Möbius Functions and Characteristic Polynomials of Subspace Lattices Under the Finite Classical Groups *†

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Abstract

In this paper, some lattices generated by the orbits of the subspaces under finite classical groups are considered. the characteristic polynomials of these lattices are obtained by using the effective approach by Aigner in [2] , and their expressions are also determined.

1 Introduction

Möbius functions and the characteristic polynomial of poset and lattices have been discussed detailedly in [1, 2]. Some functions of lattice listing in [3] of which Möbius function were proved afterwards. In [2], some Möbius functions were listed by two methods, that is to say, their Möbius functions are derived by using the property of lattice and definition of Möbius functions, or levels of the characteristic polynomial. This paper mainly gives that lattices generated by the orbits of the subspaces under finite classical groups (see [3-5]), and their Möbius functions are given by the second mention above.

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This paper follows the terms of [2, 3], and quotes the works showing in [2, 4].

Definition 1.1[1–3] Let P be a poset with 0 and $\mathbb{N}_0$ be a nonnegative integer set. Consequently, the following functions

$$\begin{aligned} r : P &\longrightarrow \mathbb{N}_0 \\ a &\longmapsto r(a) \end{aligned}$$

are called *the rank function of P* if in the following (i) and (ii) exist.

(i) $r(0) = 0$,

(ii) For $a, b \in P$, plus $a <\cdot b$, then $r(b) = r(a) + 1$.

In the poset P with 0 if $l(P) = n$, then n is called rank of P and is written that $r(P) := l(P)$.

Suppose that L is a lattice, if all elements a, b, c of L meet the conditions of $c \le a \Rightarrow a \wedge (b \vee c) = (a \wedge b) \vee c$, then L will be defined as modular lattice. From theorem 2.27 in [2], L is a modular lattice if and only if L possesses rank function r, as well as there exist modular equation for all $x, y \in L$

$$r(x \wedge y) + r(x \vee y) = r(x) + r(y). \tag{1}$$

Suppose that L is a geometry lattice (see [5, 6]), as well as $a \in L$. If all $x \in L$, there exists that $r(a \wedge x) + r(a \vee x) = r(a) + r(x)$, then a can be defined as the modular element of L, we denoted it as aM. If the geometry lattice L is a modular lattice, then L is called the modular geometry lattice.

Suppose $\mathbb{F}_q$ is a field with q elements, n is an integer ≥ 1, and $\mathbb{F}_q^n$ is the n-dimensional vector space over $\mathbb{F}_q$, as well as $\mathbb{L}(\mathbb{F}_q^n)$ is the family that consists of all subspaces of $\mathbb{F}_q^n$. If $U, T \in \mathbb{L}(\mathbb{F}_q^n)$, $U \subset T$, can be defined as $U \le T$, then $\mathbb{L}(\mathbb{F}_q^n)$ is a modular geometry lattice, and $X \in \mathbb{L}(\mathbb{F}_q^n), r(X) = \dim X, r(\mathbb{L}(\mathbb{F}_q^n)) = n$.

Definition 1.2[1–3] Suppose that P is a finite poset containing 0 and 1 as well as rank function r and Möbius function μ, the following polynomial can be defined as characteristic polynomial of P.

$$\chi(P, x) = \sum_{a \in P} \mu(0, a) x^{r(1) - r(a)}.$$

The coefficient $w_k = \sum\limits_{r(a)=k} \mu(0, a)$ of $x^{r(1)-k}$ is called the k-th level number of first kind, the cardinality of the k-th level $W_k = \sum\limits_{r(a)=k} 1$ the k-th level number of the second.

2 Main Results

Using a method in [2] and levels of the characteristic polynomial, we give the following Theorem.

Theorem 2.1 The characteristic polynomial of $\mathbb{L}(\mathbb{F}_q^n)$ is as follow:

$$\chi(\mathbb{L}(\mathbb{F}_q^n), x) = \prod_{i=0}^{n-1}(x - q^i), \tag{2}$$

and there exists the following equations in $\mathbb{L}(\mathbb{F}_q^n)$

$$\begin{aligned} &w_k = (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q, \ W_k = \begin{bmatrix} n \\ k \end{bmatrix}_q, \ \mu(\mathbb{L}(\mathbb{F}_q^n)) = (-1)^n q^{\binom{n}{2}}, \\ &\mu(0, a) = (-1)^k q^{\binom{k}{2}}, a \in \mathbb{L}(\mathbb{F}_q^n), \text{ dim } a = k. \end{aligned} \tag{3}$$

Proof Form [3, p21], we can get that $\mathbb{L}(\mathbb{F}_q^n)$ is a geometry lattice and the modular equality holds. Let a_i be i-dimensional subspace of $\mathbb{L}(\mathbb{F}_q^n)$, $(1 \le i \le n)$. Clearly, $0 < \cdot a_1 < \cdot a_2 < \cdots < \cdot a_n = \mathbb{F}_q^n$ is the maximal chain in $\mathbb{L}(\mathbb{F}_q^n)$, and $r(a_i) = \text{dim } a_i$. Therefore, a_{i-1} is a modular element in $[0, a_i]$. Because the 1-dimensional subspace of $\mathbb{F}_q^n$ is one of points in $\mathbb{L}(\mathbb{F}_q^n)$ and the number of points which are in $[0, a_i]$ but not in $[0, a_{i-1}]$ is $\begin{bmatrix} i \\ 1 \end{bmatrix}_q - \begin{bmatrix} i-1 \\ 1 \end{bmatrix} = q^{i-1}$. On the basis of Stanley Theorem in [2], (2) holds. From q-binomial theorem, there exists:

$$\prod_{i=0}^{n-1}(x - q^i) = \sum_{k=0}^{n}(-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^{n-k}.$$

Since $W_k = \begin{bmatrix} n \\ k \end{bmatrix}_q$, $w_k = (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q$, $\mu(0, a) = w_k / W_k$, we have (3).

Now let's give that Möbius functions and characteristic polynomials of subspace lattice under the action of finite classical group.

Let G_n be one of the classical groups of degree n over $\mathbb{F}_q$, i.e., $G_n = GL_n(\mathbb{F}_q)$, $Sp_{2\nu}(\mathbb{F}_q)$ (where $n = 2\nu$ is even), $U_n(\mathbb{F}_{q^2})$, $O_{2\nu+\delta}(\mathbb{F}_q)$ $(n = 2\nu + \delta, \delta = 1, 2)$, or $Ps_n(\mathbb{F}_q)$ (where q is even) (see [4]). Define G_n acts on $\mathbb{F}_q^n$ as follows:

$$\begin{gathered} \mathbb{F}_q^n \times G_n \longrightarrow \mathbb{F}_q^n \\ ((x_1, x_2, \cdots, x_n), T) \longmapsto (x_1, x_2, \cdots, x_n)T. \end{gathered}$$

(In case $G_n = U_n(\mathbb{F}_{q^2})$, $\mathbb{F}_q^n$ should be replaced by $\mathbb{F}_{q^2}^n$). This action induces an action of G_n on the set of subspaces of $\mathbb{F}_q^n$, i.e., if P is an m-dimensional subspace of $\mathbb{F}_q^n$ $(1 \le m \le n)$, and usually use the same letter P to denote a matrix representation of the m-dimensional subspace P, then it is carried by $T \in \mathbb{F}_q^n$ into the m-dimensional subspace PT. The set of subspaces of

$\mathbb{F}_q^n$ are partitioned into orbit sets under G_n. Clearly, $\{0\}$ and $\{\mathbb{F}_q^n\}$ are two orbit, but they are less interesting. In the following orbits of subspaces under G_n distinct from these two are our main concern. Let $\mathcal{M}$ be any orbit of subspaces under G_n. Denote by $\mathcal{L}(\mathcal{M})$ the set of subspaces which are intersection of subspaces of $\mathcal{M}$. We make the convention that the intersection of an empty set of subspace is $\mathbb{F}_q^n$. Then $\mathbb{F}_q^n \in \mathcal{L}(\mathcal{M})$. Partially order $\mathcal{L}(\mathcal{M})$ by reverse inclusion, i.e., for $P, Q \in \mathcal{L}(\mathcal{M})$, we define $P \leq Q$ if $P \supset Q$, so that $\mathcal{L}(\mathcal{M})$ has $\mathbb{F}_q^n$ as its minimum element and $\mathcal{M}$ as its set of atoms. The poset $\mathcal{L}(\mathcal{M})$ is a finite lattice, is called the lattice generated by $\mathcal{M}$.

For any $X \in \mathcal{L}(\mathcal{M})$,

$$r(X) = \begin{cases} m+1-\dim X, & \text{if } X \neq \mathbb{F}_q^n, \\ 0, & if\ X = \mathbb{F}_q^n. \end{cases}$$

Then r is the rank function of $\mathcal{L}(\mathcal{M})$, $r(\mathcal{L}(\mathcal{M}) = m+1$.

In this paper, we just listed the calculation in the $G_n = GL_n(\mathbb{F}_q)$, $Sp_{2\nu}(\mathbb{F}_q)$, and $U_n(\mathbb{F}_{q^2})$. The rest can be found in other references as well. With regards to the characteristic polynomial of $\mathcal{L}(\mathcal{M})$, the calculation of which is wrong in [5,7]. At present, we recalculate it as below.

(1) The case of $G_n = GL_n(\mathbb{F}_q)$

Let $\mathcal{M} = (m, n)$ be a set that consists of all m-dimensional subspaces of $\mathbb{F}_q^n$, when $1 \leq m \leq n$, $\mathcal{M} = \mathcal{M}(m,n)$ is an orbit under the action of $GL_n(\mathbb{F}_q)$. Denoted by $\mathcal{L}(m,\ n)$ the lattice of subspaces generated by $\mathcal{M}(m,\ n)$ (see [6,7]). By Theorem 2.13 in [3], when $0 \leq m < n$, $\mathcal{L}(m,\ n)$ consists of $\mathbb{F}_q$ and all subspaces of dimension $\leq m$.

Theorem 2.2 Suppose that $0 \leq m \leq n$, then

$$\chi(\mathcal{L}(m,\ n),\ x) = g_{m+1}(x) + \left(\sum_{j=0}^{m} \begin{bmatrix} m+1 \\ j \end{bmatrix}_q - \sum_{j=0}^{m} \begin{bmatrix} l \\ j \end{bmatrix}_q\right) g_j(x). \qquad (4)$$

where $g_0 = 1$, $g_h(x) = (x-1)(x-q)\cdots(x-q^{h-1})$ $(h = m+1, j)$ is Gauss polynomial.

Proof For any $X \in \mathcal{L}(m,n)$,

$$r(X) = \begin{cases} m+1-\dim X, & \text{if} X \neq \mathbb{F}_q^n, \\ 0, & \text{if} X \neq \mathbb{F}_q^n. \end{cases} \qquad (5)$$

We can get that $r : \mathcal{L}(m,\ n) \longrightarrow \mathbb{N}_0$ is the rank function of $\mathcal{L}(m,\ n)$.

Suppose that V_i is the $m+1$-dimensional subspace of $\mathbb{F}_q^n$, $i = 1, 2, \cdots$, $\mathcal{A}_i$ is a set consisting of m-dimensional subspace of V_i, and $\mathcal{L}(\mathcal{A}_i)$ is a lattice generated by $\mathcal{A}_i$, the partially by reverse inclusion, as well as $\mathbb{F}_q^{m+1}$ is a lattice generated by the subspaces of $\mathbb{F}_q^{m+1}$ by reverse inclusion. We write $V_0 = \mathbb{F}_q^n$, $\mathcal{L}_0 = \mathcal{L}(m,n)$, and $\mathcal{L}_i = \mathcal{L}(\mathcal{A}_i)$ as above. For $P \in \mathcal{L}_0$ and $P \in \mathcal{L}_i$, we define

$$\mathcal{L}_0^P = \{Q \in \mathcal{L}_0 | Q \subset P\} = \{Q \in \mathcal{L}_0 | Q \geq P\}$$

and

$$\mathcal{L}_i^P = \{Q \in \mathcal{L}_i | Q \subset P\} = \{Q \in \mathcal{L}_i | Q \geq P\},$$

respectively. Clearly, $\mathcal{L}_0^{V_0} = \mathcal{L}_0$, $\mathcal{L}_i^{V_i} = \mathcal{L}_i$. For $P \in \mathcal{L}_0 \backslash \{V_0\}$, then $P \in \mathcal{L}_j$, where j is a fixed number among $1, 2, \cdots, j$. According to Theorem 2.13 in [3], there exists $\mathcal{L}_0^P = \mathcal{L}_1^P$. For any $P \in \mathcal{L}_i \backslash \{V_i\}$, by Proposition 2.4 of [3], there exists $\chi(\mathcal{L}_i^P, x) = g_{\dim P}(x)$, where $g_{\dim P}(x) = (x-1)(x-q)\cdots(x-q^{\dim P-1})$ is Gauss polynomial. Since $V_i \simeq \mathbb{F}_q^{m+1}$, there exists $\mathcal{L}(\mathcal{A}_i) \simeq \mathcal{L}(\mathbb{F}_q^{m+1})$. In consideration of isomorphism lattice possess the same rank function and characteristic polynomial, therefore, both $\mathcal{L}(\mathcal{A}_i)$ and $\mathcal{L}(\mathbb{F}_q^{m+1})$ possess the same rank function r.

$$r(X) = \begin{cases} m+1-\dim X, & \text{if } X \in \mathcal{L}_i \backslash \{V_i\}, \\ 0, & if\ X = V_i, \end{cases}$$

and the characteristic polynomial $\chi(\mathcal{L}_i, x) = \chi(\mathcal{L}(\mathbb{F}_q^{m+1}), x)$, and $\mathcal{L}_0$ possesses maximum element 0 and minimum element V_0, and $\mathcal{L}_1$ possesses maximum element 0 and minimum element V_1, resulting in the characteristic polynomial of $\mathcal{L}_0$ and $\mathcal{L}_1$, are $\chi(\mathcal{L}^{V_0}, x) = \sum_{P \in \mathcal{L}^{V_0}} \mu(V_0, P) x^{r'(0)-r'(P)}$and $\chi(\mathcal{L}_1^{V_1}, x) = \sum_{P \in \mathcal{L}_1^{V_1}} \mu(V_1, P) x^{r'(0)-r'(P)}$,respectively. As regards above two equations, we perform Möbius inversion, to get

$$x^{m+1} = \sum_{P \in \mathcal{L}_0^{V_0}} \chi(\mathcal{L}_0^P, x) = \sum_{P \in \mathcal{L}_0} \chi(\mathcal{L}_0^P, x)$$

and

$$x^{m+1} = \sum_{P \in \mathcal{L}_1^{V_1}} \chi(\mathcal{L}_1^P, x) = \sum_{P \in \mathcal{L}_1} \chi(\mathcal{L}_1^P, x).$$

Hence

$$\begin{aligned} \chi(\mathcal{L}(m,n), x) = \chi(\mathcal{L}_0^{V_0}, x) &= x^{m+1} - \sum_{P \in \mathcal{L}_0 \backslash \{V_0\}} \chi(\mathcal{L}_0^P, x) \\ &= \sum_{P \in \mathcal{L}_1} \chi(\mathcal{L}_1^P, x) - \sum_{P \in \mathcal{L}_0 \backslash \{V_0\}} \chi(\mathcal{L}_0^P, x). \end{aligned}$$

Because $\chi(\mathcal{L}_1^{V_1}, x) = g_{m+1}(x)$, and there exists $\mathcal{L}_1^P = \mathcal{L}_0^P$ when $P \in \mathcal{L}_0 \backslash \{V_0\}$. Therefore,

$$\chi(\mathcal{L}(m, l), x) = \chi(\mathcal{L}_0^{V_0}, x) = g_{m+1}(x) + \sum_{P \in \mathcal{L}_1 \backslash \{V_1\}} \chi(\mathcal{L}_1^P, x) - \sum_{P \in \mathcal{L}_0 \backslash \{V_0\}} \chi(\mathcal{L}_0^P, x).$$

Because $\mathcal{L}_0$ and $\mathbb{F}_q^n$ possess the same j-dimensional subspaces, where $0 \leq j \leq m < n$, and the number of j-dimensional subspaces in $\mathcal{L}_0$ and $\mathcal{L}_1$, is $\begin{bmatrix} n \\ j \end{bmatrix}_q$ and $\begin{bmatrix} m+1 \\ j \end{bmatrix}_q$, respectively. Therefore, (4) holds.

Remarks: The dual of poset $\mathbb{L}(\mathbb{F}_q^n, \leq)$ is denoted by the poset $\mathbb{L}^*(\mathbb{F}_q^n, \leq^*)$. Since $\mathcal{L}(n-1, n) = \mathbb{L}^*(\mathbb{F}_q^n)$, and $\mathbb{L}^*(\mathbb{F}_q^n) \simeq \mathbb{L}(\mathbb{F}_q^n)$, we have $\chi(\mathcal{L}(n-1, n), x) = \chi(\mathcal{L}(\mathbb{F}_q), x) = g_n(x)$.

Corollary 2.3 Suppose that $0 \leq m < n$. Then in $\mathcal{L}(m, n)$, $w_i (0 \leq i \leq m)$ and w_{m+1} are

$$w_i = (-1)^i q^{\binom{i}{2}} \begin{bmatrix} m+1 \\ i \end{bmatrix} - \sum_{j=0}^{m} \left(\sum_{k=1}^{n-m-1} q^{n-k+1-j} \begin{bmatrix} n-k \\ j-1 \end{bmatrix}_q \right) \left((-1)^i q^{\binom{i}{2}} \begin{bmatrix} j \\ i \end{bmatrix}_q \right)$$
$$w_{m+1} = (-1)^{m+1} q^{\binom{m+1}{2}}, \tag{6}$$

respectively, and

$$\mu(\mathcal{L}(m, n)) = (-1)^{m+1} q^{\binom{m+1}{2}}. \tag{7}$$

Proof According to recurrence formula (see [8], p127) $\begin{bmatrix} n+1 \\ m \end{bmatrix}_q = \begin{bmatrix} n \\ m \end{bmatrix}_q + q^{n+1-m} \begin{bmatrix} n \\ m-1 \end{bmatrix}_q$, we obtain $\begin{bmatrix} n \\ j \end{bmatrix}_q - \begin{bmatrix} m+1 \\ j \end{bmatrix}_q = \sum_{k=1}^{n-m-1} q^{n-k+1-j} \begin{bmatrix} n-k \\ j-1 \end{bmatrix}_q$. Therefore

$$\chi(\mathcal{L}(m, n), x) = g_{m+1}(x) - \sum_{j=0}^{m} (\sum_{k=1}^{n-m-1} q^{n-k+1-j} \begin{bmatrix} n-k \\ j-1 \end{bmatrix}_q) g_j(x).$$

On the basis of Gauss polynomial

$$g_{m+1}(x) = \sum_{i=0}^{m+1} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} m+1 \\ i \end{bmatrix}_q x^i, \ g_j(x) = \sum_{i=0}^{j} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} j \\ i \end{bmatrix}_q x^i,$$

there exists

$$\chi(\mathcal{L}(m, n), x) = \sum_{i=0}^{m+1} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} m+1 \\ i \end{bmatrix}_q x^i$$
$$- \sum_{j=0}^{m} \left(\sum_{k=1}^{n-m-1} q^{n-k+1-j} \begin{bmatrix} n-k \\ j-1 \end{bmatrix}_q \right) \left(\sum_{i=0}^{j} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} j \\ i \end{bmatrix}_q x^i \right).$$

Let $a \in \mathcal{L}(m,n)$, $\dim a = i$, in case of $a \neq \mathbb{F}_q^n$, $x^{r(0)-r(a)} = x^i$; in case of $a = \mathbb{F}_q^n$, $x^{r(0)-r(a)} = x^{m+1}$. In addition, the coefficient of x^i is

$$(-1)^i q^{\binom{i}{2}} \begin{bmatrix} m+1 \\ i \end{bmatrix}_q - \sum_{j=0}^{m} \Big(\sum_{k=1}^{n-m-1} q^{n-k+1-j} \begin{bmatrix} n-k \\ j-1 \end{bmatrix}_q \Big) \Big((-1)^i q^{\binom{i}{2}} \begin{bmatrix} j \\ i \end{bmatrix}_q \Big),$$

and w_i is equal to the coefficient of x^i $(0 \leq i \leq m)$, w_{m+1} is equal to the coefficient of x^{m+1}, hence (6) holds. Since $\mu(\mathcal{L}(m,n)) = w_{m+1}$, (7) holds.

(2) The cases of $Sp_{2\nu}(\mathbb{F}_q)$ $(n = 2\nu)$

We Suppose that $n = 2\nu$, where ν is a positive integer. Let $K = \begin{pmatrix} 0 & I^{(\nu)} \\ -I^{(\nu)} & 0 \end{pmatrix}$. The symplectic group of degree 2ν over $\mathbb{F}_q$, denoted by $Sp_{2\nu}(\mathbb{F}_q)$, consists of all $2\nu \times 2\nu$ matrices T such that $TK\,{}^tT = K$, where ${}^tT$ denotes the transpose of T. An m-dimensional subspace is said to be of type (m,s) if $PK\,{}^tP$ is of rank $2s$. It is known that the suspaces of type (m,s) exist if and only

$$2s \leq m \leq \nu + s \tag{8}$$

(see [4]) and that the set of subspaces of the same type form a orbit under $Sp_{2\nu}(\mathbb{F}_q)$. Denote the orbit by $\mathcal{M} = \mathcal{M}(m,s;2\nu)$ and the lattice generated by $\mathcal{M}(m,s;2\nu)$ by $\mathcal{L}(m,s;2\nu)$[6]. According to Theorem 3.6 of [3], when $1 < n = 2\nu$ and (m,s) satisfies (8) and $m \neq 2\nu$, then $\mathcal{L}_R(m,s;2\nu)$ consists of $\mathbb{F}_q^{2\nu}$ and all subspaces type (m_1,s_1) which satisfies (8) and

$$m - m_1 \geq s - s_1 \geq 0. \tag{9}$$

Theorem 2.4 Suppose that $1 \leq m < 2\nu$, (m,s) satisfy (8), then

$$\chi(\mathcal{L}_R(m,\ s,\ 2\nu),\ x) = g_{m+1}(x) + \Big(\sum_{m_1=0}^{m} \begin{bmatrix} m+1 \\ m_1 \end{bmatrix}_q - \sum_{*} N(m_1,\ s_1;\ 2\nu) \Big) g_{m_1}(x), \tag{10}$$

where (*) represents the condition that satisfy formula (8) and (9), and $N(m_1,\ s_1;\ 2\nu) = |\mathcal{M}(m_1,s_1;2\nu)|$ is the number of subspaces of type $(m_1,\ s_1)$ in $\mathbb{F}_q^{2\nu}$, and $g_h(x)$ $(h = m+1, m_1)$ is Gauss polynomial.

Proof For any $X \in \mathcal{L}(m,\ s;\ 2\nu)$, we define

$$r(X) = \begin{cases} m+1-\dim X, & \text{if } X \neq \mathbb{F}_q^{2\nu}, \\ 0, & \text{if } X = \mathbb{F}_q^{2\nu}. \end{cases}$$

Then $r : \mathcal{L}(m,\ s;\ 2\nu) \longrightarrow \mathbb{N}_0$ is the rank function of lattice $\mathcal{L}(m,\ n;\ 2\nu)$(see [3]).

Suppose that V_i is the $m+1$-dimensional subspace of $\mathbb{F}_q^{2\nu}$, where $i = 1, 2, \cdots$, $\mathcal{A}_i$ is a set which consists of m-dimensional subspace of V_i. Moreover, $\mathcal{L}(\mathcal{A}_i)$ is a set which is a lattices generated by $\mathcal{A}_i$ according to reverse inclusion. We write $V_0 = \mathbb{F}_q^{2\nu}$, $\mathcal{L}_0 = \mathcal{L}(m, s; 2\nu)$, and $\mathcal{L}_i = \mathcal{L}(\mathcal{A}_i)$ for convenience.

Same as the deduction of Theorem 2.2, there exists

$$\chi(\mathcal{L}(m,\ s;\ 2\nu),\ x) = g_{m+1}(x) + \sum_{P\in\mathcal{L}_1\backslash\{V_1\}} \chi(\mathcal{L}_1^P),\ x) - \sum_{P\in\mathcal{L}_0\backslash\{V_0\}} \chi(\mathcal{L}_0^P,\ x).$$

It is asserted that the subspace satisfied (8) and (9) is same in $\mathcal{L}_0 = \mathcal{L}(m,\ s;\ 2\nu)$ and $\mathbb{F}_q^{2\nu}$. In fact, let $\mathcal{M}(m,\ s;\ 2\nu)$ be the set of subspaces of type (m_1, s_1) which satisfied (8), (9), and $m_1 \neq 2\nu$ in $\mathbb{F}_q^{2\nu}$. By Theorem 3.5 in [3], the results as follows:

$$\mathcal{M}(m_1,\ s_1;\ 2\nu) \subseteq \mathcal{L}(m_1,\ s_1;\ 2\nu) \subseteq \mathcal{L}(m,\ s;\ 2\nu).$$

We ascertain (m_1, s_1) $(m_1 \neq 2\nu)$ which satisfy (8) and (9), then we get the number of subspaces of type (m_1, s_1) in $\mathcal{L}_R(m, s; 2\nu)$ is $N(m_1,\ s_1;\ 2\nu)$ according to Theorem 3.18 of [4]. On the basis of the deduction of Theorem 2.2, the number of m-dimensional subspaces in $\mathcal{L}_1\backslash\{V_1\}$ is $\begin{bmatrix} m+1 \\ m_1 \end{bmatrix}_q$, hence (10) holds where (*) represents (m_1, s_1) that satisfies formula (8) and (9).

Because $\mathcal{L}(2\nu-1,\ \nu-1,\ 2\nu) = \mathcal{L}^*(\mathbb{F}_q)$, we have(see [3])

Corollary 2.5 Suppose that $1 \leq m < 2\nu$, (m, s) satisfies (8). Then w_i $(0 \leq i \leq m)$ and w_{m+1} in $\mathcal{L}(m,\ s,\ 2\nu)$ are, respectively

$$w_i = (-1)^i q^{\binom{i}{2}} \begin{bmatrix} m+1 \\ i \end{bmatrix} + \Big(\sum_{m_1=0}^{m} \begin{bmatrix} m+1 \\ m_1 \end{bmatrix}_q - \sum_{*} N(m_1,\ s_1;\ 2\nu) \Big) \Big((-1)^i q^{\binom{i}{2}} \begin{bmatrix} m_1 \\ i \end{bmatrix}_q \Big),$$
$$w_{m+1} = (-1)^{m+1} q^{\binom{m+1}{2}} \tag{11}$$

and

$$\mu(\mathcal{L}(m,\ s,\ 2\nu) = (-1)^{m+1} q^{\binom{m+1}{2}}. \tag{12}$$

Proof From formula (10) and the expression of Gauss polynomial, we have

$$\chi(\mathcal{L}(m,\ s,\ 2\nu),\ x) = \sum_{i=0}^{m+1} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} m+1 \\ i \end{bmatrix} x^i$$
$$+ \Big(\sum_{m_1=0}^{m} \begin{bmatrix} m+1 \\ i \end{bmatrix}_q - \sum_{*} N(m_1,\ s_1;\ 2\nu) \Big) \Big((-1)^i q^{\binom{i}{2}} \begin{bmatrix} m_1 \\ i \end{bmatrix}_q \Big) x^i,$$

With regards to $0 \leq i \leq m$, the coefficient of x^i is w_i and the coefficient of x^{m+1} is w_{m+1}. Therefore (11) holds. Since $\mu(\mathcal{L}(m,\ s,\ 2\nu) = w_{m+1}$, (12) holds.

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