submatrix of A such that B has a submatrix D with first row and first column of D to be nonzero. Let $(g_1(x), g_2(x), \cdots, g_m(x))$ be the generator m-tuple of row-cyclic array code C. Then $deg(g_i(x)) = s - k_i$ for all $i = 1, 2, \cdots, m$. Choose $b_i(x)$ such that $b_i(x) \neq 0$. Then such a $b_i(x)$ is a classical burst of order r or less. Let the first nonzero component of the vector corresponding to $b_i(x)$ be the coefficient of x^j under the correspondence θ' i.e.

$$(a_0, a_1, \cdots, a_{s-1}) \longleftrightarrow a_0 + a_1 x + \cdots + a_{s-1} x^{s-1}.$$

Then, the polynomial $b_i(x)$ can be written as

$$\begin{aligned} b_i(x) &= a_j x^j + a_{j+1} x^{j+1} + \cdots + a_{j+r-1} x^{j+r-1} \\ &= x^j (a_j + a_{j+1} x + \cdots + a_{j+r-1} x^{r-1}) \\ &= x^j p(x) \end{aligned}$$

where $\deg p(x) \leq r - 1 = \min_{i=1}^{m}\{s - k_i\} - 1 \leq s - k_i - 1 < s - k_i = \deg g_i(x)$.

Now $g_i(x)$ does not divide x^j and also $g_i(x)$ does not divide $p(x)$ as deg $(p(x)) <$ deg $g_i(x)$.

Therefore, $g_i(x)$ does not divide $b_i(x)$.

This implies that $b_i(x)$ is not a code polynomial in classical code C_i which further implies that $A = \begin{pmatrix} & b_1(x) & \\ 0 & b_2(x) & 0 \\ & \vdots & \\ & b_m(x) & \end{pmatrix}$ is not an array of code polynomials in $C = \bigoplus_{i=1}^{m} C_i$. Hence, the row-cyclic array code C detects every CT-burst of order $m \times \min_{i=1}^{m}\{s - k_i\}$ or less. □

Another upper bound on the order of CT-bursts that can be detected by a row-cyclic array code is obtained in the following theorem:

Theorem 3.2. *Let $C = \bigoplus_{i=1}^{m} C_i$ be an $[m \times s, \sum_{i=1}^{m} k_i, \min_{i=1}^{m} d_i]$ row-cyclic array code. Then no code array is a CT-burst of order $m \times t$ where $t \leq s - k_1$. Therefore, every $[m \times s, \sum_{i=1}^{m} k_i, \min_{i=1}^{m} d_i]$ row-cyclic array code $C = \bigoplus_{i=1}^{m} C_i$ detects every CT-burst of order $m \times t$ $(t \leq s - k_1)$.*

Proof. Let

$$\begin{aligned} A &= (0 \quad B \quad 0) = \begin{pmatrix} & b_1 & \\ 0 & b_2 & 0 \\ & \vdots & \\ & b_m & \end{pmatrix} \\ &= \begin{pmatrix} & b_1(x) & \\ 0 & b_2(x) & 0 \\ & \vdots & \\ & b_m(x) & \end{pmatrix} \in A_s^{(m)} \end{aligned}$$

denote a CT-burst of order $m \times t$ where $t \leq s - k_1$ and B is an $m \times t$ submatrix of A with first row and first column of B nonzero. Let $(g_1(x), g_2(x), \cdots, g_m(x))$ be the generator m-tuple of the row-cyclic array code C. Then $deg(g_i(x)) = s - k_i$ for all $i = 1, 2, \cdots, m$. Clearly $b_1(x) \neq 0$. As in Theorem 3.1, we have

$$\begin{aligned} b_1(x) &= a_j x^j + a_{j+1} x^{j+1} + \cdots + a_{j+t-1} x^{j+t-1} \\ &= x^j (a_j + a_{j+1} x + \cdots + a_{j+t-1} x^{t-1}) \\ &= x^j p(x) \end{aligned}$$

where

$$\deg p(x) \leq t - 1 \leq s - k_1 - 1 < s - k_1 = \deg g_1(x).$$

Now $g_1(x)$ does not divide x^j and also $g_1(x)$ does not divide $p(x)$ as deg $(p(x)) <$ deg $g_1(x)$.

Therefore, $g_1(x)$ does not divide $b_1(x)$.

This implies that $b_1(x)$ is not a code polynomial in classical code C_1.

This implies that $A = \begin{pmatrix} & b_1(x) & \\ 0 & b_2(x) & 0 \\ & \vdots & \\ & b_m(x) & \end{pmatrix}$ is not an array of code polynomials in $C = \bigoplus_{i=1}^{m} C_i$. Hence, the row-cyclic array code C detects every CT-burst of order $m \times t$ where $t \leq s - k_1$. □

Remark 3.1. Clearly the bound obtained in Theorem 3.2 is better than the one obtained in Theorem 3.1 as $\min_{i=1}^{m}\{s - k_i\} \leq s - k_1$ with the only limitation that order of nonzero submatrix B in CT-burst A is $m \times t$ and not $m \times t$ or less where $(t \leq s - k_1)$. We may also take the order as $m \times t$ or less in Theorem 3.2 but with the constraint that $b_1(x) \neq 0$ i.e. first row of CT-burst A is nonzero.

Now, we obtain the ratio of CT-bursts of order $m \times r$ where $r > (s - k_1)$ that go undetected in row-cyclic array codes. We spilt the problem into tow parts viz. when $r = s - k_1 + 1$ and when $r > s - k_1 + 1$ and obtain the desired ratio in the following two theorems. In the subsequent theorems, $|J|$ denote the cardinality of a set J.

Theorem 3.3. *Let $C = \bigoplus_{i=1}^{m} C_i$ be a row-cyclic array code over F_q where each C_i is a $[s, k_i, d_i]$ classical cyclic code equipped with m-metric and having generator polynomial $g_i(x)$. Then the ratio of CT-bursts of order $m \times r$ (where $r = s - k_1 + 1$) that go undetected in a row-cyclic array code C is given by*

$$\frac{(s - r + 1)\left(q^{|J_2|(r-s) + \sum_{i \in J_2} k_i} \times (q - 1) \times q^{(|J_1| - 1)}\right)}{T_{m \times s}^{m \times r}(F_q)} \tag{3}$$

where J_1 and J_2 are subsets of $N = \{1, 2, \cdots, m\}$ such that $i \in J_1 \Leftrightarrow r - 1 =$

$s-k_i$ and $i \in J_2 \Leftrightarrow r-1 > s-k_i$ and $T^{m\times r}_{m\times s}(F_q)$ is given by (1).

Proof. Let $J_3 = N/(J_1 \cup J_2)$. Then J_1, J_2, J_3 are pairwise disjoint and $N = J_1 \cup J_2 \cup J_3$ (i.e. $|J_1| + |J_2| + |J_3| = m$). Clearly, $1 \in J_1$. Consider a CT-burst A of order $m \times (s-k_1+1)$. We can write A as

$$A = (0 \quad B \quad 0) = \begin{pmatrix} & b_1 & \\ 0 & b_2 & 0 \\ & \vdots & \\ & b_m & \end{pmatrix}$$

$$= \begin{pmatrix} & b_1(x) & \\ 0 & b_2(x) & 0 \\ & \vdots & \\ & b_m(x) & \end{pmatrix} \quad \text{(under the identification } \theta).$$

where $B = \begin{pmatrix} b_1(x) \\ b_2(x) \\ \vdots \\ b_m(x) \end{pmatrix}$ is an $m \times r$ submatrix of A such that first row and first column of B is nonzero.

Now, the CT-burst A will go undetected if

$$g_i(x) \text{ divides } b_i(x) \ \forall \ i \in N.$$

Without any loss of generality, we may assume that deg $b_i(x) \leq r-1$ for all $i \in N$. Let $i \in N$. We find possible number of ways of choosing $b_i(x)$.

There are three mutually exclusive cases to consider:

Case 1. When $i \in J_3$.

In this case, $r-1 < s-k_i$ and i cannot be 1. Since deg $b_i(x) \leq r-1$ and deg $g_i(x) = (s-k_i)$ and $r-1 < s-k_i$, therefore $g_i(x)$ divides $b_i(x)$ iff $b_i(x) = 0$.

Thus there is only one way of choosing $b_i(x)$.

Hence, possible number of ways of choosing $b_i(x)$for all $i \in J_3$

$$= (1)^{|J_3|} = 1. \tag{4}$$

Case 2. When $i \in J_1$.

In this case $r-1 = s-k_i$ and i can be 1. Now $g_i(x)$ divides $b_i(x)$ iff $b_i(x) = g_i(x)q_i(x)$ for some $q_i(x)$.

Since deg $g_i(x) = s-k_i$ and deg $b_i(x) \leq r-1$, therefore deg $q_i(x) \leq (r-1)-(s-k_i) = 0$.

But the degree of a polynomial cannot be negative, thus, deg $q_i(x) = 0$.

Thus, total number of ways of choosing $q_i(x) = \begin{cases} q-1 & \text{if} \quad i=1 \\ q \quad \text{if} & i \in J_1/\{1\}. \end{cases}$

Therefore, total number of ways of choosing $q_i(x)$ and hence $b_i(x)$ $\forall\, i \in J_1$

$$= (q-1)q^{|J_1|-1}. \tag{5}$$

Case 3. When $i \in J_2$.

In this case, $r-1 > s-k_i$ and i can not be 1. Also, $0 \leq$ deg $q_i(x) \leq (r-1)-(s-k_i)$. Denote $(r-1)-(s-k_i)$ by P.

Now, number of possibilities for $q_i(x)$

$$\begin{aligned} &= \text{number of polynomials of dgree upto } P \\ &= q + (q-1)q + (q-1)q^2 + \cdots + (q-1)q^P \\ &= q^{P+1} = q^{r-s-k_i}. \end{aligned}$$

Therefore, total number of possible ways of choosing $q_i(x)$ and hence $b_i(x)$ $\forall\, i \in J_2$

$$\begin{aligned} &= \prod_{i \in J_2} q^{r-s+k_i} \\ &= q^{|J_2|(r-s)+\sum_{i \in J_2} k_i} \end{aligned} \tag{6}$$

Combining the three cases, i.e. multiplying (4), (5) and (6) and using the fact that the CT-burst A or order $m \times r$ can have first $(s-r+1)$ positions as the starting positions, we get total number of CT-bursts of order $m \times (s-k_1+1)$ that go undetected in the row-cyclic array code C and is given by

$$(s-r+1)\left(1 \times (q-1) \times q^{(|J_1|-1)} \times q^{(|J_2|(r-s)+\sum_{i \in J_2} k_i}\right)$$

$$= \quad (s-r+1)\Big(q^{(|J_2|(r-s)+\sum_{i\in J_2} k_i} \times (q-1) \times q^{(|J_1|-1)}\Big). \qquad (7)$$

Also, total number of CT-bursts of order $m \times r$ (where $r = s - k_1 + 1$) viz. $T^{m\times r}_{m\times s}(F_q)$ is given by (1). Therefore, the required ratio is obtained on dividing (7) by (1).

Example 3.1. Let C be the binary $[2 \times 2, 1+1]$ row-cyclic array code of order 2×2 generated by $(g_1(x), g_2(x)) = (1+x, 1+x)$. Then $C = C_1 \oplus C_2$ where C_1 and C_2 are classical cyclic codes of length 2 each generated by $1+x$.

Here $k_1 = k_2 = 1$ and $s = 1$.

Therefore, $s - k_1 = s - k_2 = 1$. Let $r = 2$. Then $2 = r = s - k_1 + 1$.

Here $N = \{1,2\}$, $J_1 = \{1,2\}$, $J_2 = \phi$, $|J_1| = 2, |J_2| = 0$.

The ratio computed in (3) for this example turns out to be 2/10. The ratio is justified by the fact that there are 10 CT-bursts of order 2×2 in $\mathrm{Mat}_{2\times 2}(F_2)$ (since $T^{2\times 2}_{2\times 2}(F_q) = 10$) given by

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix},$$

and out of these 10 CT-bursts, 2 CT-bursts viz. $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ are undetected by the row-cyclic array code C.

Example 3.2. Let $C = C_1 \oplus C_2$ be a row-cyclic array code of order generated by $(g_1(x), g_2(x)) = (1, 1+x)$. It is clear that C_1 and C_2 are classical cyclic codes of length 2 generated by 1 and $1+x$ respectively.

Here $k_1 = 2, k_2 = 1$ and $s = 2$.

Therefore, $s - k_1 = 0$ and $s - k_2 = 1$.

Let $r = 1$. Then $1 = r = s - k_1 + 1$ and $r - 1 < s - k_2$.

Here $N = \{1,2\}, J_1 = \{1\}, J_2 = \phi$. (Note that $J_3 = \{2\}$).

The ratio computed in (3) for this example turns out to be 2/4 and is justified by the fact that there are 4 CT-bursts of order 2×1 in $\mathrm{Mat}_{2\times 2}(F_2)$ viz.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

and out of these 4 CT-bursts, 2 CT-burst viz. $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ go undetected in the row-cyclic array code C. □

Now, we obtain the ratio of the undetected CT-burst array errrors of order $m \times r$ where $r > s - k_1 + 1$.

Theorem 3.4. *Let $C = \bigoplus_{i=1}^{m} C_i$ be a row-cyclic array code over F_q where each C_i is a $[s, k_i, d_i]$ classical cyclic code equipped with m-metric and having generator polynomial $g_i(x)$. Then the ratio of the CT-bursts of order $m \times r$ (where $r > s - k_1 + 1$) that go undetected in a row-cyclic array code C is given by*

$$\frac{(s-r+1)(D-E)}{T_{m\times s}^{m\times r}(F_q)}, \tag{8}$$

where

(i) $D = \left(q^{|J_2 - 1|(r-s) + \sum_{\substack{i \in J_2 \\ i \neq 1}} k_i + |J_1|} \times (q^{r-s+k_1} - 1)\right)$,

(ii) $E = \left(q^{|J_2 - 1|(r-s-1) + \sum_{\substack{i \in J_2 \\ i \neq 1}} k_i} \times (q^{(r-1)-(s-k_1)} - 1)\right)$,

(iii) *J_1 and J_2 are subsets of $N = \{1, 2, \cdots, m\}$ such that $i \in J_1 \Leftrightarrow r - 1 = s - k_i$ and $i \in J_2 \Leftrightarrow r - 1 > s - k_i$ and $T_{m\times s}^{m\times r}(F_q)$ is given by (1).*

Proof. Let $J_3 = N/(J_1 \cup J_2)$. Then J_1, J_2 and J_3 are pairwise disjoint, $N = J_1 \cup J_2 \cup J_3$ and $|J_1| + |J_2| + |J_3| = m$. Clearly, $1 \in J_2$. Consider a CT-burst $A = \begin{pmatrix} & b_1(x) & \\ 0 & b_2(x) & 0 \\ & \vdots & \\ & b_m(x) & \end{pmatrix}$ of order $m \times r$ where $r > s - k_1 + 1$. As in

Theorem 3.3, the CT-burst A will go undetected if $g_i(x)$ divides $b_i(x) \forall\ i \in N$ where deg $b_i(x) \leq r-1\ \forall\ i$. Let $i \in N$. We find various ways of choosing $b_i(x)$.

There are three cases to consider:

Case 1. When $i \in J_3$.

This case is same as Case 1 of Theorem 3.3. Therefore, the total number of possible ways of choosing $b_i(x)\ \forall\ i \in J_3$

$$= (1)^{|J_3|} = 1. \tag{9}$$

Case 2. When $i \in J_1$.

In this case $r-1 = s-k_i$ and i can not be 1. Now $g_i(x)$ divides $b_i(x)$ iff $b_i(x) = g_i(x)q_i(x)$ for some $q_i(x)$.

Since deg $b_i(x) \leq r-1$ and deg $g_i(x) = s-k_i \Rightarrow$ deg $q_i(x) \leq (r-1)-(s-k_i) = 0 \Rightarrow$ deg $q_i(x) = 0$.

This gives the number of possible ways of choosing $q_i(x) = q$.

Therefore, the total number of possible ways of choosing $q_i(x)$ and hence for $b_i(x)\ \forall\ i \in J_1$

$$= q^{|J_1|}. \tag{10}$$

Case 3. When $i \in J_2$.

In this case, we have $r-1 > s-k_i$ and i can take value 1. Also, $0 \leq$ deg $q_i(x) \leq (r-1)-(s-k_i)$.

Now, the number of possible ways of choosing $q_i(x) (i \neq 1)$

$$\begin{aligned} &= \text{number of polynomials of dgree upto } (r-1)-(s-k_i) \\ &= q^{r-s+k_i}. \end{aligned}$$

Also, the number of possibilities for $q_1(x) = (q^{r-s+k_1} - 1)$.

(Note that we have subtracted 1 from q^{r-s+k_1} to take care of the fact that $q_1(x)$ is a polynomial of degree 0 and it has to be a nonzero constant).

Therefore, the total number of possible ways of choosing $q_i(x)$ and hence $b_i(x)\ \forall\ i \in J_2$

$$= \prod_{\substack{i\in J_2\\ i\neq 1}} q^{r-s+k_i} \times (q^{r-s+k_1} - 1)$$

$$= \left(q^{(|J_2|-1)(r-s)+\sum\limits_{\substack{i\in J_2\\ i\neq 1}} k_i}\right) \times (q^{r-s+k_1} - 1). \tag{11}$$

Combining the three cases, i.e. multiplying (9), (10) and (11), we get the total number of ways of choosing $b_i(x)\ \forall\ N$ and is given by

$$1 \times q^{|J_1|} \times \left(q^{(|J_2|-1)(r-s)+\sum\limits_{\substack{i\in J_2\\ i\neq 1}} k_i}\right) \times (q^{r-s+k_1} - 1)$$

$$= \left(q^{(|J_2|-1)(r-s)+\sum\limits_{\substack{i\in J_2\\ i\neq 1}} k_i+|J_1|}\right) \times (q^{r-s+k_1} - 1). \tag{12}$$

Amongst all these possible ways, we eliminate the number of ways which give rise to the first column of submatrix $B = \begin{pmatrix} b_1(x) \\ b_2(x) \\ \vdots \\ b_m(x) \end{pmatrix}$ as zero. This will occur when $q_i(x) = 0\ \forall i\ \in J_1$ and constant term of $q_i(x) = 0\ \forall\ i \in J_2$.

The number of ways in which $b_i(x)$ can be chosen such that $q_i(x) = 0\ \forall\ i \in J_1$ and constant term of $q_i(x) = 0\ \forall\ i \in J_2$ is given by

$$1 \times \left(q^{(|J_2|-1)(r-s-1)+\sum\limits_{\substack{i\in J_2\\ i\neq 1}} k_i}\right) \times (q^{(r-1)-(s-k_1)} - 1). \tag{13}$$

Substracting (13) from (12) and the fact that the CT-burst A of order $m\times r$ can have first $(s-r+1)$ positions as the starting positions, we get the total number of CT-bursts of order r (where $r > s - k_1 + 1$) that go undetected in the row-cyclic array code C and is given by

$$(s - r + 1) \times ((12) - (13)). \tag{14}$$

Also, the total number of CT-bursts of order $m \times r$ viz. $T^{m\times r}_{m\times s}(F_q)$ is given by (1). Therefore, the desired ratio is obtained on dividing (14) by (1). □

Example 3.3. Let $C = C_1 \oplus C_2$ be a row-cyclic array code generated by $(g_1(x), g_2(x)) = (1, 1+x)$. It is clear that C_1 and C_2 are classical cyclic codes of length 2 generated by 1 and $1+x$ respectively.

Here $N = \{1, 2\}$, $k_1 = 2, k_2 = 1$ and $s = 2$.

Therefore, $s - k_1 = 0$ and $s - k_2 = 1$.

Let $r = 2$. Then $2 = r > s - k_1 + 1 = 1$.

Here $J_1 = \{2\}$ as $r - 1 = s - k_2$ and $J_2 = \{1\}$ as $r - 1 > s - k_1$.

The ratio computed in (8) for this example turns out to be 5/10 and is justified by the fact that out of the 10 CT-bursts of order 2×2 in $\mathrm{Mat}_{2\times2}(F_2)$ listed in Example 3.1, there are five CT-bursts that go undetected and these undetected bursts are given by

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

4. Decoding Algorithm for CT-Burst Error Correction

In this section, we give decoding algorithm for CT-burst error correction in row-cyclic array codes.

Algorithm.

Let $C = \bigoplus_{i=1}^{m} C_i$ be a q-ary $[m \times s, \sum_{i=1}^{m} k_i, \min_{i=1}^{m} d_i]$ row-cyclic array code having generator m-tuple of polynomials $(g_1(x), g_2(x), \cdots, g_m(x))$ and correcting all CT-burst errors of order mr or less $(1 \le r \le s)$. Let $w(x) = (w_1(x), w_2(x), \cdots, w_m(x))$ be a received array with an error pattern $e(x) = (e_1(x), e_2(x), \cdots, e_m(x))$ such that $e(x)$ is a CT-burst of order mr or less $(1 \le r \le s)$. The goal is to determine $e(x)$. This is obtained in the following four steps:

Step 1. Compute the syndrome m-tuple $(S_j^{(1)}(x), S_j^{(2)}(x), \cdots, S_j^{(m)}(x))$ for $j = 0, 1, 2, \cdots$ where for all $i = i$ *to* m, $S_j^{(i)}(x)$ is given by

$$S_j^{(i)}(x) = \text{syndrome of } x^j w_i(x).$$

Step 2. Find the m-tuple of nonnegative integers $(l_1, l_2, \cdots, l_m)$ such that syndrome for $x^{l_i} w_i(x) (1 \le i \le m)$ is a classical CT-burst of length r.

Step 3. Compute the remainder m-tuple $e(x) = (e_1(x), \cdots, e_m(x))$ where for all $i = i$ *to* $m, e_i(x)$ is given by

$$e_i(x) = x^{s-l_i} S_{l_i}^{(i)}(x)(\text{mod } (x^s - 1)).$$

Step 4. Decode $(w_1(x), \cdots, w_m(x))$ to $(w_1(x) - e_1(x), \cdots, w_m(x) - e_m(x))$.

Proof of Algorithm. First of all, we show the existence of m-tuple of nonnegative integers $(l_1, l_2, \cdots, l_m)$ in Step 2. By the assumption, there exists an error pattern $e(x) = (e_1(x), \cdots, e_m(x))$ such that $e(x)$ is a CT-burst of order mr or less which in turn implies that each $e_i(x)(1 \leq i \leq m)$ has a cyclic run of zeros of length $s-r$. (A cyclic run of zeros of length l of an s-tuple is a succession of l cyclically consecutive zero components). Thus there exists an m-tuple $(l_1, l_2, \cdots, l_m)$ such that cyclic array shift of the error $(e_1(x), \cdots, e_m(x))$ through $(l_1, l_2, \cdots, l_m)$ positions (or equivalently, cyclic shift of error $e_i(x)$ through l_i positions $(1 \leq i \leq m)$ in classical sense) has all its nonzero components confined to first r columns of e (Note that we are identyfying $e(x) \leftrightarrow e$ under the map θ). The cyclic shift of error $e_i(x)$ through l_i positions $(1 \leq i \leq m)$ is in fact the remainder of $x^{l_i} w_i(x)(\text{mod } (x^s - 1))$ divided by $g_i(x)$.

Also, for all $i = 1$ *to* m

$$\begin{aligned} S_{l_i}^{(i)}(x) &= (x^{l_i} w_i(x)(\text{mod } (x^s - 1))(\text{mod } g_i(x)) \\ &= (x^{l_i} w_i(x)(\text{mod } g_i(x)). \end{aligned}$$

Therefore, each $S_{l_i}^{(i)}(x)(1 \leq i \leq m)$ is a classical CT-burst of length r. Now, for all $i = 1$ *to* m, the word

$$t_i(x) = (x^{s-l_i} S_{l_i}^{(i)}(x))(\text{mod } (x^s - 1))$$

is a cyclic shift of $(S_{l_i}^{(i)}, 0)$ through $s - l_i$ positions, where $S_{l_i}^{(i)}$ is a vector in $F_q^{s-k_i}$ corresponding to the polynomial $S_{l_i}^{(i)}$. It is clear that each $t_i(x)$ is a classical CT-burst of order r. Also, for all $i = 1$ *to* m, we have

$$\begin{aligned} x^{l_i}(w_i(x) - t_i(x)) &= x^{l_i}(w_i(x) - x^{s-l_i} S_{l_i}^{(i)}(x)) \\ &= x^{l_i} w_i(x) - x^s S_{l_i}^{(i)}(x) \\ &= S_{l_i}^{(i)}(x) - x^s S_{l_i}^{(i)}(x) \end{aligned}$$

$$\begin{aligned} &= (1-x^s)S_{l_i}^{(i)}(x) \\ &\equiv 0(\text{mod } (g_i(x))). \end{aligned} \quad (15)$$

Since $g_i(x)$ and x^{l_i} are coprime to each other, therefore from (15), we get

$$g_i(x)|(w_i(x) - t_i(x)) \quad \forall\, i = 1, 2, \cdots, m$$

$$\Rightarrow w_i(x) - t_i(x) \in C_i \quad i = 1 \text{ to } m.$$

Also $w_i(x) - e_i(x) \in C_i$ implies $e_i(x) - t_i(x) \in C_i$ which further implies that $e_i(x)$ and $t_i(x)$ belong to the same coset (mod $g_i(x)$). Since both $e_i(x)$ and $t_i(x)$ are the classical CT-bursts of length r and each C_i is r CT-burst error correcting classical cyclic code (since $C = \bigoplus_{i=1}^{m} C_i$ corrects all bursts of order $m \times r$), we get

$$e_i(x) = t_i(x) = (x^{s-l_i} S_{l_i}^{(i)}(x))(\text{mod } (x^s - 1)).$$

□

Remark 4.1 The above algorithm also holds for the correction of all CT-bursts of order pr or less $(1 \leq p \leq m, 1 \leq r \leq s)$.

Example 4.1. Consider the binary row-cyclic array code $C = \bigoplus_{i=1}^{2} C_i$ where C_1 and C_2 are [7,4,4] classical cyclic codes in F_2^7 equipped with m-metric and generated by $g_1(x) = 1 + x^2 + x^3$ and $g_2(x) = 1 + x + x^3$ respectively. Then parameters of row cyclic code C are $[2 \times 7, 4 + 4, 4]$. The row-cyclic array code C corrects all CT-bursts of order 2×1 or less as seen from the fact that syndrome 2-tuples of all CT-burst array errors of order 2×1 or less are all distance as shown in Table 4.1.

Table 4.1

CT-bursts of order 2×1 or less in $Mat_{2\times 7}(F_2)$	Syndrome 2-tuple
$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	(100, 100)
$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	(010, 010)
$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$	(001, 001)
$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$	(101, 110)
$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$	(111, 011)
$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$	(110, 111)
$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	(011, 101)
$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	(100, 000)
$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	(010, 000)
$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	(001, 000)
$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	(101, 000)
$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	(111, 000)
$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	(110, 000)
$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	(011, 000)

Table Contd.

CT-bursts of order 2×2 or less in $\text{Mat}_{2\times 7}(F_2)$	Syndrome 2-tuple
$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	(000, 100)
$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	(000, 010)
$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$	(000, 001)
$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$	(000, 110)
$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$	(000, 011)
$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$	(000, 111)
$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	(000, 101)

The syndrome 2-tuple $S = (S_1, S_2)$ for a CT-burst $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ of order 2×1 or less for the code C have been found by using the relation $S = bH^T$ where H is the parity check matrix of the code C and is given by

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix},$$

where

$$H_1 = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

and

$$H_2 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

Now, consider the received array

$$w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \in \text{Mat}_{2\times 7}(F_2).$$

Under the identification $\theta : \text{Mat}_{m\times s}(F_2) \longleftrightarrow A_s^{(m)}$, w can be identified as

$$w = \begin{pmatrix} 1 + x^2 + x^3 + x^4 \\ 1 + x + x^3 + x^4 \end{pmatrix} = \begin{pmatrix} w_1(x) \\ w_2(x) \end{pmatrix}.$$

We Compute the syndrome $S_j^{(i)}(x)$ of $x^j w_i(x)(1 \le i \le 2)$ until $S_j^{(i)}$ is a classical CT-burst of length 1 or less.

Table 4.2

j	$S_j^{(1)}(x)$	$S_j^{(2)}(x)$
0	$1+x+x^2$	$x+x^2$
1	$1+x$	$1+x+x^2$
2	$x+x^2$	$1+x^2$
3	1	1

Therefore, $l_1 = l_2 = 3$ i.e. $(l_1, l_2) = (3, 3)$.

Decode $w_1(x) = (1011100) = 1 + x^2 + x^3 + x^4$ to $w_1(x) - t_1(x)$ where

$$\begin{aligned} t_1(x) = e_1(x) &= x^{s-l_1} S_{l_1}^{(1)}(x)(\text{mod } (x^s - 1)) \\ &= x^{7-3} S_3^{(1)}(x)(\text{mod } (x^7 - 1)) \\ &= x^4 \end{aligned}$$

Thus $w_1(x)$ is decoded to

$$w_1(x) - t_1(x) = 1 + x^2 + x^3 + x^4 - x^4 = 1 + x^2 + x^3 = 1011000$$

Similarly, decode $w_2(x) = 1101100 = 1+x+x^3+x^4$ to $w_2(x) - t_2(x)$ where

$$\begin{aligned} t_2(x) = e_2(x) &= x^{s-l_2} S_{l_2}^{(2)}(x)(\text{mod } (x^s - 1)) \\ &= x^{7-3} S_3^{(2)}(x)(\text{mod } (x^7 - 1)) \\ &= x^4 \end{aligned}$$

Therefore, $w_2(x)$ is decoded to

$$w_2(x) - t_2(x) = 1 + x + x^3 + x^4 - x^4 = 1 + x + x^3 = 1101000.$$

Hence

$$w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

is decoded to $\begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$.

Remark 4.2. Since the $[2 \times 7, 4+4, 4]$ row-cyclic array code C of Example 4.1 corrects all CT-bursts of order 2×1 or less, therefore the code C must satisfy the Rieger's bound for an $[m \times s, k]$ m-metric array code correcting all CT-bursts or order pr $(1 \leq p, m, 1 \leq r \leq s)$ obtained in [6] and is given by

$$ms - k \geq 2pr$$

or

$$ms - \sum_{i=1}^{m} k_i \geq 2pr \quad (\text{as } k = \sum_{i=1}^{m} k_i \text{ for row-cyclic array codes})$$

which is true as

$$14 - 8 \geq 2 \times 2 \times 1$$

or

$$6 \geq 4.$$

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