A new version for Co-PI index of a graph

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Abstract: The Co-PI index have been introduced by Hasani et al. recently. In this paper, we present a new version for the Co-PI index, and the Cartesian product, Corona product and join of graphs under this new index are computed.
Keywords: Co-PI index; Cartesian product; Corona product; Join.

1. Introduction

Throughout this paper we consider only finite connected graphs without loops and multiple edges. Let $G = (V, E)$ be a simple graph, V and E denote its vertex set and edge set, while $|V|$ and $|E|$ be its order and size, respectively. The **degree** of a vertex u in G is the number of edges that are incident to it, denoted by $\deg_G(u)$ or $\deg(u)$ for short if there is no confuse. If each vertex in a graph has the same degree, then we call the graph is **regular**. The **distance** between two vertices u and v is the number of edges in a shortest path connecting them, denoted by $\mathrm{dist}_G(u, v)$ or short for $\mathrm{dist}(u, v)$. The maximum value of such numbers, $\mathrm{diam}(G)$, is said to be the **diameter** of G.

Let $e = uv \in E$, we denote $\mathbf{N}_u(e)$ be the set of vertices lying closer to

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u than to v, $\mathbf{N}_v(e)$ be the set of vertices lying closer to v than to u, and $\mathrm{N}_0(e)$ be that of vertices with the same distance from u to v, respectively. More formally

$$\mathbf{N}_u(e) = \mathbf{N}_u(e : G) = \{x \in V(G) | \mathrm{dist}_G(x, u) < \mathrm{dist}_G(x, v)\},$$

$$\mathbf{N}_v(e) = \mathbf{N}_v(e : G) = \{x \in V(G) | \mathrm{dist}_G(x, v) < \mathrm{dist}_G(x, u)\},$$

$$\mathrm{N}_0(e) = \mathrm{N}_0(e : G) = \{x \in V(G) | \mathrm{dist}_G(x, u) = \mathrm{dist}_G(x, v)\}.$$

The number of such vertices is then $n_u(e) = n_u(e : G) = |\mathbf{N}_u(e)|$, $n_v(e) = n_v(e : G) = |\mathbf{N}_v(e)|$ and $n_0(e) = n_0(e : G) = |\mathrm{N}_0(e)|$. Other terminology and notation needed will be introduced as it naturally occurs in the following and we use [1] for those not defined here.

Topological indices are numerical parameters of a graph which are invariants under graph isomorphisms. Hundreds of topological indices have been considered in quantitative structure-activity relationship (QSAR) and quantitative structure-property relationship (QSPR) researches and various applications have been found. The Wiener index was the oldest one to be used in chemistry, which was introduced in 1947 by Wiener as the path number for the characterization of alkanes, and was defined as the count of all shortest distances in a graph [2, 3]. Later, Gutman generalized the Wiener index for cyclic graphs, which is the well-known Szeged index when applied to cyclic structures [4, 5]. The fact that the Szeged index mainly takes into account how the vertices are distributed in a graph, so it is natural to introduce an index that takes into account the distribution of edges. The PI index is a Szeged-like index that takes into account the distribution of edges and is an unique topological index related to parallelism of edges, see [6-9] for details. In literature [10] a vertex version of this graph invariant was introduced, by which it is possible to find an explicit formula for the PI index of the Cartesian product of graphs. We encourage the interest readers to consult [11-15] for the mathematical properties of the PI and vertex PI indices of graphs and [8, 16, 17] for some chemical applications and computational techniques.

Recently, Hasani et al. introduced a new index similar to the vertex version of PI index, which was named the Co-PI index of a graph and defined as [18]:

$$Co-PI_v(G)=\sum_{e\in E(G)}|n_u(e)-n_v(e)|.$$

They presented explicit expressions of $TUC_4C_S(S)$-nanotubes for the first vertex of Co-PI index, and $TUC_4C_8(R)$-nanotubes were discussed for this new index in literature [19].

This paper is organized as follows. In Section 2 we present an equivalent definition of Co-PI index. In Section 3 we explore explicit formulae of graphs operations for the Co-PI index under the new definition.

2. An equivalent definition of Co-PI index

The **transmission** $\mathrm{T}_G(u)$ of a vertex u is the sum of distances from it to all the other vertices in G :

$$\mathrm{T}(u)=\mathrm{T}_G(u)=:\sum_{v\in V}\mathrm{dist}_G(u,v),$$

and G is said to be **transmission-regular** if all its vertices have the same transmission.

Easily to see that the complete graph K_n is transmission-regular since each vertex has transmission $n-1$. Similarly, the balanced complete bipartite graph $K_{s,s}$ and the cycle C_n are also transmission-regular ones.

The reader should note that a regular graph may not be transmission-regular, a transmission-regular graph does not need to be a regular one. But we still have the following:

Theorem 2.1. Let G be a connected graph with diameter two. Then G is transmission-regular if and only if G is regular.

Proof. ($\Leftarrow$) Let G be a k-regular graph. Let u be an arbitrary vertex and $x_1,x_2,\cdots,x_k$ be its adjacent vertices, then the transmission $\sum_{i=1}^{k}\mathrm{dist}(u,x_i)$

of u corresponding to x_i is k. Since the diameter of G is two, the contribution to $\mathrm{T}(u)$ from vertices z_j, the neighbor of $x_i \neq u$, is $\sum_{j=1}^{k-1} \mathrm{dist}(u, z_j) = 2(k-1)$ for $1 \leq i \leq k$ and $1 \leq j \leq k-1$. Hence, we have $\mathrm{T}(u) = k + 2k(k-1) = 2k^2 - k$. This implies that G is transmission-regular.

($\Rightarrow$) By contradiction. Assume that there at least exist two vertices s and t with distinct degrees, without loss of generality, let $\deg(s) = p$, $\deg(t) = q$ and $p \neq q$. We distinguish the following cases:

Case 1. If s is adjacent to t. It is obvious that $\sum_{l=1}^{p} \mathrm{dist}(s, x_l)$ with respect to $x_1, x_2, \cdots, x_p$, the neighbors of s, is equal to p. The rest vertices have distances two from s, among which there are $\deg(x_l) - 1$ vertices with respect to vertex x_l, $1 \leq l \leq p$, having distance two from s. Hence $\mathrm{T}(s) = p + 2\sum_{l=1}^{p-1}[\deg(x_l) - 1] + 2[\deg(t) - 1]$. On the other hand, among all vertices except for those adjacent to t has distance two from t, thus we have $\mathrm{T}(t) = q + 2\sum_{l=1}^{p-1}[\deg(x_l) - 1] + 2[\deg(s) - 1]$. Since G is transmission-regular, then $\mathrm{T}(s) = \mathrm{T}(t)$, which implies $p = q$, a contradiction.

Case 2. If s is non-adjacent to t. The proof follows much in the same way as in the previous case, and we omit it here.

This completes the proof of Theorem 2.1. □

Theorem 2.2. Let G be a connected graph and $e = uv$ an arbitrary edge. Then $|n_u(e) - n_v(e)| = |\mathrm{T}(u) - \mathrm{T}(v)|$.

Proof. For sake of simplicity, let $\mathrm{N}_u(e) = \{u_1, u_2, \cdots, u_p\}$, $\mathrm{N}_v(e) = \{v_1, v_2, \cdots, v_q\}$ and $\mathrm{N}_0(e) = \{w_1, w_2, \cdots, w_r\}$. Then

$$\mathrm{T}(u) - \mathrm{T}(v) = \sum_{s \in V} \mathrm{dist}(u, s) - \sum_{t \in V} \mathrm{dist}(v, t)$$

can be written as follows:

$$\left[\sum_{i=1}^{p} \mathrm{dist}(u, u_i) + \sum_{i=1}^{q} \mathrm{dist}(u, v_i) + \sum_{i=1}^{r} \mathrm{dist}(u, w_i)\right]$$
$$- \left[\sum_{j=1}^{q} \mathrm{dist}(v, v_j) + \sum_{j=1}^{p} \mathrm{dist}(v, u_j) + \sum_{j=1}^{r} \mathrm{dist}(v, w_j)\right].$$

Note that $\text{dist}(u, v_i) = 1 + \text{dist}(v, v_i)$ and $\text{dist}(v, u_j) = 1 + \text{dist}(u, u_j)$ for $1 \leq i \leq p, 1 \leq j \leq q$. Then we obtain

$$\left[\sum_{i=1}^{p} \text{dist}(u, u_i) + \sum_{i=1}^{q} (1 + \text{dist}(v, v_i)) + \sum_{i=1}^{r} \text{dist}(u, w_i)\right]$$
$$- \left[\sum_{j=1}^{q} \text{dist}(v, v_j) + \sum_{j=1}^{p} (1 + \text{dist}(u, u_j)) + \sum_{j=1}^{r} \text{dist}(v, w_j)\right].$$

By some elementarily computations, we have

$$\left[q + \left(\sum_{i=1}^{p} \text{dist}(u, u_i) + \sum_{i=1}^{q} \text{dist}(v, v_i) + \sum_{i=1}^{r} \text{dist}(u, w_i)\right)\right]$$
$$- \left[p + \left(\sum_{j=1}^{q} \text{dist}(v, v_j) + \sum_{j=1}^{p} \text{dist}(u, u_j) + \sum_{j=1}^{r} \text{dist}(v, w_j)\right)\right].$$

This completes the proof of Theorem 2.2. □

From Theorem 2.2, we can claim that the Co-PI index is equivalent to the following definition:

$$Co - PI_v(G) = \sum_{e \in E(G)} |\text{T}(u) - \text{T}(v)|,$$

which is more convenient than the original definition when we counting the number of vertices lying closer to u than to v.

The following results clearly follows from Theorem 2.1 and 2.2.

Corollary 2.3. Let G be a connected graph. Then $Co - PI_v(G) \geq 0$, with equality if and only if G is transmission-regualr.

Corollary 2.4. Let G be a connected graph with diameter two. Then $Co - PI_v(G) \geq 0$, with equality if and only if G is regualr.

Before closing this section, we present the explict formulae of the path P_n for Co-PI index, which can be verified by direct computation.

Theorem 2.5. $Co - PI_v(P_n) = \frac{n(n-2)}{2}$ if n is even, and $Co - PI_v(P_n) = \frac{(n-1)^2}{2}$ if n is odd.

3. Operation graphs for the Co-PI index

The Wiener index of Cartesian product graphs was studied by Yeh and Gutman in literature [20] and Graovac and Pisanski in literature [3] independently. Khalifeh et al. [10] computed an explicit formula for the PI index of Cartesian product of graphs, which is an extension of the main result by Klavzar for bipartite graphs [21].

Here we continue this progress to compute the Co-PI index of three composite graphs, each of them will be treated in a separate subsection.

3.1. Cartesian product

The **Cartesian product** $G_1 \square G_2$ of graphs G_1 and G_2 is a graph with vertex set $V_1 \times V_2$, and two vertices (u_1, u_2) and (v_1, v_2) adjoint by an edge

$$(u_1, u_2) \sim (v_1, v_2) \Leftrightarrow \begin{cases} u_1 = v_1 \text{ and } u_2 \sim v_2 \text{ in } G_2, \\ \text{or} \\ u_2 = v_2 \text{ and } u_1 \sim v_1 \text{ in } G_1. \end{cases}$$

The following is an auxiliary result. We encourage the interest reader to consult the book of Imrich and Klavzar [22] for details.

Lemma 3.1. Let G_1 and G_2 be two graphs. Then the following hold:

- $|V(G_1 \square G_2)| = |V_1| \cdot |V_2|$ and $|E(G_1 \square G_2)| = |E_1| \cdot |V_2| + |V_1| \cdot |E_2|$.
- $G_1 \square G_2$ is connected if and only if G_1 and G_2 are all connected.
- $\text{dist}_{G_1 \square G_2}((u_1, u_2), (v_1, v_2)) = \text{dist}_{G_1}(u_1, v_1) + \text{dist}_{G_2}(u_2, v_2)$ for two vertices (u_1, u_2) and (v_1, v_2).

- The Cartesian product is associative and commutative.

The main results of this section say that:

Theorem 3.2. Let $G = G_1 \square G_2$ be the Cartesian product of two connected graphs G_1 and G_2. Then

$$Co-PI_v(G_1 \square G_2) = |V_1|^2 Co-PI_v(G_2) + |V_2|^2 Co-PI_v(G_1).$$

Proof. Note that there are only two types of edges in G—corresponding to copies of G_1 and that of G_2, respectively. Then the Co-PI index of G can be written as the sum:

$$\sum_{a \in V(G_1)} \sum_{st \in E(G_2)} \left|\mathrm{T}((a,s)) - \mathrm{T}((a,t))\right| + \sum_{s \in V(G_2)} \sum_{ab \in E(G_1)} \left|\mathrm{T}((a,s)) - \mathrm{T}((b,s))\right|.$$

Since $\mathrm{T}((a,s)) - \mathrm{T}((a,t))$ equals to

$$\left[\sum_{(x,y) \in V(G)} \mathrm{dist}_G((a,s),(x,y)) - \sum_{(z,w) \in V(G)} \mathrm{dist}_G((a,t),(z,w))\right]$$

$$= \left[\sum_{a \neq x; s \neq y} \mathrm{dist}_{G_2}(s,y) + \sum_{a = x; s \neq y} \mathrm{dist}_{G_2}(s,y) + \sum_{a \neq x; s = y} \mathrm{dist}_{G_2}(s,y)\right]$$

$$- \left[\sum_{a \neq z; t \neq w} \mathrm{dist}_{G_2}(t,w) + \sum_{a = z; t \neq w} \mathrm{dist}_{G_2}(t,w) + \sum_{a \neq z; t = w} \mathrm{dist}_{G_1}(t,w)\right]$$

$$+ \left[\sum_{a \neq x; s \neq y} \mathrm{dist}_{G_1}(a,x) + \sum_{a = x; s \neq y} \mathrm{dist}_{G_1}(a,x) + \sum_{a \neq x; s = y} \mathrm{dist}_{G_2}(a,x)\right]$$

$$- \left[\sum_{a \neq z; t \neq w} \mathrm{dist}_{G_1}(a,z) + \sum_{a = z; t \neq w} \mathrm{dist}_{G_1}(a,z) + \sum_{a \neq z; t = w} \mathrm{dist}_{G_1}(a,z)\right]$$

$$= \left[|V_1| \cdot |\mathrm{T}(s) - |V_1| \cdot \mathrm{T}(t)|\right] + \left[|V_2| \cdot \mathrm{T}(a) - |V_2| \cdot \mathrm{T}(a)\right].$$

This implies that

$$|\mathrm{T}((a,s)) - \mathrm{T}((a,t))| = |V_1| \cdot |\mathrm{T}(s) - \mathrm{T}(t)|.$$

The same reasoning applies to the edges of the type $(a,s)(b,s)$, i.e.,

$$|\mathrm{T}((a,s)) - \mathrm{T}((b,s))| = |V_2| \cdot |\mathrm{T}(a) - \mathrm{T}(b)|.$$

By simple computations we have

$$Co-PI_v(G_1\Box G_2)=|V_1|^2Co-PI_v(G_2)+|V_2|^2Co-PI_v(G_1).$$

This completes the proof of the Theorem 3.2. □

We compare Theorem 3.2 with the following respective formulae.

- (Graovac et al. [3]) $W(G_1\Box G_2)=|V_1|^2W(G_2)+|V_2|^2W(G_1)$.
- (Klavzar et al. [5]) $S_Z(G_1\Box G_2)=|V_1|^3S_Z(G_2)+|V_2|^3S_Z(G_1)$.
- (Khalifeh et al. [10]) $PI_v(G_1\Box G_2)=|V_1|^2PI_v(G_2)+|V_2|^2PI_v(G_1)$.

Example 1. Let $S=P_r\Box P_s$ be the **rectangular-grid**. Then

$$Co-PI_v(P_r\Box P_s)=\begin{cases}\frac{s(s-2)}{2}r^2+\frac{r(r-2)}{2}s^2, & \text{if } r \text{ is even, } s \text{ is even.}\\ \frac{(s-1)^2}{2}r^2+\frac{r(r-2)}{2}s^2, & \text{if } r \text{ is even, } s \text{ is odd.}\\ \frac{(s-1)^2}{2}r^2+\frac{(r-1)^2}{2}s^2, & \text{if } r \text{ is odd, } s \text{ is odd.}\\ \frac{(s-1)^2}{2}r^2+\frac{r(r-2)}{2}s^2, & \text{if } r \text{ is odd, } s \text{ is even.}\end{cases}$$

Example 2. Let $R=P_r\Box C_l$ be the **C_4-nanotube**. Then $Co-PI_v(R)=\frac{r(r-2)}{2}l^2$ if r is even, and $Co-PI_v(R)=\frac{(r-1)^2}{2}l^2$ if r is odd.

The Cartesian product is one of the most important graph operations, we use it as a representative of the whole group of graph operations and present the results in more details.

Let $\bigotimes_{i=1}^n G_i$ denote the Cartesian product of n graphs $G_1,G_2,\cdots,G_n$ with order at least two, then we have the following:

Theorem 3.3. Let $G_1, G_2, \cdots, G_n$ be n graphs with order at least two. Then

$$Co-PI_v\left(\bigotimes_{i=1}^{n} G_i\right) = \sum_{i=1}^{n}\left(Co-PI_v(G_i)\prod_{j\neq i}^{n}|V_j|^2\right).$$

Proof. By induction on n. If $n = 2$, the proof will be obtained directly from Theorem 3.2. Now we show the theorem holds when $n = N+1$. By Lemma 3.1, the Cartesian product is associative, then $Co-PI_v(\bigotimes_{i=1}^{N+1} G_i)$ can be represented as

$$\begin{aligned}
&Co-PI_v\left(G_{N+1}\square\bigotimes_{i=1}^{N} G_i\right)\\
=&|V_{N+1}|^2 Co-PI_v\left(\bigotimes_{i=1}^{N} G_i\right) + \left|V\left(\bigotimes_{i=1}^{N} G_i\right)\right|^2 Co-PI_v(G_{N+1})\\
=&|V_{N+1}|^2\left(\sum_{i=1}^{N} Co-PI_v(G_i)\prod_{j\neq i}^{N}|V_j|^2\right) + Co-PI_v(G_{N+1})\prod_{i=1}^{N}|V_i|^2\\
=&\sum_{i=1}^{N+1}\left(Co-PI_v(G_i)\prod_{j\neq i}^{N+1}|V_j|^2\right).
\end{aligned}$$

This completes the proof Theorem 3.3. □

The Wiener index, Szeged index and vertex-PI index of Cartesian product of n graphs are given by

- (Graovac et al. [3]) $W(\bigotimes_{i=1}^{n} G_i) = \sum_{i=1}^{n}\left(W(G_i)\prod_{j\neq i}^{n}|V_j|^2\right)$.
- (Klavzar et al. [5]) $S_Z(\bigotimes_{i=1}^{n} G_i) = \sum_{i=1}^{n}\left(S_Z(G_i)\prod_{j\neq i}^{n}|V_j|^3\right)$.
- (Khalifeh et al. [10]) $PI_v(\bigotimes_{i=1}^{n} G_i) = \sum_{i=1}^{n}\left(PI_v(G_i)\prod_{j\neq i}^{n}|V_j|^2\right)$.

3.2. Corona product

The **Corona product** $G_1 \circ G_2$ of two graphs G_1 and G_2 is the graph formed by taking $|G_1|$ copies of G_2 and joining each vertex of the i-th copy with vertex v_i of G_1.

It is obvious that $|V(G_1 \circ G_2)| = |V_1| + |V_1| \cdot |V_2|$ and $|E(G_1 \circ G_2)| = |E_1| + |V_1| \cdot |V_2| + |V_1| \cdot |E_2|$.

Theorem 3.4. Let $G = G_1 \circ G_2$ be the Corona product of two connected graphs G_1 and G_2. If G_2 is regular, then

$$\begin{aligned} Co - PI_v(G_1 \circ G_2) =& (|V_2| + 1)Co - PI_v(G_1) \\ &+ |V_1| \cdot |V_2| \left(|V_1| + |V_1| \cdot |V_2| - \frac{2|E_2|}{|V_2|} - 2 \right). \end{aligned}$$

Proof. The edges of G can be partitioned into three types:

$\mathbb{E}_1 = \{e \in E(G_1 \circ G_2) | e \in E_1\}$.

$\mathbb{E}_2 = \{e \in E(G_1 \circ G_2) | e \in E_2^i$, where E_2^i is the edges of i-th copy of $G_2\}$.

$\mathbb{E}_3 = \{e \in E(G_1 \circ G_2) | e = uv$ such that $u \in V_1$ and $v \in V_2\}$.

Thus the Co-PI index of G is equal to the sum of the contributions from the above three partitions:

$$\begin{aligned} \sum_{e \in \mathbb{E}_1} \big| n_u(e:G) - n_v(e:G) \big| &+ \sum_{e \in \mathbb{E}_2} \big| n_u(e:G) - n_v(e:G) \big| \\ &+ \sum_{e \in \mathbb{E}_3} \big| n_u(e:G) - n_v(e:G) \big|. \qquad (*) \end{aligned}$$

If $e = uv \in \mathbb{E}_1$, then $n_u(e:G) = [|V_2| + 1] n_u(e:G_1)$ and analogous statements hold for vertex v, i.e., $n_v(e:G) = [|V_2| + 1] n_v(e:G_1)$.

If $e = uv \in \mathbb{E}_2$, then $n_u(e:G) = \deg_{G_2}(u) - t_{G_2}(e)$ and $n_v(e:G) = \deg_{G_2}(v) - t_{G_2}(e)$, where $t_{G_2}(e)$ denotes the number of triangles containing e in G_2. Noticing G_2 is regular, then the contribution from the intermediate term of $(*)$ is equal to zero.

If $e = uv \in \mathbb{E}_3$, then $n_u(e:G) n_v(e:G) = |V_1| + |V_1| \cdot |V_2| - [\deg_{G_2}(u) + 1]$ and $n_u(e:G) + n_v(e:G) = |V_1| + |V_1| \cdot |V_2| - \deg_{G_2}(u)$.

By simple computation, we obtain

$$Co-PI_v(G_1\circ G_2) =(|V_2|+1)Co-PI_v(G_1) \\ +|V_1|\cdot|V_2|\left(|V_1|+|V_1|\cdot|V_2|-\frac{2|E_2|}{|V_2|}-2\right).$$

This completes the proof of Theorem 3.4. □

Example 3. Let $C_n\circ\overline{K}_m$ be the **thorny cycle**. Then $Co\text{--}PI_v(C_n\circ\overline{K}_m) = m^2n^2+mn^2-2mn.$

3.3. Join (Sum)

The **join** G_1+G_2 of graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph union $G_1\cup G_2$ together with all edges joining V_1 and V_2. The join is sometimes also called a **sum**.

Theorem 3.5. Let $G=G_1+G_2$ be the join of two regular graphs G_1 and G_2. Then

$$Co-PI_v(G_1+G_2) =Co-PI_v(G_1)+Co-PI_v(G_2) \\ +|V_1|\cdot|V_2|\cdot\left||V_2|-|V_1|+\frac{2|E_1|}{|V_1|}-\frac{2|E_2|}{|V_2|}\right|.$$

The proof follows much in the same way as in previous cases, and we omit it.

Now using Theorem 3.5, we count the following examples.

Example 4. $Co-PI_v(K_{s,t}) = Co-PI_v(\overline{K}_s+\overline{K}_t) = st|s-t|$.

Example 5. $Co-PI_v(S_n) = Co-PI_v(\overline{K}_{n-1}+K_1) = n^2-3n+2.$

As an immediate consequence, we have the following

Corollary 3.6. Let G be a connected graph. Then $Co\text{--}PI_v(\underbrace{G+\cdots+G}_{N\ times}) = NCo-PI_v(G).$

Acknowledgements

The authors would like to thank the referees for their valuable comments, suggestions and corrections which resulted in a great improvement of the original manuscript.

This work has been supported by the National Natural Science Foundation of China (11071016, 11171129).

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