Upper Bounds of Four Types of Graph Labelings

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Abstract

In this paper we give upper bounds of the number of edges in four types of labeled graphs of known orders.

0 Introduction

A graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions.
Over the past four decades in excess of 1200 papers have spawned a bewildering array of graph labeling methods. Despite the unabated procession of papers, there are few general results on graph labelings. Indeed, the papers focus on particular families of graphs and methods, and feature ad hoc arguments [1].

In this paper we give samples of upper bounds of the number of edges of four types of labeled graphs with given orders.
These upper bounds could be useful for eliminating many families of graphs, that could not be labeled with a certain type of labeling, which sounds good, considering the fact that: to disprove that a family doesn't possess a certain labeling is much harder to prove that this family possesses it.

Also this general dealing could draw us out of proving or disproving the labeling of a certain family to say instead of it that $K_n \backslash ie$ (= K_n *after deleting* i *edges*) cannot be labeled with a type of graph labelings for certain values of n and i.
We introduce four types of graph labelings, for each of them we give its definition followed by a theorem, which gives the upper bound, and some spontaneous results.

Throughout this paper we use the basic notations and terminology in graph theory as in [2].

1 Prime cordial labeling

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Definition 1.1 [1, 6, 7]: A prime cordial labeling of a graph $G = (V(G), E(G))$ is a bijection f from $V(G)$ to $\{1, 2, \ldots, |V(G)|\}$, such that if each edge uv is assigned the label 1 if $gcd(f(u), f(v)) = 1$ and 0 if $gcd(f(u), f(v)) > 1$, then the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1.

Theorem 1.2 [3]: $\phi(1) + \phi(2) + \phi(3) + \cdots + \phi(n) = \frac{3n^2}{\pi^2} + O(nlogn)$
Where ϕ is Euler's function: $\phi : \mathbb{N} \to \mathbb{N}; \phi(t) = |\{s \in \mathbb{N}; s < t, \gcd(s, t) = 1\}|$

In *Theorem 1.3* we give an upper bound for the number of edges of a graph G to be a prime cordial graph.

Theorem 1.3: Let G be a graph of order n and number of edges $|E(G)|$, which is greater than $u_1 = n(n-1) - 6n^2/\pi^2 + 3$, then the graph G can't be a prime cordial graph.

Proof: We count the all edges that could be labeled 1 (so we can count those which could be labeled 0 since the edge which is not labeled 1 is labeled 0).
To count the edges labeled 1 we are going to use Euler's function ϕ, depending on it we define another function Φ as follows:

$$\Phi : \mathbb{N} \to \mathbb{N}; \Phi(n) = \sum_{i=2}^{n} \phi(i)$$

Now, when G is a graph of order n, then $\Phi(n)$ is exactly the number of all possible edges that could be labeled 1.

On the other hand, the number of possible edges could be labeled 0 is:

$$\frac{n(n-1)}{2} - \Phi(n).$$

We have also by *Theorem 1.2*: $\sum_{i=1}^{n} \phi(i) = \frac{3n^2}{\pi^2} + O(nlogn)$, so we can induce that: $\Phi(n) = \sum_{i=2}^{n} \phi(i) = \sum_{i=1}^{n} \phi(i) - 1 = \frac{3n^2}{\pi^2} + O(nlogn) - 1 \geq \frac{3n^2}{\pi^2} - 1$.

i.e. $\Phi(n) \geq \frac{3n^2}{\pi^2} - 1$, which means also that: $\frac{n(n-1)}{2} - \Phi(n) \leq \frac{n(n-1)}{2} - \frac{3n^2}{\pi^2} + 1$.
Now by comparing the two amounts: $\frac{3n^2}{\pi^2} - 1$ and $\frac{n(n-1)}{2} - \frac{3n^2}{\pi^2} + 1$, we can find that:
$\frac{3n^2}{\pi^2} - 1 > \frac{n(n-1)}{2} - \frac{3n^2}{\pi^2} + 1, n \geq 3 \Rightarrow \Phi(n) > n(n-1)/2 - \Phi(n), n \geq 3$.

So by *Definition 1.1* , a prime cordial graph could maximally contain the following number of edges:

$$n(n-1) - 2\Phi(n) + 1 \leq u_1 = n(n-1) - \frac{6n^2}{\pi^2} + 3$$

Note 1.4: Since $u_1$cannot be an integer, for every n, then if we set $U_1 = \lceil u_1 \rceil$, and we can deduce the following: If G is a graph of order n with $|E(G)| \geq U_1$, then the graph G is not a prime cordial graph.

Note 1.5: For small n we will get better upper bound by calculating $n(n-1) - 2\Phi(n) + 1$ as an exact upper bound, and this works for $n = 3$ and 4,(*the cases in which* u_1*does not work*).

Results 1.6: The graphs $K_n \backslash ie$, where $n \geq 5, i = 0,1,2, \dots, \frac{n(n-1)}{2} - U_1$, are not prime cordial graphs. And this proves the conjecture:(*All complete graphs* K_n, $n > 2$ *are not prime cordial graphs*) [6].

In the following plots we can see: U_1"brown points", $n(n-1) - 2\Phi(n) + 1$ "blue points" and $n(n-1)/2$ "Green points", for $2 \leq n \leq 50$:

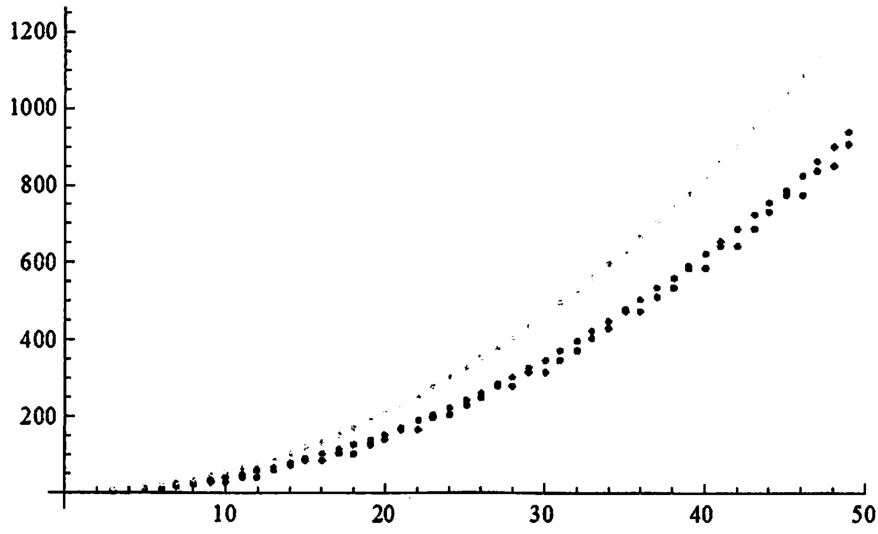

Figure 1

2 k-equitable graph labeling

Definition 2.1: Cahit [5] introduced the concept of k-equitable graphs. A 2-equitable graph is called cordial. It is a weaker verision of both graceful and harmonious graphs. For a binary labeling f from the vertex set $V(G)$ of a graph to the set: $\{0,1,\dots,k-1\}$ and an edge $e = xy$ of G, we define the induced mapping f^* as $f^*(e) = |f(x) - f(y)|$. The labeling f is called a k-equitable labeling if the number of vertices with label i and the number of vertices with label j differ by at most 1, and the number of edges with label i and the number of edges with label j differ by at most 1. By $v_f(i)$ we mean the number of vertices with label i and by $e_f(i)$ we mean the number of edges with label i. Thus For an equitable labeling we must have $|v_f(i) - v_f(j)| \leq 1, |e_f(i) - e_f(j)| \leq 1$ for all $0 \leq i, j \leq k - 1$.

Theorem 2.2: A necessary condition for a graph G of order n to be a k-equitable graph, is that its number of edges $|E(G)| \leq u_2$, where $u_2 = k\left\lceil\frac{n}{k}\right\rceil^2 + k - 1$.

Proof: The maximum number of the vertex labels $v_f(i)$ is $\left\lceil\frac{n}{k}\right\rceil$, $0 \leq i \leq k$, precisely, for the edge label $k - 1$, it can be frequented by the number $\left\lceil\frac{n}{k}\right\rceil^2$, because all possible cases of getting this edge label are by connecting all vertices of label 0 and all vertices of label $k - 1$, i.e. multiplying $\left\lceil\frac{n}{k}\right\rceil$ twice. Now the total number of edges can't exceed the number:

$$\left\lceil\frac{n}{k}\right\rceil^2 + (k-1)\left(\left\lceil\frac{n}{k}\right\rceil^2 + 1\right) = k\left\lceil\frac{n}{k}\right\rceil^2 + k - 1$$

Considering that we could have maximally $\left(\left\lceil\frac{n}{k}\right\rceil^2 + 1\right)$ labeled edges of the labels: $\{0, 1, 2, \ldots, k-2\}$.

Note 2.3: *Figure 2* shows the upper bound "the green layers" with the number of edges in complete graph "the blue plot" for $n = 2,3, \ldots, 11$ and $k = 2,3, \ldots, 8$.

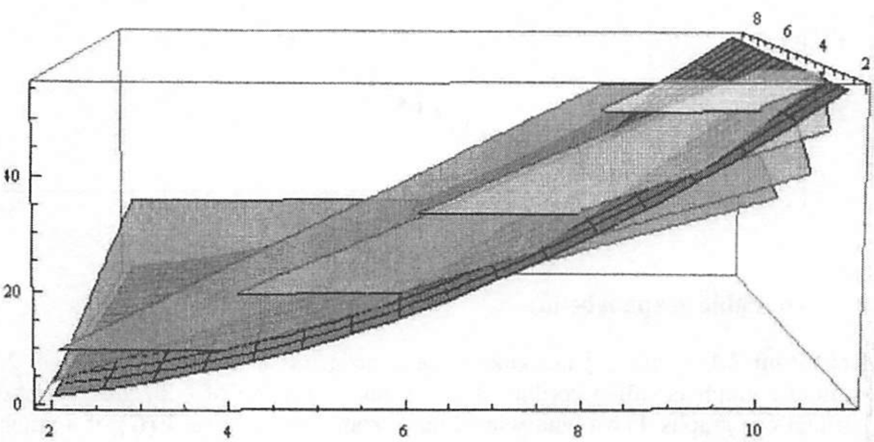

Figure 2

Note 2.4: The red numbers in the following table denote when the upper bounds work, while the black ones are greater than the number of edges in the corresponding complete graphs and hence they don't work.

n	$\frac{n(n-1)}{2}$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$
$n=2$	1	3	5	7	9	11	13	15	17
$n=3$	3	9	5	7	9	11	13	15	17
$n=4$	6	9	14	7	9	11	13	15	17
$n=5$	10	19	14	19	9	11	13	15	17
$n=6$	15	19	14	19	24	11	13	15	17
$n=7$	21	33	29	19	24	29	13	15	17
$n=8$	28	33	29	19	24	29	34	15	17
$n=9$	36	51	29	39	24	29	34	39	17
$n=10$	45	51	50	39	24	29	34	39	44
$n=11$	55	73	50	39	49	29	34	39	44

3 *(k, d)*-odd mean labeling

Definition 3.1: Gayathri and Amuthavalli [1] say a graph $G(p,q)$ has a (k,d)-odd mean labeling if there exists an injection f from the vertices of G to $\{0,1,2,\ldots,2k-1+2(q-1)d\}$ such that the induced map f^* defined on the edges of G by $f^*(uv)=\lceil\frac{f(u)+f(v)}{2}\rceil$ is a bijection from the edges of G to $\{2k-1, 2k-1+2d, 2k-1+4d,\ldots,2k-1+2(q-1)d\}$.

Theorem 3.2: Let $G(p,q)$ be a graph of order $p=2t+1$, $t\geq 2$, and let $u>2t(t-1)+k/d+1$. If the number of the edges $q\geq u$, then the graph G is not a (k,d)-odd mean graph.

Proof: Let $q=u$, the following $\left(\frac{t(t-1)}{2}+1\right)$ edge labels: $2k-1+2d(u-1), 2k-1+2d(u-2),\ldots,2k-1+2d\left(u-\left(\frac{t(t-1)}{2}+1\right)\right)$, need at least $t+1$ vertex labels with minimum vertex label, which could be found from the solution "y" of the equation:

$\left\lceil\frac{y+\text{maximum vertex label}}{2}\right\rceil=$ minmum edge label above $\Rightarrow\left\lceil\frac{y+2k-1+2d(u-1)}{2}\right\rceil=2k-1+2d\left(u-\left(\frac{t(t-1)}{2}+1\right)\right)$, which is

$$y\geq 2\left(2k-1+2d\left(u-\left(\frac{t(t-1)}{2}+1\right)\right)\right)-(2k-1+2d(u-1)+1)$$

$\Rightarrow y \geq 2k - 2 + 2du - 2dt(t-1) - 2d$

$\xrightarrow[u>2t(t-1)+k/d+1]{} y > 2(2k - 1 + t(t-1)d).$

On the other hand the following $\left(\frac{t(t-1)}{2} + 1\right)$ edge labels: $2k-1, 2k-1+2d, \ldots, 2k-1+2\left(\frac{t(t-1)}{2}\right)d$ need at least $t+1$ vertex labels with maximum vertex label, which could be found from the solution "x" of the equation: $\left\lceil\frac{x+\text{minimum vertex label}}{2}\right\rceil$ = maximum edge label above. i.e. $\left\lceil\frac{x+0}{2}\right\rceil = 2k-1+2\left(\frac{t(t-1)}{2}\right)d$, is $x \leq 2\left[2k-1+2\left(\frac{t(t-1)}{2}\right)d\right]$. Since $y > x$ this means we are in need to $(t+1)+(t+1) = 2t+2$ distinct vertex labels to get those edge labels i.e. the graph of $2t+1$ vertex can't be odd mean with such number of edges $q = u$.

In case $q > u$ then the solution "y" of the equation: $\left\lceil\frac{y+2k-1+2d(q-1)}{2}\right\rceil = 2k-1+2d\left(q-\left(\frac{t(t-1)}{2}+1\right)\right)$ is still greater than $2(2k-1+t(t-1)d)$, again we need at least $2t+2$ vertex label and the result follows.□

Theorem 3.3: Let $G(p,q)$ be a graph of order $p = 2t, t \geq 2$, and let $u > 2(t-1)^2 + k/d + 1$. Now if the number of the edges $q \geq u$, then the graph G is not (k,d)-odd mean graph.

Proof: Let $q = u$, similar to the proof of *Theorem 3.2* we have:

$y \geq 2k - 2 + 2du - 2dt(t-1) - 2d$

$\xrightarrow[u>2(t-1)^2+k/d+1]{} y > 2\left(2k-1+2\left(\frac{(t-1)(t-2)}{2}\right)d\right).$

On the other hand the following $\left(\frac{(t-1)(t-2)}{2} + 1\right)$ edge labels: $2k-1, 2k-1+2d, \ldots, 2k-1+2\left(\frac{(t-1)(t-2)}{2}\right)d$ need at least t vertex labels with maximum vertex label, which could be found from the solution "x" of the equation: $\left\lceil\frac{x+\text{minimum vertex label}}{2}\right\rceil$ = maximum edge label above. i.e. $\left\lceil\frac{x+0}{2}\right\rceil = 2k-1+2\left(\frac{(t-1)(t-2)}{2}\right)d$, is $x \leq 2\left[2k-1+2\left(\frac{(t-1)(t-2)}{2}\right)d\right]$. Since $y > x$ this means we are in need to $(t+1)+t = 2t+1$ distinct vertex labels to get those edge labels i.e. the graph of $2t$ vertex can't be odd mean with such number of edges $q = u$.

In case $q > u$ then the solution "y" of the equation: $\left\lceil\frac{y+2k-1+2d(q-1)}{2}\right\rceil = 2k-1+2d\left(q-\left(\frac{t(t-1)}{2}+1\right)\right)$ is still greater than $2\left[2k-1+2\left(\frac{(t-1)(t-2)}{2}\right)d\right]$, again we need at least $2t+1$ vertex label and the result follows.□

Note 3.4: For $k < d, u = 2t(t-1)+2$, where $p = 2t+1$ and $u = 2(t-1)^2+2$, where $p = 2t$.

Results 3.5: The graph $K_p \backslash ie$ is not a (k,d)-odd mean graph, where $k < d$:
1) When p is odd and $i = 0, 1, 2, \dots, (3p-7)/2$.
2) When p is even and $i = 0, 1, 2, \dots, (3p-8)/2$.

4 $Z_2 \oplus Z_2$-cordial labeling

Introduction 4.0: Mark Hovey [4] has introduced A-cordial labeling as a generalization of harmonious and cordial labeling.

We denote the elements of $Z_2 \oplus Z_2$ by $\hat{1}, a, b, c$ with $\hat{1}$ being the identity element.

Definition 4.1: For any abelian group A, a graph $G = (V(G), E(G))$ is said to be A-cordial if there is a labeling of $V(G)$ with elements of A so that for all a, b in A when the edge ab is labeled with $f(a) + f(b)$, then $v_f(a)$ and $v_f(b)$ differ by at most 1 and $e_f(a)$ and $e_f(b)$ differ by at most 1. That is, $|v_f(a) - v_f(b)| \le 1$ and $|e_f(a) - e_f(b)| \le 1 \ \forall\, a, b \in A$, where $v_f(x)$ and $e_f(x)$ are respectively the number of vertices labeled with x and the number of edges labeled with x.

Theorem 4.2: A $Z_2 \oplus Z_2$-cordial graph of order $p \equiv 0, 1, 2, 3 \mod 4$, could maximally contain the following number of edges: $\frac{p(p-4)}{2} + 3, \frac{(p-1)(p-3)}{2} + 3, \frac{(p-2)^2}{2} + 3, \frac{(p-1)(p-3)}{2} + 3$ respectively.

Proof: For $p \equiv 0 \mod 4$ let $p = 4k$, the number of vertices of each of $\{a, b, c, \hat{1}\}$ is k so the number of all possible edge labels of the form $\hat{1}$ are equal to: $4\binom{k}{2} = 2k(k-1) = \frac{p(p-4)}{8}$, *(since the edge label:* $\hat{1} = a \oplus a = b \oplus b = c \oplus c = \hat{1} \oplus \hat{1}$). By *Definition 4.1* we could maximally have the number: $\frac{p(p-4)}{8} + 1$ of labeled edges of the other labels a, b and c, so the maximum number of edges of a $Z_2 \oplus Z_2$- cordial graph of order $p = 4k$ is $\frac{p(p-4)}{8} + 3\left(\frac{p(p-4)}{8} + 1\right) = \frac{p(p-4)}{2} + 3$.
Similar arguments in cases $p \equiv 1, 2, 3 \mod 4$, with setting $p = 4k+1, 4k+2, 4k+3$, show that all possible edge labels of the form $\hat{1}$ are equal to: $\binom{k+1}{2} + 3\binom{k}{2}$, $2\binom{k+1}{2} + 2\binom{k}{2}$, $3\binom{k+1}{2} + \binom{k}{2}$, respectively, so by returning to *Definition 4.1* in these cases, we can calculate the maximum number of edges of a $Z_2 \oplus Z_2$-cordial graph to be: $\frac{(p-1)(p-3)}{2} + 3$, $\frac{(p-2)^2}{2} + 3$, $\frac{(p-1)(p-3)}{2} + 3$, for $p \equiv 1, 2, 3 \mod 4$ respectively.

Results 4.3: The graph $K_p \backslash ie$ is not a $Z_2 \oplus Z_2$- cordial graph:

1) when $p \equiv 0 \; mod \; 4$ and $i = 0, 1, 2, \dots, \frac{3p-8}{2}$.
2) when $p \equiv 1 \; mod \; 4$ and $i = 0, 1, 2, \dots, \frac{3p-11}{2}$.
3) when $p \equiv 2 \; mod \; 4$ and $i = 0, 1, 2, \dots, \frac{3p-12}{2}$.
4) when $p \equiv 3 \; mod \; 4$ and $i = 0, 1, 2, \dots, \frac{3p-11}{2}$.

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