On the basis number of the wreath product of ladders

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Abstract

The basis number of a graph G is defined to be the least non negative integer d such that there is a basis $\mathcal{B}$ of the cycle space of G such that each edge of G is contained in at most d members of $\mathcal{B}$. In this paper, we determine the basis number of the wreath product of different ladders.

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1 Introduction.

The graphs considered in this paper are finite, undirected, simple and connected. Most of the notations that follow can be found in [9]. For a given graph G, we denote the vertex set of G by $V(G)$ and the edge set by $E(G)$. The set $\mathcal{E}$ of all subsets of $E(G)$ forms an $|E(G)|$-dimensional vector space over Z_2 with vector addition $X \oplus Y = (X \backslash Y) \cup (Y \backslash X)$ and scalar multiplication $1 \cdot X = X$ and $0 \cdot X = \emptyset$ for all $X, Y \in \mathcal{E}$. The cycle space, $\mathcal{C}(G)$, of a graph G is the vector subspace of $(\mathcal{E}, \oplus, \cdot)$ spanned by the cycles of G (see [10], [11]). Note that the non-zero elements of $\mathcal{C}(G)$ are cycles and edge disjoint union of cycles. It is known that the dimension of the cycle space is the *cyclomatic number* or the *first Betti number* $\dim \mathcal{C}(G) = |E(G)| - |V(G)| + r$ where r is the number of components of G.

A basis $\mathcal{B}$ for $\mathcal{C}(G)$ is called a *cycle basis* of G. A cycle basis $\mathcal{B}$ of G is called a d-fold if each edge of G occurs in at most d of the cycles in

$\mathcal{B}$. *The basis number,* $b(G)$, of G is the least non-negative integer d such that $\mathcal{C}(G)$ has a *d-fold* basis. The *required cycle basis* is a cycle basis with $b(G)$-fold. Let G and H be two graphs, $\varphi : G \longrightarrow H$ be an isomorphism and $\mathcal{B}$ be a (required) basis of $\mathcal{C}(G)$. Then $\mathcal{B}' = \{\varphi(c) | c \in \mathcal{B}\}$ is called the *corresponding (required)* basis of $\mathcal{B}$ in H. The first important result concerning the basis number is the following lemma of MacLane [22]

Theorem 1.1.(MacLane). *The graph G is planar if and only if* $b(G) \leq 2$.

We say that two vertices u and v of the graph G are *isomorphic* if and only if there is $\alpha \in Aut(G)$ such that $\alpha(u) = v$. $B \subseteq V(G)$ is said to be an *isomorphism class* if and only if B is the maximal set in which each pair of vertices are isomorphic.

Let G and H be two graphs. (1) The Cartesian product $G \Box H$ has the vertex set $V(G \Box H) = V(G) \times V(H)$ and the edge set $E(G \Box H) = \{(u_1, v_1)(u_2, v_2) | u_1 u_2 \in E(G) \text{ and } v_1 = v_2, \text{ or } v_1 v_2 \in E(H) \text{ and } u_1 = u_2\}$. (2) The direct product $G \times H$ is the graph with the vertex set $V(G \times H) = V(G) \times V(H)$ and the edge set $E(G \times H) = \{(u_1, u_2)(v_1, v_2) | u_1 v_1 \in E(G) \text{ and } u_2 v_2 \in E(H)\}$. (3) The wreath product $G \ltimes H$ has the vertex set $V(G \ltimes H) = V(G) \times V(H)$ and the edge set $E(G \ltimes H) = \{(u_1, v_1)(u_2, v_2) | u_1 = u_2 \text{ and } v_1 v_2 \in H, \text{ or } u_1 u_2 \in G \text{ and there is } \alpha \in \text{Aut}(H) \text{ such that } \alpha(v_1) = v_2\}$ (See [1], [11] and [13]).

Many authors have studied the basis number and the required cycle bases of graph products. The Cartesian product of any two graphs was studied by Ali and Marougi [4] and Alsardary and Wojciechowski [7]. Schmeichel [23], Ali [2], [3] and Jaradat [14], [19] and [20] gave upper bounds for the basis number of the semi-strong and the direct products of some special graphs. An upper bound of the basis number of the strong product of graphs was obtained by Jaradat [16]. The lexicographic product was studied by Ali and Marougi [5], Jaradat and Alzoubi [15], Jaradat [21] and Jaradat et al. [17] how provided upper bounds of the basis number of the lexicographic product of two graphs and determined the basis numbers of some classes of the same product.

The problem of constructing a required cycle basis and determining the basis number for the wreath product is difficult. This is primarily so because the structure of the graph obtained by the wreath product depends on the automorphism groups of the second factor. Even the problem of finding a required cycle bases and determining the basis numbers of the wreath products of special classes of graphs is surprisingly nontrivial and specialized see Jaradat [18], Al-Qeyyam and Jaradat [6] and Bataineh and et al. [8].

In this paper, we continue investigating the basis numbers of the wreath product of some graphs. In fact, we determine the basis number of the wreath product of different ladders. Our method in this paper does not only allow the systematic treatment of the basis number of the wreath product of some special classes of graphs, but also has found applications on the minimum cycle bases of the wreath product of some classes of graphs. Moreover, it gives a way to treat more general cases which will appear in the forthcoming papers.

In the rest of this paper, $f_B(e)$ stand for the number of elements of $B \subseteq \mathcal{C}(G)$ containing the edge e and $E(B) = \cup_{C \in B} E(C)$. Moreover, ab stand for an edge.

2 A certain basis for $\mathcal{C}(K_{4,4})$.

In this section, we present a basis for the complete bipartite graph $K_{4,4}$ which will play a major role in our work. Let $\{x_1, x_2, x_3, x_4\}$ and $\{y_1, y_2, y_3, y_4\}$ be the independent partition sets of vertices of $K_{4,4}$. Let

$$\mathcal{S} = \left\{ \mathcal{S}_i^j = x_i y_j x_{i+1} y_{j+1} x_i | 1 \le i, j \le 3 \right\}.$$

Then, by Theorem 2.4 of Schemiechel, $\mathcal{S}$ is a basis for $\mathcal{C}(K_{4,4})$. Now define the following cycles:

$$\begin{aligned} \mathcal{S}_1^{(*,1)} &= x_1 y_1 x_4 y_4 x_1, \\ \mathcal{S}_2^{(*,2)} &= x_4 y_3 x_2 y_4 x_4, \\ \mathcal{S}_3^{(*,3)} &= x_1 y_2 x_3 y_1 x_1. \end{aligned}$$

Lemma 2.1. $\mathcal{S}^* = \mathcal{S} \cup \{\mathcal{S}_1^{(*,1)}, \mathcal{S}_2^{(*,2)}, \mathcal{S}_3^{(*,3)}\} - \{\mathcal{S}_1^1, \mathcal{S}_2^2, \mathcal{S}_3^3\}$ *is a basis for* $\mathcal{C}(K_{4,4})$.

Proof. Since $\mathcal{S}$ is a basis for $\mathcal{C}(K_{4,4})$ and $|\mathcal{S}| = |\mathcal{S}^*|$, it suffices to show that span$(\mathcal{S}^*) = \mathcal{C}(K_{4,4})$ that is each cycle of $\{\mathcal{S}_1^1, \mathcal{S}_2^2, \mathcal{S}_3^3\}$ can be written as a linear combination of cycles of $\mathcal{S}^*$. Now,

$$\begin{aligned} \mathcal{S}_3^{(*,3)} \oplus \mathcal{S}_2^1 &= x_1 y_2 x_3 y_1 x_1 \oplus x_2 y_1 x_3 y_2 x_2 \\ &= x_1 y_1 x_2 y_2 x_1 \\ &= \mathcal{S}_1^1 \end{aligned}$$

and

$$\begin{aligned} \mathcal{S}_2^{(*,2)} \oplus \mathcal{S}_2^3 &= x_4 y_3 x_2 y_4 x_4 \oplus x_2 y_3 x_3 y_4 x_2 \\ &= x_3 y_3 x_4 y_4 x_3 \\ &= \mathcal{S}_3^3 \end{aligned}$$

Set

$$l_1 = \mathcal{S}_1^2 \oplus \mathcal{S}_3^{(*,3)} \oplus \mathcal{S}_1^3 \text{ and } l_2 = \mathcal{S}_2^{(*,2)} \oplus \mathcal{S}_3^2 \oplus \mathcal{S}_3^1.$$

Then

$$\begin{aligned} l_1 &= x_1y_2x_2y_3x_1 \oplus x_1y_2x_3y_1x_1 \oplus x_1y_3x_2y_4x_1 \\ &= x_1y_4x_2y_2x_3y_1x_1, \end{aligned}$$

and

$$\begin{aligned} l_2 &= x_4y_3x_2y_4x_4 \oplus x_3y_2x_4y_3x_3 \oplus x_3y_1x_4y_2x_3 \\ &= x_2y_3x_3y_1x_4y_4x_2. \end{aligned}$$

Thus,

$$\begin{aligned} l_1 \oplus l_2 \oplus \mathcal{S}_1^{(*,1)} &= x_1y_4x_2y_2x_3y_1x_1 \oplus x_2y_3x_3y_1x_4y_4x_2 \oplus x_1y_1x_4y_4x_1 \\ &= x_2y_2x_3y_3x_2 \\ &= \mathcal{S}_2^2. \end{aligned}$$

■

Remark 2.2. *Let $e \in E(K_{4,4})$. (1) If $e = x_3y_4$ or x_2y_1, then $f_{\mathcal{S}^*}(e) = 1$. (2) If $e = x_iy_i, i = 1,2,3,4$ or $x_1y_j, j = 2,3,4$ or $x_4y_j, j = 1,2,3$, then $f_{\mathcal{S}^*}(e) = 2$. (3) If $e = x_3y_1$ or x_2y_4, then $f_{\mathcal{S}^*}(e) = 3$. (4) If $e = x_3y_2$ or x_2y_3, then $f_{\mathcal{S}^*}(e) = 4$.*

3 A certain structure of L_η.

In this section, we present a certain structure of L_η which will serve us in attacking our main result. The following result will be useful in the coming results.

Theorem 3.1. *(Jaradat [13]) Let G and H be any two graphs, then $G \ltimes H = (G \Box H) \cup (G \times H^*) = (G \Box N_H) \cup (N_G \Box H) \cup (G \times H^*)$ where N_H* and *N_G are the null graphs with vertex sets $V(H)$* and *$V(G)$, respectively, and H^* is a vertex disjoint union of complete graphs, say $H^* = \cup_{i=1}^r K_{n_i}$ such that $V(H) = \cup_{i=1}^r V(K_{n_i})$* and *$V(K_{n_i})$ is an isomorphism class of the vertices of H for each i.* ■

For two paths $P_k = u_1u_2\dots u_\eta$ and $P_2 = xy$, the ladder $L_\eta = P_2 \Box P_\eta$. Thus,

$$Aut(L_\eta) = \{(\iota_2, \iota_\eta), (\iota_2, \alpha_\eta), (\alpha_2, \iota_\eta), (\alpha_2, \alpha_\eta)\}$$

where the components are appropriate automorphism of the factors (i.e., ι_2, ι_η are the identity automorphism, $\alpha_2(x) = y$ and $\alpha_2(y) = x$ and $\alpha_\eta(u_i) = \eta - i + 1$ for each $i = 1, 2, \ldots, \eta$). Therefore, From the $Aut(L_\eta)$, one can partition the vertices of L_η into isomorphism classes as follows:

Case 1. $\eta = 2m + 1$. Then $V(L_\eta) = (\cup_{i=1}^{m} S_i) \cup T_{m+1}$ where

$$\begin{aligned} S_i &= \{(x, u_i), (x, u_{\eta-i+1}), (y, u_{\eta-i+1}), (y, u_i)\}, \\ T_{m+1} &= \{(x, u_{m+1}), (y, v_{m+1})\}. \end{aligned}$$

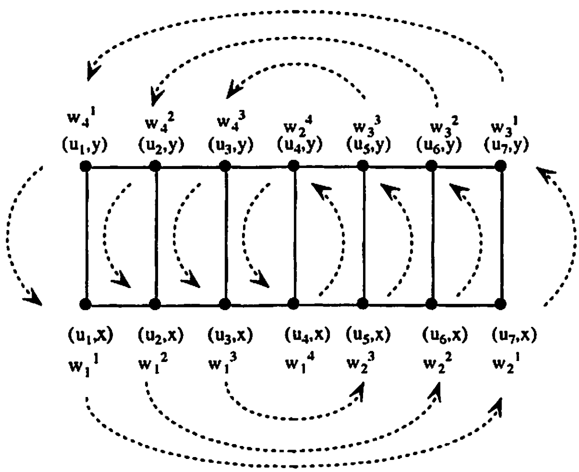

Figure 1: The relabeling of the vertices of L_7.

For simplifying the construction of the basis in the main result, we relabel the vertices of L_{2m+1} as follows (see Figure 1): $S_i = \{w_1^i, w_2^i, w_3^i, w_4^i\}$ where $w_1^i = (x, u_i), w_2^i = (x, u_{\eta-i+1}), w_3^i = (y, u_{\eta-i+1})$ and $w_4^i = (y, u_i)$ for each $i = 1, 2, \ldots, m$. Also, $T_{m+1} = \{w_1^{m+1}, w_2^{m+1}\}$ where $w_1^{m+1} = (x, u_{m+1})$ and $w_2^{m+1} = (y, u_{m+1})$. Therefore, the edge set of L_{2m+1} consists of the following seven sets of edges:

$$
\begin{aligned}
E_1 &= \{w_2^i w_3^i : i = 1, 2, \ldots, m\}, \\
E_2 &= \{w_1^i w_4^i : i = 1, 2, \ldots, m\}, \\
E_3 &= \{w_1^i w_1^{i+1} : i = 1, 2, \ldots, m\}, \\
E_4 &= \{w_2^i w_2^{i+1} : i = 1, 2, \ldots, m-1\}, \\
E_5 &= \{w_3^i w_3^{i+1} : i = 1, 2, \ldots, m-1\}, \\
E_6 &= \{w_4^i w_4^{i+1} : i = 1, 2, \ldots, m-1\}, \\
E_7 &= = \{w_1^{m+1} w_2^{m+1}, w_2^m w_1^{m+1}, w_3^m w_2^{m+1}, w_4^m w_2^{m+1}\}.
\end{aligned}
$$

The following remarks are easy to see.

Remark 3.2. $E_1 \cup E_2 \cup E_3 \cup E_4 \cup \{w_1^{m+1} w_2^{m+1}, w_2^m w_1^{m+1}\}$ *forms an edge set of a tree.*

Remark 3.3. $E_1 \cup E_2 \cup E_5 \cup E_6 \cup \{w_1^{m+1} w_2^{m+1}, w_3^m w_2^{m+1}, w_4^m w_2^{m+1}\}$ *forms an edge set of a tree.*

By Theorem 3.1, and the structure of L_{2m+1}, one can see that $L_{2m+1}^* = (\cup_{i=1}^m K_4^i) \cup K_2^{m+1}$. Thus, $ab \ltimes L_{2m+1}$ is decomposable into $\cup_{i=1}^m (ab \times K_4^i) \cup (ab \times K_2^{m+1}) \cup (\cup_{i=1}^7 (ab \square E_i))$ where K_4^i is the complete graph with vertex set $S_i = \{w_1^i, w_2^i, w_3^i, w_4^i\}$, and K_2^{m+1} is the complete graph with vertex set $T_{m+1} = \{w_1^{m+1}, w_2^{m+1}\}$.

Case 2. $\eta = 2m$. Then $V(L_\eta) = (\cup_{i=1}^m S_i)$ where S_i is as defined above for each $i = 1, 2, \ldots, m$. Therefore, the edge set of L_{2k} consists of the following edge sets: $A_1 = E_1$, $A_2 = E_2, A_4 = E_4, A_5 = E_5, A_6 = E_6$ and

$$
\begin{aligned}
A_3 &= \{w_1^i w_1^{i+1} : i = 1, 2, \ldots, m-1\}, \\
A_7 &= \{w_1^m w_2^m, w_3^m w_4^m\}.
\end{aligned}
$$

where E_1, E_2, E_4, E_5 and E_6 are as in Case 1. As in Case 1, one can see that $L_{2m}^* = \cup_{i=1}^m K_4^i$. Thus, $ab \ltimes L_{2m}$ is decomposable into $\cup_{i=1}^m (ab \times K_4^i) \cup (\cup_{i=1}^7 (ab \square A_i))$.

4 The basis number of $L_n \ltimes L_\eta$.

In this section, we investigate the basis number of the wreath product of two ladders. According to the parity of η, we split this section into two parts $\eta = 2m+1$ and $\eta = 2m$. We start with $\eta = 2m+1$. Note that for each i, $(ab \times K_4^i) \cup (ab \square N_i)$ is isomorphic to $K_{4,4}$ and $(ab \times K_2^i) \cup (ab \square M_i)$

is isomorphic to $K_{2,2}$ where N_i and M_i are the null graphs with vertex set S_i and T_i, respectively. For simplicity, throughout this work, we set

$$K_{4,4}^{(ab,i)} = \left(ab \times K_4^i\right) \cup (ab\Box N_i)$$

and

$$K_{2,2}^{(ab,i)} = \left(ab \times K_2^i\right) \cup (ab\Box M_i).$$

Therefore, $ab \ltimes L_{2m+1}$ is decomposable into $\left(\cup_{i=1}^{m} K_{4,4}^{(ab,i)}\right) \cup K_{2,2}^{(ab,m+1)} \cup \left(\cup_{i=1}^{7}(\{a,b\}\Box E_i)\right)$. Throughout this work, we assume that S_{ab}^i is the corresponding basis of $\mathcal{S}^*$ in $K_{4,4}^{(ab,i)}$ for each i where $\mathcal{S}^*$ is as in Lemma 2.1. Also, let

$$T_{ab}^i = \{(a,w_1^i)(b,w_1^i)(a,w_2^i)(b,w_2^i)(a,w_1^i)\}.$$

Note that T_{ab}^i is a basis for $\mathcal{C}\left(K_{2,2}^{(ab,i)}\right)$. For $i = 1,2,\ldots,m$, we set

$$\mathcal{A}_{(ab,1)}^i = \{(a,w_1^i)(b,w_2^i)(b,w_3^i)(a,w_1^i)\},$$

and for $j = 2,3$, we set

$$\mathcal{A}_{(ab,j)}^i = \{(a,w_1^i)(a,w_4^i)(b,w_j^i)(a,w_1^i)\}.$$

Moreover,

$$\begin{aligned}
\mathcal{A}_{(ab,4)} &= \{(a,w_1^{m+1})(b,w_2^{m+1})(a,w_2^{m+1})(a,w_1^{m+1})\},\\
\mathcal{A}_{(ab,5)} &= \{(a,w_2^{m+1})(b,w_1^{m+1})(b,w_2^{m+1})(a,w_1^{m+1})(a,w_2^{m+1})\},\\
\mathcal{A}_{(ab,6)} &= \{(a,w_1^{m+1})(b,w_2^{m+1})(b,w_1^{m+1})(a,w_1^{m+1})\},\\
\mathcal{A}_{(ab,7)} &= \{(a,w_1^{m+1})(a,w_2^{m+1})(b,w_1^{m+1})(a,w_1^{m+1})\}.
\end{aligned}$$

The following result will be used frequently [17].

Lemma 4.1. (Jaradat, et al.) *Let A and B be two linearly independent sets of cycles and $E(A) \cap E(B)$ induces a forest in G (we allow the possibility that $E(A) \cap E(B) = \varnothing$). Then $A \cup B$ is linearly independent.* ■

Lemma 4.2. $\mathcal{S}_{ab}^{**} = \cup_{i=1}^{m}\left(S_{ab}^i \cup \left\{\mathcal{A}_{(ab,1)}^i, \mathcal{A}_{(ba,1)}^i, \mathcal{A}_{(ba,2)}^i, \mathcal{A}_{(ab,3)}^i\right\}\right) \cup T_{ab}^{(m+1)} \cup \mathcal{A}_{(ab,4)} \cup \mathcal{A}_{(ab,5)}$ *is a linearly independent subset of $\mathcal{C}(ab \ltimes L_{2m+1})$.*

Proof. Since S_{ab}^i is a basis for $\mathcal{C}\left(K_{4,4}^{(ab,i)}\right)$, as a result S_{ab}^i is linearly independent for each $1 \le i \le m$. $\mathcal{A}_{(ab,1)}^i$ contains the edge $(b,w_2^i)(b,w_3^i)$

which does not appear in any cycle of S^i_{ab}. Thus, $S^i_{ab} \cup \{\mathcal{A}^i_{(ab,1)}\}$ is linearly independent. $\mathcal{A}^i_{(ba,1)}$ contains the edge $(a, w^i_2)(a, w^i_3)$ which does not appear in any cycle of $S^i_{ab} \cup \{\mathcal{A}^i_{(ab,1)}\}$. Thus, $S^i_{ab} \cup \{\mathcal{A}^i_{(ab,1)}, \mathcal{A}^i_{(ba,1)}\}$ is linearly independent. Similarly, $\mathcal{A}^i_{(ba,2)}$ contains the edge $(b, w^i_1)(b, w^i_4)$ and $\mathcal{A}^i_{(ab,3)}$ contains the edge $(a, w^i_1)(a, w^i_4)$ and each of these two edges does not appear in any other cycles of $S^i_{ab} \cup \left\{\mathcal{A}^i_{(ab,1)}, \mathcal{A}^i_{(ba,1)}, \mathcal{A}^i_{(ba,2)}, \mathcal{A}^i_{(ab,3)}\right\}$. Hence, $S^i_{ab} \cup \left\{\mathcal{A}^i_{(ab,1)}, \mathcal{A}^i_{(ba,1)}, \mathcal{A}^i_{(ba,2)}, \mathcal{A}^i_{(ab,3)}\right\}$ is linearly independent. Since

$$\begin{aligned} S^i_{ab} \cup \left\{\mathcal{A}^i_{(ab,1)}, \mathcal{A}^i_{(ba,1)}, \mathcal{A}^i_{(ba,2)}, \mathcal{A}^i_{(ab,3)}\right\} \subseteq \; & \mathcal{C}\left(\left(K^{(ab,i)}_{4,4}\right) \cup\right. \\ & \left.(\{a,b\}\Box\{w^i_1 w^i_4, w^i_2 w^i_3\})\right) \end{aligned}$$

for each i and since

$$\begin{aligned} & E\left(K^{(ab,i)}_{4,4} \cup (\{a,b\}\Box\{w^i_1 w^i_4, w^i_2 w^i_3\})\right) \\ & \cap E\left(K^{(ab,j)}_{4,4} \cup (\{a,b\}\Box\{w^j_1 w^j_4, w^j_2 w^j_3\})\right) \\ = \; & \varnothing \end{aligned}$$

whenever $i \neq j$, by Lemma 4.1 we have that $\cup_{i=1}^m \left(S^i_{ab} \cup \left\{\mathcal{A}^i_{(ab,1)}, \mathcal{A}^i_{(ba,1)}, \mathcal{A}^i_{(ba,2)}, \mathcal{A}^i_{(ab,3)}\right\}\right)$ is linearly independent. Now, The cycle $\mathcal{A}_{(ab,4)}$ contains the edge $(a, w^{m+1}_1)(a, w^{m+1}_2)$ which does not appear in the cycle $T^{(m+1)}_{ab}$. So, $\{T^{(m+1)}_{ab}, \mathcal{A}_{(ab,4)}\}$ is linearly independent. Similarly, $\mathcal{A}_{(ab,5)}$ contains the edge $(b, w^{m+1}_1)(b, w^{m+1}_2)$ which does not appear in any cycle of $\{T^{(m+1)}_{ab}, \mathcal{A}_{(ab,4)}\}$. So, $\{T^{(m+1)}_{ab}, \mathcal{A}_{(ab,4)}, \mathcal{A}_{(ab,5)}\}$ is linearly independent. Since

$$\{T^{(m+1)}_{ab}, \mathcal{A}_{(ab,4)}, \mathcal{A}_{(ab,5)}\} \subseteq \mathcal{C}\left(\left(K^{(ab,m+1)}_{2,2}\right) \cup (\{a,b\}\Box w^{m+1}_1 w^{m+1}_2)\right)$$

and

$$\begin{aligned} & E\left(\cup_{i=1}^m \left[K^{(ab,i)}_{4,4} \cup (\{a,b\}\Box\{w^i_1 w^i_4, w^i_2 w^i_3\})\right]\right) \\ & \cap E\left(K^{(ab,m+1)}_{2,2} \cup (\{a,b\}\Box w^{m+1}_1 w^{m+1}_2)\right) \\ = \; & \varnothing, \end{aligned}$$

by Lemma 4.1, S^{**}_{ab} is linearly independent. ■

By using, word by word, the proof of the above lemma after replacing $\mathcal{A}_{(ab,4)}$ and $\mathcal{A}_{(ab,5)}$ by $\mathcal{A}_{(ab,6)}$ and $\mathcal{A}_{(ab,7)}$ we get the following result.

Lemma 4.3. *$\mathcal{S}_{ab}^{***} = \cup_{i=1}^{m} \left(\mathcal{S}_{ab}^{i} \cup \left\{ \mathcal{A}_{(ab,1)}^{i}, \mathcal{A}_{(ba,1)}^{i}, \mathcal{A}_{(ba,2)}^{i}, \mathcal{A}_{(ab,3)}^{i} \right\} \right) \cup \mathcal{T}_{ab}^{(m+1)} \cup \mathcal{A}_{(ab,6)} \cup \mathcal{A}_{(ab,7)}$ is linearly independent subset of $(Cab \ltimes L_{2m+1})$.* ■

Now, for $i = 1, 2, \ldots, m-1$ and $j = 1, 2, 3, 4$ we set

$$\mathcal{T}_{(ab,j)}^{i} = \{(a, w_j^i)(b, w_j^i)(b, w_j^{i+1})(a, w_j^{i+1})(a, w_j^i)\},$$

and

$$\mathcal{T}_{(ab,1)}^{m} = \{(a, w_1^m)(b, w_1^m)(b, w_1^{m+1})(a, w_1^{m+1})(a, w_1^m)\}.$$

Also, for $i = 1, 2, \ldots, m-1$, and $j = 1, 3$, we set

$$\mathcal{K}_{(ab,j)}^{i} = \{(a, w_j^i)(a, w_j^{i+1})(b, w_{j+1}^{i+1})(b, w_{j+1}^{i})(a, w_j^i)\}.$$

Moreover,

$$\mathcal{X}_{ab} = (a, w_1^m)(a, w_1^{m+1})(b, w_1^{m+1})(b, w_2^m)(a, w_1^m),$$

$$\mathcal{Z}_{ab} = (a, w_3^m)(a, w_2^{m+1})(b, w_2^{m+1})(b, w_4^m)(a, w_3^m).$$

Let

$$\begin{aligned} \mathcal{T}_{(ab,12)}^{i} &= \{\mathcal{T}_{(ab,1)}^{i}, \mathcal{T}_{(ab,2)}^{i}\},\ \mathcal{T}_{(ab,34)}^{i} = \{\mathcal{T}_{(ab,3)}^{i}, \mathcal{T}_{(ab,4)}^{i}\}, \\ \mathcal{K}_{ab}^{(r,*)} &= \left(\cup_{i=1 \text{ and } i \text{ is odd}}^{r} \mathcal{K}_{(ab,1)}^{i} \right) \cup \left(\cup_{i=1 \text{ and } i \text{ is even}}^{r} \mathcal{K}_{(ba,1)}^{i} \right), \end{aligned}$$

and

$$\mathcal{K}_{ab}^{(r,**)} = \left(\cup_{i=1 \text{ and } i \text{ is odd}}^{r} \mathcal{K}_{ab(3)}^{(i)} \right) \cup \left(\cup_{i=1 \text{ and } i \text{ is even}}^{r} \mathcal{K}_{ba(3)}^{(i)} \right).$$

Lemma 4.4.

$$\mathcal{B}_{ab}^{*} = \begin{cases} \mathcal{B}_{ab} \cup \{\mathcal{X}_{ab}\} & \text{if } m \text{ is odd} \\ \mathcal{B}_{ab} \cup \{\mathcal{X}_{ba}\} & \text{if } m \text{ is even} \end{cases}$$

is a linearly independent subset of $C(ab \ltimes L_{2m+1})$ where $\mathcal{B}_{ab} = \mathcal{S}_{ab}^{**} \cup \left(\cup_{i=1}^{m-1} \mathcal{T}_{(ab,12)}^{i} \right) \cup \mathcal{T}_{(ab,1)}^{m} \cup \mathcal{K}_{ab}^{(m-1,*)}$. ■

Proof. Note that, for each odd i between 1 and $m-1$, the cycle $\mathcal{K}_{(ab,1)}^{i}$ contains an edge of the form $(a, w_1^i)(a, w_1^{i+1})$ which does not occur in any other cycle of $\mathcal{S}_{ab}^{**} \cup \left(\cup_{i=1 \text{ and } i \text{ is odd}}^{m-1} \mathcal{K}_{(ab,1)}^{i} \right)$. Hence, $\mathcal{S}_{ab}^{**} \cup \left(\cup_{i=1 \text{ and } i \text{ is odd}}^{m-1} \mathcal{K}_{(ab,1)}^{i} \right)$ is linearly independent. Similarly, for each even i between 1 and $m-1$, the cycle $\mathcal{K}_{(ba,1)}^{i}$ contains an edge of the form $(b, w_1^i)(b, w_1^{i+1})$

which does not occur in any other cycle of $\mathcal{S}_{ab}^{**} \cup \mathcal{K}_{ab}^{(m-1,*)}$. Thus, $\mathcal{S}_{ab}^{**} \cup \mathcal{K}_{ab}^{(m-1,*)}$ is linearly independent. Now, we use the mathematical induction on m to show that $\cup_{i=1}^{m-1}\mathcal{T}_{(ab,12)}^{i}$ is independent with $\mathcal{S}_{ab}^{**} \cup \mathcal{K}_{ab}^{(m-1,*)}$. If $m = 2$, then $\cup_{i=1}^{m-1}\mathcal{T}_{(ab,12)}^{i} = \mathcal{T}_{ab(12)}^{(1)} = \{\mathcal{T}_{(ab,1)}^{1}, \mathcal{T}_{(ab,2)}^{1}\}$. $\mathcal{T}_{(ab,1)}^{1}$ contains the edge $(b, w_1^1)(b, w_1^2)$ which does not occur in $\mathcal{S}_{ab}^{**} \cup \mathcal{K}_{ab}^{(m-1,*)}$. Similarly, $\mathcal{T}_{(ab,2)}^{1}$ contains $(a, w_2^1)(a, w_2^2)$ which does not occur in any cycle of $\mathcal{S}_{ab}^{**} \cup \mathcal{K}_{ab}^{(m-1,*)} \cup \{\mathcal{T}_{(ab,1)}^{1}\}$. Thus, $\mathcal{S}_{ab}^{**} \cup \mathcal{K}_{ab}^{(m-1,*)} \cup \{\mathcal{T}_{(ab,1)}^{1}, \mathcal{T}_{(ab,2)}^{1}\}$ is linearly independent. Assume that $m \geq 3$ and it is true for less than m. Now, $\cup_{i=1}^{m-1}\mathcal{T}_{(ab,12)}^{i} = \left(\cup_{i=1}^{m-2}\mathcal{T}_{(ab,12)}^{i}\right) \cup \mathcal{T}_{(ab,1)}^{m-1} \cup \mathcal{T}_{(ab,2)}^{m-1}$. Note that $\mathcal{T}_{(ab,1)}^{m-1}$ contains $(a, w_{(1,m-1)})(a, w_1^m)$ and $(b, w_1^{m-1})(b, w_1^m)$. Moreover, $(b, w_1^{m-1})(b, w_1^m)$ does not occur in any cycle of $\mathcal{S}_{ab}^{**} \cup \mathcal{K}_{ab}^{(m-1,*)} \cup \left(\cup_{i=1}^{m-2}\mathcal{T}_{(ab,12)}^{i}\right)$ if m is even and $(a, w_{(1,m-1)})(a, w_1^m)$ does not occur in any cycle of $\mathcal{S}_{ab}^{**} \cup \mathcal{K}_{ab}^{(m-1,*)} \cup \left(\cup_{i=1}^{m-2}\mathcal{T}_{(ab,12)}^{i}\right)$ if m is odd. Similarly, $\mathcal{T}_{(ab,2)}^{m-1}$ contains $(a, w_{(2,m-1)})(a, w_2^m)$ and $(b, w_{(2,m-1)})(b, w_2^m)$. Moreover, each cycle of $\mathcal{S}_{ab}^{**} \cup \mathcal{K}_{ab}^{(m-1,*)} \cup \left(\cup_{i=1}^{m-2} \mathcal{T}_{(ab,12)}^{i}\right) \cup \{\mathcal{T}_{(ab,1)}^{m-1}\}$ does not contain the edge $(a, w_{(2,m-1)})(a, w_2^m)$ if m is even and does not contain the edge $(b, w_{(2,m-1)})(b, w_2^m)$ if m is odd. Thus, $\mathcal{S}_{ab}^{**} \cup \left(\cup_{i=1}^{m-1}\mathcal{T}_{(ab,12)}^{i}\right) \cup \mathcal{K}_{ab}^{(m-1,*)}$ is linearly independent. Finally, $\mathcal{T}_{(ab,1)}^{m}$ contains $(a, w_1^m)(a, w_1^{m+1})$ and $(b, w_1^m)(b, w_1^{m+1})$. Moreover, no cycle of $\mathcal{S}_{ab}^{**} \cup \left(\cup_{i=1}^{m-1}\mathcal{T}_{(ab,12)}^{i}\right) \cup \mathcal{K}_{ab}^{(m-1,*)}$ contains $(a, w_1^m)(a, w_1^{m+1})$ if m is even or $(b, w_1^m)(b, w_1^{m+1})$ if m is odd. Thus, $\mathcal{B}_{ab}$ is linearly independent. To this end, since $\mathcal{X}_{ab}$ contains the edge $(b, w_2^m)(b, w_1^{m+1})$ which does not appear in any cycle of $\mathcal{B}_{ab}$ if m is odd, and $\mathcal{X}_{ba}$ contains the edge $(a, w_2^m)(a, w_1^{m+1})$ which does not appear in any cycle of $\mathcal{B}_{ab}$ if m is even, as a result $\mathcal{B}_{ab}^{*}$ is linearly independent. ■

By a similar arguments as in the above lemma we can prove the following results:

Lemma 4.5.

$$\mathcal{A}_{ab}^{*} = \begin{cases} \mathcal{A}_{ab} \cup \{\mathcal{Z}_{ab}\} & \text{if } m \text{ is odd} \\ \mathcal{A}_{ab} \cup \{\mathcal{Z}_{ba}\} & \text{if } m \text{ is even} \end{cases}$$

is linearly independent where $\mathcal{A}_{ab} = \mathcal{S}_{ab}^{***} \cup \left(\cup_{i=1}^{m-1}\mathcal{T}_{(ab,34)}^{i}\right) \cup \mathcal{K}_{ab}^{(m-1,**)}$. ■

Now define the following cycle,

$$\mathcal{V}_{ab} = \{(a, w_2^m)(b, w_2^m)(b, w_1^{m+1})(a, w_1^{m+1})(a, w_2^m)\},$$

and for $j = 3, 4$

$$\mathcal{R}_{(ab,j)} = \{(a,w_j^m)(b,w_j^m)(b,w_2^{m+1})(a,w_2^{m+1})(a,w_j^m)\}.$$

Lemma 4.6. *$\mathcal{B}_{ab}^{**} = \mathcal{B}_{ab}^{*} \cup \{\mathcal{V}_{ab}\}$ is a linearly independent set of $\mathcal{C}(ab \ltimes L_{2m+1})$.*

Proof. Since $\mathcal{V}_{ab}$ contains the edge $(a,w_2^m)(a,w_1^{m+1})$ if m is odd and $(b,w_2^m)(b,w_1^{m+1})$ if m is even which does not appear in any cycle of $\mathcal{B}_{ab}^{**}$, as a result $\mathcal{B}_{ab}^{**}$ is linearly independent. ■

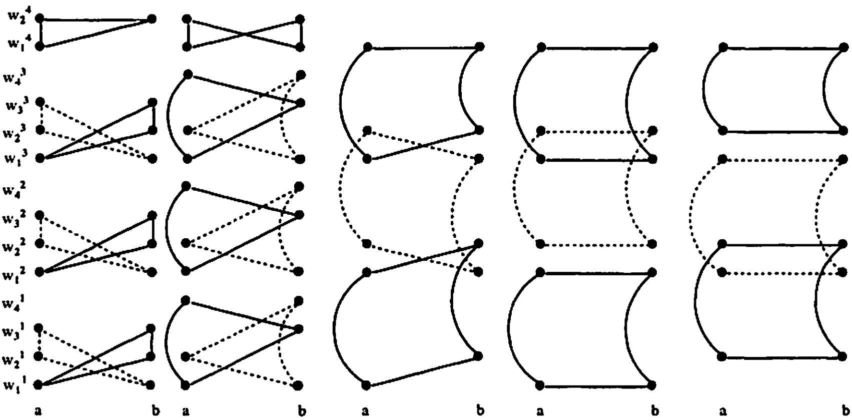

Figure 2: The figure represents cycles of $\mathcal{B}_{ab}^{**} - \left(\cup_{i=1}^{m}\left(S_{ab}^{i}\cup\right) \cup T_{ab}^{(m+1)}\right)$ for $\eta = 7$

Remark 4.7. *Using Remark 2.2 and by the help of Figure 2, one can see the following: Let $e \in E(\mathcal{B}_{ab}^{**})$. Then*

(1) if $e = (a,w_1^1)(b,w_1^1)$, then $f_{\mathcal{B}_{ab}^{**}}(e) \le 3$.

(2) if $e = (a,w_2^{m+1})(b,w_2^{m+1})$, then $f_{\mathcal{B}_{ab}^{**}}(e) = 2$.

(3) If $e = (a,w_j^i)(b,w_j^i)$ where $1 \le i \le m$ and $j = 3,4$, then $f_{\mathcal{B}_{ab}^{**}}(e) = 2$.

(4) If $e = (a,w_1^i)(a,w_1^{i+1})$ or $(b,w_2^i)(b,w_2^{i+1})$, $1 \le i \le m-1$, then $f_{\mathcal{B}_{ab}^{**}}(e) \le 2$ if i is odd and ≤ 1 if i is even. Similarly, for $e = (a,w_2^i)(a, w_2^{i+1})$ or $(b,w_1^i)(b,w_1^{i+1})$, $1 \le i \le m-1$.

(5) If $e = (a,w_1^m)(a,w_1^{m+1})$ or $(b,w_2^m)(b,w_1^{m+1})$, then $f_{\mathcal{B}_{ab}^{**}}(e) \le 2$ if m is odd and ≤ 1 if m is even. Similarly, for $e = (b,w_1^m)(b, w_1^{m+1})$ or $(a,w_2^m)(a,w_1^{m+1})$.

(6) If $e = (a,w_2^i)(a,w_3^i)$ or $(a,w_1^i)(a,w_4^i)$ for $1 \le i \le m$, then $f_{\mathcal{B}_{ab}^{**}}(e) = 1$. Similarly for $e = (b,w_2^i)(b,w_3^i)$ or $(b,w_1^i)(b,w_4^i)$ for $1 \le i \le m$.

(7) If $e = (a,w_1^{m+1})(a,w_2^{m+1})$, then $f_{\mathcal{B}_{ab}^{**}}(e) = 2$.

(8) If $e=(b,w_1^{m+1})(b,w_2^{m+1})$, *then* $f_{B_{ab}^{**}}(e)=1$.

(9) If $e=(a,w_1^{m+1})(b,w_2^{m+1})$ *or* $(a,w_2^{m+1})(b,w_1^{m+1})$, *then* $f_{B_{ab}^{**}}(e)\le 3$.

(10) If $e\in E(B_{ab}^{**})$ *which is not as in any of the above forms, then* $f_{B_{ab}^{**}}(e)\le 4$.

By a similar argument as in the above lemma, we have the following result.

Lemma 4.8. $\mathcal{A}_{ab}^{**}=\mathcal{A}_{ab}^{*}\cup\{\mathcal{R}_{(ab,3)},\mathcal{R}_{(ab,4)}\}$ *is a linearly independent set of* $C(ab\ltimes L_{2m+1})$. ■

Remark 4.9. *As in the above remark one can see the following. Let* $e\in E(A_{ab}^{**})$. *Then*

(1) If $e=(a,w_j^i)(b,w_j^i)$ *where* $1\le i\le m$ *and* $j=1,2$, *then* $f_{\mathcal{A}_{ab}^{**}}(e)\le 2$.

(2) If $e=(a,w_1^{m+1})(b,w_1^{m+1})$, *then* $f_{\mathcal{A}_{ab}^{**}}(e)\le 3$.

(3) If $(a,w_2^{m+1})(b,w_2^{m+1})$, *then* $f_{\mathcal{A}_{ab}^{**}}(e)\le 4$.

(4) If $e=(a,w_3^i)(a,w_3^{i+1})$ *or* $(b,w_4^i)(b,w_4^{i+1})$, $1\le i\le m-1$, *then* $f_{\mathcal{A}_{ab}^{**}}(e)\le 2$ *if i is odd and* ≤ 1 *if i is even. Similarly for* $e=(a,w_4^i)(a,w_4^{i+1})$ *or* $(b,w_3^i)(b,w_3^{i+1})$, $1\le i\le m-1$.

(5) If $e=(a,w_3^m)(a,w_2^{m+1})$ *or* $(b,w_4^m)(b,w_2^{m+1})$, *then* $f_{\mathcal{A}_{ab}^{**}}(e)\le 2$ *if m is odd and* ≤ 1 *if m is even.. Similarly for* $e=(a,w_4^m)(a,w_2^{m+1})$ *or* $(b,w_3^m)(b,w_2^{m+1})$.

(6) If $e=(a,w_2^i)(a,w_3^i)$ *or* $(a,w_1^i)(a,w_4^i)$ *for* $1\le i\le m$, *then* $f_{\mathcal{B}_{ab}^{**}}(e)=1$. *Similarly for* $e=(b,w_2^i)(b,w_3^i)$ *or* $(b,w_1^i)(b,w_4^i)$ *for* $1\le i\le m$.

(7) If $e=(a,w_1^1)(b,w_4^1)$ *or* $(a,w_1^1)(b,w_1^1)$, *then* $f_{\mathcal{B}_{ab}^{**}}(e)=2$.

(8) If $e\in E(B_{ab}^{**})$ *and not of the above forms, then* $f_{\mathcal{A}_{ab}^{**}}(e)\le 4$.

The proof of the following results follows from noting that the elements of $\mathcal{B}_{L_n\square w_1^1}$ and $\mathcal{B}_{a_{2n}\square L_{2m+1}}$ are the bounded faces of the planar graphs $L_n\square w_1^1$ and $a_{2n}\square L_{2m+1}$.

Lemma 4.10. $\mathcal{B}_{L_n\square w_1^1}=\{(a_i,w_1^1)(a_{2n-i+1},w_1^1)(a_{2n-i},w_1^1)(a_{i+1},w_1^1)(a_i,w_1^1)|i=1,2,n-1\}$ *is a basis for* $C(L_n\square w_1^1)$. ■

Lemma 4.11. $\mathcal{B}_{a_{2n}\square L_{2m+1}}=\{(a_{2n},w_1^i)(a_{2n},w_1^{i+1})(a_{2n},w_4^{i+1})(a_{2n},w_4^i)(a_{2n},w_1^i)|i=1,2,\dots,m-1\}\cup\{(a_{2n},w_2^i)(a_{2n},w_2^{i+1})(a_{2n},w_3^{i+1})(a_{2n},w_3^i)(a_{2n},w_2^i)|i=1,2,\dots,m-1\}\cup\{(a_{2n},w_1^m)(a_{2n},w_1^{m+1})(a_{2n},w_2^{m+1})(a_{2n},w_4^m)(a_{2n},w_1^m),(a_{2n},w_2^m)(a_{2n},w_1^{m+1})(a_{2n},w_2^{m+1})(a_{2n},w_3^m)(a_{2n},w_2^m)\}$ *is a basis for* $a_{2n}\square L_{2m+1}$. ■

Throughout the rest of this work the ladder L_n will be considered as a cycle $C_{2n} = a_1a_2\dots a_{2n}a_1$ in addition to the edge set $\{a_ja_{2n-j+1} : j = 2,3,\dots,n-1\}$. Let

$$\mathcal{P} = \left(\cup_{i=1}^{m-1} \cup_{k=1}^{2n-1} \mathcal{T}^i_{(a_ka_{k+1},34)}\right) \cup \left(\cup_{k=1}^{2n-1} \left(\mathcal{R}_{(a_ka_{k+1},3)} \cup \mathcal{R}_{(a_ka_{k+1},4)}\right)\right)$$

Lemma 4.12. *$\mathcal{P}\cup\left(\cup_{k=1}^{2n-1}\mathcal{B}^{**}_{a_ka_{k+1}}\right)\cup\mathcal{B}^{**}_{a_{2n}a_1}\cup\mathcal{B}_{L_n\Box w_1^1}\cup\mathcal{B}^{*}_{a_{2n}\Box L_{2m+1}}$ is a linearly independent subset of $C(L_n\ltimes L_{2m+1})$ where $\mathcal{B}^{*}_{a_{2n}\Box L_{2m+1}} = \mathcal{B}_{a_{2n}\Box L_{2m+1}}$ $\cup\{(a_{2n},w_2^{m+1})(a_1,w_1^{m+1})(a_1,w_2^{m+1})\ (a_{2n},w_1^{m+1})(a_{2n},w_1^m)\ (a_{2m},w_4^m)(a_{2n},$ $w_2^{m+1})\} - \{(a_{2n},w_1^m)(a_{2n},w_1^{m+1})(a_{2n},w_2^{m+1})(a_{2n},w_4^m)(a_{2n},\ w_1^m)\}$.*

Proof. $\cup_{k=1}^{2n-1}\mathcal{B}^{**}_{a_ka_{k+1}}$ is linearly independent follows from Lemma 4.1 and noting that for $k<j$,

$$\begin{aligned} &E(\mathcal{B}^{**}_{a_ka_{k+1}})\cap E(\mathcal{B}^{**}_{a_ja_{j+1}})\\ \subseteq\ &\{a_j\}\Box\left(E_1\cup E_2\cup E_3\cup E_4\cup\{w_1^{m+1}w_2^{m+1}, w_2^m w_1^{m+1}\}\right)\end{aligned}$$

which is an edge set of a tree, Remark 4.2. Since $P_{2n} = a_1a_2\dots a_{2n}$ is a path of L_n, any linear combination of cycles of $\mathcal{B}_{L_n\Box w_1^1}$ contains an edge of $E((L_n - P_{2n})\Box w_1^1)$ which is not in any cycle of $\cup_{k=1}^{2n-1}\mathcal{B}^{**}_{a_ka_{k+1}}$. Therefore, $\left(\cup_{k=1}^{2n-1}\mathcal{B}^{**}_{a_ka_{k+1}}\right)\cup\mathcal{B}_{L_n\Box w_1^1}$ is linearly independent. Note that $E(\left(\cup_{k=1}^{2n-1}\mathcal{B}^{**}_{a_ka_{k+1}}\right)\cup\mathcal{B}_{L_n\Box w_1^1})\cap E(\mathcal{B}^{**}_{a_{2n}a_1})$ is a subset of

$$\begin{aligned} &(\{a_{2n},a_1\}\Box\left(E_1\cup E_2\cup E_3\cup E_4\cup\{w_1^{m+1}w_2^{m+1}, w_2^m w_1^{m+1}\}\right))\\ &\cup\{(a_1,w_1^1)(a_{2n},w_1^1)\}\end{aligned}$$

which is an edge set of a tree, Remark 4.4. Therefore, by Lemma 4.1, $\left(\cup_{k=1}^{2n-1}\mathcal{B}^{**}_{a_ka_{k+1}}\right)\cup\mathcal{B}^{**}_{a_{2n}a_1}\cup\mathcal{B}_{L_n\Box w_1^1}$ is linearly independent. To this end, it is easy to see that $\mathcal{B}^{*}_{a_{2n}\Box L_{2m+1}}$ is a linearly independent and any linear combination of this set must contain an edge of $\{a_{2n}\}\Box(E_5\cup\ E_7\cup \{w_3^m w_2^{m+1}, w_4^m w_2^{m+1}\})$ which does not occur in any cycle of $\cup_{k=1}^{2n-1}\mathcal{B}^{**}_{a_ka_{k+1}}\cup \mathcal{B}^{**}_{a_{2n}a_1}\cup\mathcal{B}_{L_n\Box w_1^1}$. Hence, $\left(\cup_{j=1}^{2n-1}\mathcal{B}^{**}_{a_ka_{k+1}}\right)\cup\mathcal{B}^{**}_{a_{2n}a_1}\cup\mathcal{B}_{L_n\Box w_1^1}\cup\mathcal{B}^{*}_{a_{2n}\Box L_{2m+1}}$ is linearly independent. It is clear that $\mathcal{P}$ is the basis of the planar graph

$$a_1a_2\dots a_{2n}\Box w_3^1w_3^2w_3^3\dots w_3^m w_2^{m+1}w_4^m w_4^{m-1}w_4^{m-2}\dots w_4^2w_4^1$$

obtained by pasting all the 4-cycles at the common edges of the successive cycles. Thus, $\mathcal{P}$ is a linearly independent set. Note that

$$\begin{aligned}
& E(\mathcal{P}) \cap E\left((\cup_{k=1}^{2n-1}\mathcal{B}^{**}_{a_k a_{k+1}}) \cup E(\mathcal{B}_{L_n \Box w_1^1}) \cup E(\mathcal{B}^{**}_{a_{2n}a_1}) \cup \right. \\
& \left. E(\mathcal{B}^{*}_{a_{2n}\Box L_{2m+1}})\right) \\
\subseteq\ & E(a_1 a_2 \ldots a_{2n} \Box V(L_{2m+1})) \cup \\
& E(a_{2n} \Box w_3^1 w_3^2 w_3^3 \ldots w_3^m w_2^{m+1} w_4^m w_4^{m-1} w_4^{m-2} \ldots w_4^2 w_4^1)
\end{aligned}$$

which is an edge set of a tree. Thus, by Lemma 4.1, $\mathcal{P} \cup \left(\cup_{j=1}^{2n-1}\mathcal{B}^{**}_{a_k a_{k+1}}\right) \cup \mathcal{B}_{L_n \Box w_1^1} \cup \mathcal{B}^{**}_{a_{2n}a_1} \cup \mathcal{B}^{*}_{a_{2n}\Box L_{2m+1}}$ is a linearly independent set. ■

Lemma 4.13. $\dim \mathcal{C}(L_n \ltimes L_{2m+1}) = 52mn - 32m + 10n - 7$ *and* $\dim \mathcal{C}(L_n \ltimes L_{2m}) = 52mn - 32m - 4n + 1$.

Proof. By Theorem 3.1,

$$\begin{aligned}
L_n \ltimes L_{2m+1} &= (L_n \Box L_{2m+1}) \cup (L_n \times L^*_{2m+1}) \\
&= (L_n \Box L_{2m+1}) \cup \left(\cup_{i=1}^m \left(L_k \times K_4^{(i)}\right)\right) \cup \left(L_k \times K_2^{(m+1)}\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
|E(L_n \ltimes L_{2m+1})| &= |E(L_n \Box L_{2m+1})| + |E(\cup_{i=1}^m L_n \times K_4^{(i)})| \\
&\quad + |E(L_k \times K_2^{(m+1)})| \\
&= (2n)(3(2m+1) - 2) + 2(2m+1)(3n-2) \\
&\quad + 2\sum_{i=1}^{m} 6(3n-2) + 2(3n-2) \\
&= 60mn - 32m + 14n - 8
\end{aligned}$$

Therefore,

$$\begin{aligned}
\dim \mathcal{C}(L_n \ltimes L_{2m+1}) &= 60mn - 32m + 14n - 8 - 4n(2m+1) + 1 \\
&= 52mn - 32m + 10n - 7.
\end{aligned}$$

To proof the second part, we follow the same above argument after taking into account that $L_n \times L^*_{2m} = \cup_{i=1}^m \left(L_k \times K_4^{(i)}\right)$. ■

Theorem 4.14. $\mathcal{B}(L_n \ltimes L_{2m+1}) = \mathcal{P} \cup \left(\cup_{k=1}^{2n-1}\mathcal{B}^{**}_{a_k a_{k+1}}\right) \cup \mathcal{B}^{**}_{a_{2n}a_1} \cup \mathcal{B}_{L_n \Box w_1^1} \cup \mathcal{B}^{*}_{a_{2n}\Box L_{2m+1}} \cup \left(\cup_{k=2}^{n-1}\mathcal{A}^{**}_{a_k a_{2n-k+1}}\right)$ *is a 4-fold cycle basis for* $\mathcal{C}(L_n \ltimes L_{2m+1})$.

Proof. Since $\{a_k a_{2n-k+1}\}_{k=2}^{n-1}$ is an independent set, $E(\mathcal{A}^{**}_{a_j a_{2n-j+1}}) \cap E(\mathcal{A}^{**}_{a_k a_{2n-k+1}}) = \varnothing$ whenever $k \neq j$. Therefore, $\cup_{k=2}^{n-1} \mathcal{A}^{**}_{a_k a_{2n-k+1}}$ is linearly independent. Since

$$\begin{aligned} & E(\mathcal{P} \cup (\cup_{k=1}^{2n-1} \mathcal{B}^{**}_{a_k a_{k+1}}) \cup \mathcal{B}_{L_n \Box w_1^1} \cup \mathcal{B}^{**}_{a_{2n} a_1} \cup \mathcal{B}^{*}_{a_{2n} \Box L_{2m+1}}) \cap \\ & E(\cup_{k=2}^{n-1} \mathcal{A}^{**}_{a_k a_{2n-k+1}}) \\ \subseteq\ & \{a_k, a_{2n-k+1}\}_{j=1}^{n-2} \Box (E_1 \cup E_2 \cup E_5 \cup E_6 \cup \\ & \{w_3^m w_2^{m+1}, w_4^m w_2^{m+1}, w_1^{m+1} w_2^{m+1}\}) \cup \{a_k a_{2n-k+1}\}_{k=2}^{n-1} \Box w_1^1. \end{aligned}$$

which is a forest Remark 3.3, as a result, by Lemma 4.1, $\mathcal{B}((L_n \ltimes L_{2m+1})$ is a linearly independent set. Note that

$$\begin{aligned} |\mathcal{B}^{**}_{ab}| &= |\mathcal{B}_{ab}| + 2 = (16m+3), \\ |\mathcal{B}^{*}_{a_{2n} \Box L_{2m+1}}| &= 2m, \end{aligned}$$

and

$$|\mathcal{P}| = (2n-1)(2m)$$

Thus,

$$\begin{aligned} |\mathcal{P}| + |\mathcal{B}^{*}_{a_{2n} \Box L_{2m+1}}| + |\cup_{k=1}^{2n-1} \mathcal{B}^{**}_{a_k a_{k+1}}| &= (2n-1)(2m) + 2m + \\ & \sum_{k=1}^{(2n-1)} (16m+3), \\ &= 4mn + (2n-1)(16m+3). \end{aligned}$$

Also,

$$|\mathcal{A}^{**}_{ab}| = |\mathcal{A}_{ab}| + 3 = (16m+3),$$

and

$$|\mathcal{B}_{L_n \Box w_1^1}| = n - 1.$$

Thus,

$$\begin{aligned} |\mathcal{B}(L_n \ltimes L_{2m+1})| &= |\mathcal{P}| + |\mathcal{B}^{*}_{a_{2n} \Box L_{2m+1}}| + |\cup_{k=1}^{2n-1} \mathcal{B}^{**}_{a_k a_{k+1}}| + |\mathcal{B}_{L_n \Box w_1^1}| + \\ & |\mathcal{B}^{**}_{a_{2n} a_1}| + \sum_{k=1}^{n-2} |\mathcal{A}^{**}_{a_k a_{2n-k+1}}| \\ &= 2n(2m) + (2n-1)(16m+3) + (n-1) + (16m+3) \\ & +(n-2)(16m+3) \\ &= 52mn - 32m + 10n - 7. \end{aligned}$$

To complete the proof, we count the number of cycles of $\mathcal{B}$ in which each edge of G occurs. Let $e \in E(L_n \ltimes L_{2m+1})$. Then e belongs to at least one of the following edge sets: $E(L_n \Box w_1^1), E(a_{2n} \Box L_{2m}),\ E(\mathcal{B}^{**}_{a_{2n}a_1}), \mathcal{P}$ and $E(\mathcal{B}^{**}_{a_k a_{k+1}})$ and $E(\mathcal{A}^{**}_{a_k a_{2n-k+1}})$ for some k. We consider the following cases:

Case 1. $e \in E(L_n \Box w_1^1)$. Then, by Remarks 4.7 and 4.9, we have the following:

(1) If $e = (a_k, w_1^1)(a_{k+1}, w_1^1)$, then $f_{\mathcal{B}(L_n \ltimes L_{2m+1})} = f_{\mathcal{B}^{**}_{a_k a_{k+1}}}(e) + f_{\mathcal{B}_{L_n \Box w_1^1}}(e) \leq 3 + 1$.

(2) If $e = (a_1, w_1^1)(a_{2n}, w_1^1)$, then $f_{\mathcal{B}(L_n \ltimes L_{2m+1})} = f_{\mathcal{B}^{**}_{a_{2n}a_1}}(e) + f_{\mathcal{B}_{L_n \Box w_1^1}}(e) \leq 3 + 1$.

(3) If $e = (a_k, w_1^1)(a_{2n-k+1}, w_1^1)$ for some k, then $f_{\mathcal{B}(L_n \ltimes L_{2m+1})} = f_{\mathcal{A}^{**}_{a_k a_{2n-k+1}}}(e) + f_{\mathcal{B}_{L_n \Box w_1^1}}(e) \ \leq 2 + 2$.

Case 2. $e \in E(a_{2n} \Box L_{2m}) - E(L_n \Box w_1^1)$. Then by Remark 4.7 we have the following:

(1) If $e = (a_{2n}, w_1^i)(a_{2n}, w_1^{i+1})$ where $1 \leq i \leq m$, then $f_{\mathcal{B}(L_n \ltimes L_{2m+1})}(e) = f_{\mathcal{B}^{**}_{a_{2n-1}a_{2n}}}(e) + f_{\mathcal{B}^{**}_{a_{2n}a_1}}(e) + f_{\mathcal{B}^{*}_{a_{2n} \Box L_{2m+1}}}(e) \leq 1{+}2{+}1$ for odd i and $\leq 2{+}1{+}1$ for even i.

(2) If $e = (a_{2n}, w_2^i)(a_{2n}, w_2^{i+1})$ where $1 \leq i \leq m-1$, then $f_{\mathcal{B}(L_n \ltimes L_{2m+1})}(e) = f_{\mathcal{B}^{**}_{a_{2n-1}a_{2n}}}(e) + f_{\mathcal{B}^{**}_{a_{2n}a_1}}(e) + f_{\mathcal{B}^{*}_{a_{2n} \Box L_{2m+1}}}(e) \leq 2 + 1 + 1$ for odd i and $\leq 1 + 2 + 1$ for even i.

(3) If $e = (a_{2n}, w_1^{m+1})(a_{2n}, w_2^{m+1})$, then $f_{\mathcal{B}(L_n \ltimes L_{2m+1})}(e) = f_{\mathcal{B}^{**}_{a_{2n-1}a_{2n}}}(e) + f_{\mathcal{B}^{**}_{a_{2n}a_1}}(e) + f_{\mathcal{B}^{*}_{a_{2n} \Box L_{2m+1}}}(e) \leq 1 + 2 + 1$.

(4) If $e = (a_{2n}, w_1^i)(a_{2n}, w_4^i)$ or $(a_{2n}, w_2^i)(a_{2n}, w_3^3)$ where $1 \leq i \leq m$, then $f_{\mathcal{B}(L_n \ltimes L_{2m+1})}(e) = f_{\mathcal{B}^{**}_{a_{2n-1}a_{2n}}}(e) + f_{\mathcal{B}^{**}_{a_{2n}a_1}}(e) + f_{\mathcal{B}^{*}_{a_{2n} \Box L_{2m+1}}}(e) \leq 1 + 1 + 2$.

(5) If $e = (a_{2n}, w_3^i)(a_{2n}, w_3^{i+1})$ or $(a_{2n}, w_4^i)(a_{2n}, w_4^{i+1})$ or $(a_{2n}, w_2^{m+1})(a_{2n}, w_3^m)$ or $(a_{2n}, w_2^{m+1})(a_{2n}, w_4^m)$, then $f_{\mathcal{B}(L_n \ltimes L_{2m+1})}(e) = f_{\mathcal{P}}(e) + f_{\mathcal{B}^{*}_{a_{2n} \Box L_{2m+1}}}(e) \leq 1 + 1$.

(6) If $e = (a_{2n}, w_2^m)(a_{2n}, w_1^{m+1})$, then $f_{\mathcal{B}(L_n \ltimes L_{2m+1})}(e) = f_{\mathcal{B}^{**}_{a_{2n-1}a_{2n}}}(e) + f_{\mathcal{B}^{**}_{a_{2n}a_1}}(e) + f_{\mathcal{B}^{*}_{a_{2n} \Box L_{2m+1}}}(e) \leq 2 + 1 + 1$ for odd m and $\leq 1 + 2 + 1$ for even m.

Case 3. $e \in E(\mathcal{B}^{**}_{a_k a_{k+1}}) - \big(E(L_n \Box w_1^1) \cup E(a_{2n} \Box L_{2m})\big)$ for some $1 \leq k \leq 2n - 1$. Then, by Remarks 4.7 and 4.9, we have the following:

(1) If $e = (a_k, w_3^i)(a_{k+1}, w_3^i)$ or $(a_k, w_4^i)(a_{k+1}, w_4^i)$ such that $1 \leq i \leq m$ or $(a_k, w_2^{m+1})(a_{k+1},\ w_2^{m+1})$, then $f_{\mathcal{B}(L_n \ltimes L_{2m+1})} = f_{\mathcal{B}^{**}_{a_k a_{k+1}}}(e) + f_{\mathcal{P}}(e) \leq 2 + 2$.

(2) If $e = (a_k, w_j^i)(a_{k+1}, w_s^t)$ for some i, j, t and s which is not as in (1), then $f_{\mathcal{B}(L_n \ltimes L_{2m+1})} = f_{\mathcal{B}^{**}_{a_k a_{k+1}}}(e) \leq 4$.

(3) If $e = (a_k, w_1^i)(a_k, w_1^{i+1})$ $1 \leq i \leq m$, then $f_{\mathcal{B}(L_n \ltimes L_{2m+1})}(e) =$

$f_{\mathcal{B}^{**}_{a_{k-1}a_k}}(e)+f_{\mathcal{B}^{**}_{a_k a_{k+1}}}(e) \le 1+2$ for odd i and $f_{\mathcal{B}(L_n \ltimes L_{2m+1})}(e) = f_{\mathcal{B}^{**}_{a_{k-1}a_k}}(e)$ $+f_{\mathcal{B}^{**}_{a_k a_{k+1}}}(e) \le 2+1$ for even i.

(4) If $e = (a_{k+1}, w_2^i)(a_{k+1}, w_2^{i+1})$ $1 \le i \le m-1$, then $f_{\mathcal{B}(L_n \ltimes L_{2m+1})}(e) = f_{\mathcal{B}^{**}_{a_k a_{k+1}}}(e) + f_{\mathcal{B}^{**}_{a_{k+1}a_{k+2}}}(e) \le 2+1$ for odd i and $f_{\mathcal{B}(L_n \ltimes L_{2m+1})}(e) = f_{\mathcal{B}^{**}_{a_k a_{k+1}}}(e) + f_{\mathcal{B}^{**}_{a_{k+1}a_{k+2}}}(e) \le 1+2$.

(5) If $e = (a_{k+1}, w_1^i)(a_{k+1}, w_1^{i+1})$ such that $1 \le i \le m$ or $(a_k, w_2^i)(a_k, w_2^{i+1})$ such that $1 \le i \le m-1$, then we have cases similar to (3) and (4)

(6) If $e = (a_k, w_1^{m+1})(a_k, w_2^{m+1})$, then $f_{\mathcal{B}(L_n \ltimes L_{2m+1})}(e) = f_{\mathcal{B}^{**}_{a_{k-1}a_k}}(e) + f_{\mathcal{B}^{**}_{a_k a_{k+1}}}(e) + f_{\mathcal{A}^{**}_{a_k a_{2n-k+1}}}(e) \le 1+2+1$. Similarly for $(a_{k+1}, w_1^{m+1})(a_{k+1}, w_2^{m+1})$.

(7) If $e = (a_k, w_1^i)(a_k, w_4^i)$ or $(a_k, w_2^i)(a_k, w_3^i)$ such that $1 \le i \le m$, then $f_{\mathcal{B}(L_n \ltimes L_{2m+1})}(e) = f_{\mathcal{B}^{**}_{a_{k-1}a_k}}(e) + f_{\mathcal{B}^{**}_{a_k a_{k+1}}}(e) + f_{\mathcal{A}^{**}_{a_k a_{2n-k+1}}}(e) \le 1+1+1$. Similarly for $e = (a_{k+1}, w_1^i)(a_{k+1}, w_4^i)$ or $(a_{k+1}, w_2^i)(a_{k+1}, w_3^3)$ such that $1 \le i \le m$.

(8) If $e = (a_k, w_2^m)(a_k, w_1^{m+1})$, then $f_{\mathcal{B}(L_n \ltimes L_{2m+1})}(e) = f_{\mathcal{B}^{**}_{a_{k-1}a_k}}(e) + f_{\mathcal{B}^{**}_{a_k a_{k+1}}}(e) \le 2+1$ if m is odd and $\le 1+2$ if m is even. Similarly for $(a_{k+1}, w_2^m)(a_{k+1}, w_1^{m+1})$.

Case 4. $e \in E(\mathcal{B}^{**}_{a_{2n}a_1}) - \Big(E(\cup_{k=1}^{2n-1} \mathcal{B}^{**}_{a_k a_{k+1}}) \cup E(L_n \Box w_1^1) \cup E(a_{2n} \Box\ L_{2m})\Big)$. Then, by Remarks 4.7 and 4.9, we have the following:

(1) If $e = (a_{2n}, w_1^{m+1})(a_1, w_2^{m+1})$ or $(a_{2n}, w_2^{m+1})(a_1, w_1^{m+1})$, then $f_{\mathcal{B}(L_n \ltimes L_{2m+1})}(e) = f_{\mathcal{B}^{**}_{a_{2n}a_1}}(e)\ + f_{\mathcal{B}^{*}_{a_{2n}\Box L_{2m+1}}}(e) \le 3+1$.

(2) If $e = (a_1, w_1^{m+1})(a_1, w_2^{m+1})$, then $f_{\mathcal{B}(L_n \ltimes L_{2m+1})}(e) = f_{\mathcal{B}^{**}_{a_{2n}a_1}}(e) + f_{\mathcal{B}^{**}_{a_1 a_2}}(e) + f_{\mathcal{B}^{*}_{a_{2n}\Box L_{2m+1}}}(e)\ \le 1+2+1$.

(3) If e is not as in (1) and (2), then we have cases as in Case 3.

Case 5. $e \in E(\mathcal{A}^{**}_{a_k a_{2n-k+1}}) - \Big(E(\cup_{k=1}^{2n} \mathcal{B}^{**}_{a_k a_{k+1}}) \cup E(L_n \Box w_1^1) \cup E(a_{2n} \Box L_{2m}) \cup E(\mathcal{B}^{**}_{a_{2n}a_1})\Big)$. Then, by Remarks 4.7 and 4.9, we have the following:

(1) If $e = (a_k, w_j^i)(a_{2n-k+1}, w_l^{(t)})$, then $f_{\mathcal{B}(L_n \ltimes L_{2m+1})} = f_{\mathcal{A}^{**}_{a_k a_{2n-k+1}}}(e) \le 4$.

(2) If $e = (a_k, w_3^i)(a_k, w_3^{i+1})$ such that i is odd or $(a_{2n-k+1}, w_4^i)(a_{2n-k+1}, w_4^{i+1})$ such that i is even , then $f_{\mathcal{B}(L_n \ltimes L_{2m+1})} = f_{\mathcal{A}^{**}_{a_k a_{2n-k+1}}}(e) + f_{\mathcal{P}}(e) \le 2+2$.

(3) If $e = (a_k, w_4^i)(a_k, w_4^{i+1})$ such that i is even or $(a_{2n-k+1}, w_3^i)(a_{2n-k+1}, w_3^{i+1})$ such that i is odd, then $f_{\mathcal{B}(L_n \ltimes L_{2m+1})} = f_{\mathcal{A}^{**}_{a_k a_{2n-k+1}}}(e) + f_{\mathcal{P}}(e) \le 1+2$.

(4) If $e=(a_k,w_3^m)(a_k,w_2^{m+1})$, then $f_{\mathcal{B}(L_n\ltimes L_{2m+1})}(e)=f_{\mathcal{A}^{**}_{a_ka_{2n-k+1}}}(e)+f_{\mathcal{P}}(e)\leq 2+2$ if m is odd and $\leq 1+2$ if m is even. Similarly for $(a_{2n-k+1},w_3^m)(a_{2n-k+1},w_2^{m+1})$.

(5) If $e=(a_k,w_4^m)(a_k,w_2^{m+1})$, then $f_{\mathcal{B}(L_n\ltimes L_{2m+1})}(e)=f_{\mathcal{A}^{**}_{a_ka_{2n-k+1}}}(e)+f_{\mathcal{P}}(e)\leq 2+2$ if m is even and $\leq 1+2$ if m is odd. Similarly for $(a_{2n-k+1},w_4^m)(a_{2n-k+1},w_2^{m+1})$.

Case 6. $e\in E(\mathcal{P})-E((\cup_{k=1}^{2n}\mathcal{B}^{**}_{a_ka_{k+1}})\cup(\cup_{k=2}^{n-1}\mathcal{A}^{**}_{a_ka_{2n-k+1}})\cup(L_n\Box w_1^1)\cup E(a_{2n}\Box L_{2m})\cup E(\mathcal{B}^{**}_{a_{2n}a_1}))$. Then $f_{\mathcal{B}(L_n\times L_{2m+1})}(e)=f_{\mathcal{P}}(e)\leq 2$. ■

Now, we turn into the case $\eta=2m$. Note that as in the case $\eta=2m+1$, we have that $ab\ltimes L_{2m}$ is decomposable into $\left(\cup_{i=1}^m K_{4,4}^{(ab,i)}\right)\cup\left(\cup_{i=1}^7(\{a,b\}\Box A_i)\right)$. Let

$$\begin{aligned}\mathcal{F}_{ab} &= \cup_{i=1}^m\left(S^i_{ab}\cup\left\{\mathcal{A}^i_{(ab,1)},\mathcal{A}^i_{(ba,1)},\mathcal{A}^i_{(ba,2)},\mathcal{A}^i_{(ab,3)}\right\}\right)\\ &\quad\cup\left(\cup_{i=1}^{m-1}\mathcal{T}^i_{(ab,12)}\right)\cup\mathcal{K}_{ab}^{(m-1,*)}\end{aligned}$$

$$\begin{aligned}\mathcal{M}_{ab} &= \cup_{i=1}^m\left(S^i_{ab}\cup\left\{\mathcal{A}^i_{(ab,1)},\mathcal{A}^i_{(ba,1)},\mathcal{A}^i_{(ab,2)},\mathcal{A}^i_{(ab,3)}\right\}\right)\\ &\quad\cup\left(\cup_{i=1}^{m-1}\mathcal{T}^i_{(ab,34)}\right)\cup\mathcal{K}_{ab}^{(m-1,**)}\end{aligned}$$

Then by using similar arguments as in Lemmas 4.6 and 4.8 and noting that $(a,w_1^m)(a,w_2^m)$ does not appear in any cycle of $\mathcal{F}_{ab}$ and $(a,w_3^m)(a,w_4^m)$ does not appear in any cycle of $\mathcal{M}_{ab}$ we have the following results:

Lemma 4.15. *$\mathcal{F}^*_{ab}=\mathcal{F}_{ab}\cup\{(a,w_1^m)(a,w_2^m)(b,w_2^m)(b,w_1^m)(a,w_1^m)\}$ is a linearly independent subset of $\mathcal{C}(ab\ltimes L_{2m})$.* ■

Lemma 4.16. *$\mathcal{M}^*_{ab}=\mathcal{M}_{ab}\cup\{(a,w_3^m)(a,w_4^m)(b,w_4^m)(b,w_3^m)(a,w_3^m)\}$ is linearly independent.* ■

Lemma 4.17. *$\mathcal{B}_{a_{2n}\Box L_{2m}}=\{(a_{2n},w_1^i)(a_{2n},w_1^{i+1})(a_{2n},w_4^{i+1})(a_{2n},w_4^i)(a_{2n},w_1^i)|i=1,2,\dots,m-1\}\cup\{(a_{2n},w_2^i)(a_{2n},w_2^{i+1})(a_{2n},w_3^{i+1})(a_{2n},w_3^i)(a_{2n},w_2^i)|\ i=1,2,\dots,m-1\}\cup\{(a_{2n},w_1^m)(a_{2n},w_2^m)\ (a_{2n},w_3^m)(a_{2n},w_4^m)(a_{2n},\ w_1^m)\}$ is a basis for $\mathcal{C}(a_{2n}\Box L_{2m})$.* ■

Let $\mathcal{L}=\cup_{i=1}^{m-1}\cup_{k=1}^{2n-1}\mathcal{T}^i_{(a_ka_{k+1},34)}$. Then by a similar argument as in Theorem 4.14, we have the following result.

Theorem 4.18. *$\mathcal{B}(L_n\ltimes L_{2m})=\mathcal{L}\cup\left(\cup_{k=1}^{2n-1}\mathcal{F}^*_{a_ka_{k+1}}\right)\cup\mathcal{B}_{L_n\Box w_1^1}\cup\mathcal{F}^*_{a_{2n}a_1}\cup(\mathcal{B}_{a_{2n}\Box L_{2m}})\cup\left(\cup_{j=2}^{n-1}\mathcal{M}^*_{a_ka_{2n-k+1}}\right)$ is a 4-fold cycle basis for $\mathcal{C}(L_n\ltimes L_{2m})$.*

To this end, we presented the necessary background to prove our main result.

Theorem 4.19. *For each n, η, $b(L_n \ltimes L_\eta) \leq 4$. Moreover, the equality holds for each integers n, η satisfying the following: (1) Odd $\eta \geq 3$ and $n \geq 3$. (2) Even $\eta \geq 4$* and $n \geq 4$.

Proof. By Theorems 4.14 and 4.18, it suffices to show that $\mathcal{C}(L_n \ltimes L_\eta)$ has no 3-fold basis under the stated conditions. Assume that $\mathcal{B}$ is a 3-fold basis for $\mathcal{C}(L_n \ltimes L_\eta)$ under the stated condition. Then we consider the following two cases:

Case 1. $\eta = 2m + 1$. Then we consider the following three subcases:

Subcase 1.1. $\mathcal{B}$ consists of s of 3-cycles. Then each cycle must contains an edge of $V(L_n)\Box\left(E_1 \cup E_2 \cup \{w_1^{m+1}w_2^{m+1}\}\right)$. Since the fold of each edge is at most 3, as a result $|\mathcal{B}| \leq 3s$, that is,

$$\begin{aligned} 52mn - 32m + 10n - 7 &\leq 3(2m+1)(2n) \\ &\leq 12nm + 6n. \end{aligned}$$

Thus,

$$4n(10m + 1) \leq 32m + 7.$$

Hence,

$$n \leq \frac{32m + 7}{4(10m + 1)} < 2.$$

This is a contradiction.

Subcase 1.2. $\mathcal{B}$ consists of 4-cycles. Then $4|\mathcal{B}| \leq 3|E(L_n \ltimes L_\eta)|$. And so

$$\begin{aligned} 4(52mn - 32m + 10n - 7) &\leq 3(60mn - 32m + 14n - 8) \\ 208mn - 128m + 40n - 28 &\leq 180mn - 96m + 42n - 24. \end{aligned}$$

Hence,

$$2n(14m - 1) - 32m - 4 \leq 0.$$

So,

$$\begin{aligned} n &\leq \frac{32m + 4}{2(14m - 1)} \\ &\leq \frac{3}{2}. \end{aligned}$$

This is a contradiction.

Subcase 1.3. $\mathcal{B}$ consists of t cycles of length at least 4 and s 3-cycle. Since, $3s + 4t \leq 3|E(L_n \ltimes L_\eta)|$, we have that

$$t \leq \left\lfloor \frac{3(60mn - 32m + 14n - 8) - 3s}{4} \right\rfloor.$$

Therefore,

$$|\mathcal{B}| = s + t \leq s + \frac{3(60mn - 32m + 14n - 8) - 3s}{4}.$$

Hence,

$$4|\mathcal{B}| \leq s + 3(60mn - 32m + 14n - 8).$$

By Subcase 1.1, we have

$$4(52mn - 32m + 10n - 7) \leq (12nm + 6n) + 3(60mn - 32m + 14n - 8)$$

which is equivalent to

$$8n(2m - 1) \leq 32m + 4.$$

Therefore,

$$\begin{aligned} n &\leq \frac{8(2m + 1/2)}{8(2m - 1)} \\ &< 3. \end{aligned}$$

This is a contradiction.
Case 2. $\eta = 2m$. Then we follow the same arguments as in the above three subcases taking into account that any 3-cycle must contains an edge of $V(L_n)\square(A_1 \cup A_2 \cup A_7)$ to get a similar contradiction. ■

Corollary 4.20. *For odd $\eta \geq 1$ and $n \geq 3$, $\mathcal{B}(L_n \ltimes L_\eta) = \mathcal{P} \cup (\cup_{k=1}^{2n-1} \mathcal{B}^{**}_{a_k a_{k+1}}) \cup \mathcal{B}^{**}_{a_{2n} a_1} \cup \mathcal{B}_{L_n \square w_1^1} \cup \mathcal{B}^{*}_{a_{2n} \square L_\eta} \cup (\cup_{k=2}^{n-1} \mathcal{A}^{**}_{a_k a_{2n-k+1}})$ is a required basis for the cycle space $\mathcal{C}(L_n \ltimes L_\eta)$. Further, for even $\eta \geq 4$* and *$n \geq 4$, $\mathcal{B}(L_n \ltimes L_\eta) = \mathcal{L} \cup (\cup_{k=1}^{2n-1} \mathcal{F}^{*}_{a_k a_{k+1}}) \cup \mathcal{B}_{L_n \square w_1^1} \cup \mathcal{F}^{*}_{a_{2n} a_1} \cup (\mathcal{B}_{a_{2n} \square L_\eta}) \cup (\cup_{j=2}^{n-1} \mathcal{M}^{*}_{a_k a_{2n-k+1}})$ is a required basis for the cycle space $\mathcal{C}(L_n \ltimes L_\eta)$* ■

5 Further results.

The following results will be useful in the coming result.

Proposition 5.1. ([12]) *The Cartesian product of connected graphs has a vertex transitive automorphism group if and only if every factor has a vertex transitive automorphism group.* ■

Theorem 5.2. *(Jaradat [13]) $G \ltimes H$ is isomorphic to $G[H]$ if and only if H is a vertex transitive.* ■

The circular ladder CL_η is defined to be $P_2 \square C_\eta$ where P_2 is a path of order 2 and C_η is a cycle of order η. Since both of P_2 and C_η are vertex transitive. Thus, by the above proposition, CL_η is vertex transitive. Thus, by Theorem 5.2 and Theorem 2.4 of [17] we have the following result:

Theorem 5.3. *For any $n \geq 2$ and $\eta \geq 3$, $b(L_n \ltimes CL_\eta) \leq 4$. Moreover, the equality holds if $n \geq 2$ and $\eta \geq 4$.* ■

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