On the powers of the k–Fibonacci numbers

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Abstract

In this paper we will find a combinatorial formula that relates the power of a k–Fibonacci number, $F_{k,n}^p$, to the number $F_{k,an}$. From this formula and if p is odd, we will find a new formula that allows to express the k–Fibonacci number $F_{k,(2r+1)n}$ as a combination of odd powers of $F_{k,n}$. If p is even, the formula is similar but for the even k–Lucas numbers $L_{k,2rn}$.

Keyword: k–Fibonacci numbers, k–Lucas numbers, Recurrence laws
MSC2000: 15A36; 11C20; 11B39

1 Introduction

Many generalizations of the Fibonacci sequence have been introduced and studied. Here we use the k-Fibonacci numbers defined as follows [2, 3]:

Definition 1 *For any integer number $k \geq 1$, the k–Fibonacci sequence, say $\{F_{k,n}\}_{n\in\mathbb{N}}$ is defined recurrently by:*

$$F_{k,0} = 0, \ F_{k,1} = 1, \ \text{and } F_{k,n+1} = kF_{k,n} + F_{k,n-1} \text{ for } n \geq 1.$$

For $k = 1$, the classical Fibonacci sequence is obtained $\{0, 1, 1, 2, 3, 5, \ldots\}$ and for $k = 2$, we obtain the Pascal sequence $\{0, 1, 2, 5, 12, 29, 70, \ldots\}$.

For the properties of the k–Fibonacci sequences, see [2, 3].

- *The characteristic solutions of the initial recurrence relation are $\sigma_1 = \dfrac{k+\sqrt{k^2+4}}{2}$ and $\sigma_2 = \dfrac{k-\sqrt{k^2+4}}{2}$. Among its properties, these roots verify $\sigma_1 + \sigma_2 = k$, $\sigma_1 \cdot \sigma_2 = -1$, $\sigma_1 - \sigma_2 = \sqrt{k^2+4}$.*

  $k = 1 \rightarrow \sigma_1 = \Phi$: *The Golden Ratio [7]*
  $k = 2 \rightarrow \sigma_1 = 2 + \sqrt{2}$: *The Silver Ratio*
  $k = 3 \rightarrow \sigma_1 = \frac{3+\sqrt{13}}{2}$: *The Bronze Ratio.*

- *The Binet formula:* $\forall n \geq 0$: $F_{k,n} = \dfrac{\sigma_1^n - \sigma_2^n}{\sigma_1 - \sigma_2}$,

Moreover: $\sigma_1^n + \sigma_2^n = F_{k,n-1} + F_{k,n+1}$ *[4]*

In [4] it is proved (Theorem 4) the following sum for $a \in \mathcal{N}$:

$$\sum_{j=0}^{n} F_{k,a\,j} = \frac{F_{k,a(n+1)} - (-1)^a F_{k,a\,n} - F_{k,a}}{L_{k,a}}$$

In particular, if a is odd, $a = 2r + 1$, *then*

$$\sum_{j=0}^{n} F_{k,(2r+1)j} = \frac{F_{k,(2r+1)(n+1)} + F_{k,(2r+1)n} - F_{k,2r+1}}{L_{k,2r+1}} \tag{1}$$

Theorem 1 **Sum of the alternated consecutive k-Fibonacci numbers**
The sum of the sequence $\{F_{k,h}, -F_{k,a+h}, F_{k,2a+h}, -F_{k,3a+h}, \ldots\}$ *is given by*

$$\sum_{j=0}^{n}(-1)^j F_{k,aj+h} = \frac{(-1)^{n+a}F_{k,an+h} + (-1)^n F_{k,an+h+a} - (-1)^h F_{k,a-h} + F_{k,h}}{L_{k,a} + (-1)^a + 1}$$

Proof. We will prove this theorem following a similar process that in [4].

$$\sum_{j=0}(-1)^j F_{k,aj+h} = \sum_{j=0}(-1)^j \frac{\sigma_1 - \sigma_2}{\sigma_1 - \sigma_2}$$

$$= \frac{1}{\sigma_1 - \sigma_2}\sum_{j=0}^{n}\left((-1)^j\sigma_1^{aj+h} - (-1)^j\sigma_2^{aj+h}\right)$$

$$= \frac{1}{\sigma_1 - \sigma_2}\left[\frac{(-1)^n\sigma_1^{an+h}(-\sigma_1^a) - \sigma_1^h}{-\sigma_1^a - 1} - \frac{(-1)^n\sigma_2^{an+h}(-\sigma_2^a) - \sigma_2^h}{-\sigma_2^a - 1}\right]$$

$$= \frac{1}{\sigma_1 - \sigma_2}\left[\frac{(-1)^n\sigma_1^{an+h}\cdot\sigma_1^a + \sigma_1^h}{\sigma_1^a + 1} - \frac{(-1)^n\sigma_2^{an+h}\cdot\sigma_2^a + \sigma_2^h}{\sigma_2^a + 1}\right]$$

$$= \frac{1}{(\sigma_1\sigma_2)^a + \sigma_1^a + \sigma_2^a + 1}\,\frac{1}{\sigma_1 - \sigma_2}\cdot$$

$$[(-1)^n\sigma_1^{an+h}(-1)^a + (-1)^n\sigma_1^{an+h+a} + \sigma_1^h\sigma_2^a + \sigma_1^h$$

$$- (-1)^n\sigma_2^{an+h}(-1)^a - (-1)^n\sigma_2^{an+h+a} - \sigma_1^a\sigma_2^h - \sigma_2^h]$$

$$= \frac{(-1)^{n+a}F_{k,an+h} + (-1)^n F_{k,an+h+a} - (\sigma_1\sigma_2)^h(\sigma_1^{a-h} - \sigma_2^{a-h}) + F_{k,h}}{F_{k,a-1} + F_{k,a+1} + (-1)^a + 1}$$

$$= \frac{(-1)^{n+a}F_{k,an+h} + (-1)^n F_{k,an+h+a} - (-1)^h F_{k,a-h} + F_{k,h}}{L_{k,a} + (-1)^a + 1}$$

We have used the properties $\sigma_1 \cdot \sigma_2 = -1$ *and* $\sigma_1^n + \sigma_2^n = F_{k,n-1} + F_{k,n+1} = L_{k,n}$
As particular case, for $h = 0$ *and* $a = 2r + 1$ *it is*

$$\sum_{j=0}^{n}(-1)^j F_{k,(2r+1)j} = \frac{(-1)^n F_{k,(2r+1)(n+1)} - (-1)^n F_{k,(2r+1)n} - F_{k,2r+1}}{L_{k,2r+1}} \tag{2}$$

k-Lucas numbers [1] are defined in the same form but with the initial conditions $L_{k,0} = 2$ and $L_{k,1} = k$. In this case, the Binet formula takes the form $L_{k,n} = \sigma_1^n + \sigma_2^n$ and so $L_{k,n} = F_{k,n-1} + F_{k,n+1}$

Theorem 2 **Sum of even k–Lucas numbers**
For $r \geq 1$,

$$\sum_{j=0}^{n} L_{k,2rj} = \frac{L_{k,2r(n+1)} - L_{k,2rn}}{L_{k,2r} - 2} + 1 \tag{3}$$

Proof.

$$\begin{aligned}
\sum_{j=0}^{n} L_{k,2rj} &= \sum_{j=0}^{n}\left(\sigma_1^{2rj} + \sigma_2^{2rj}\right) = \frac{\sigma_1^{2rn}\sigma_1^{2r} - 1}{\sigma_1^{2r} - 1} + \frac{\sigma_2^{2rn}\sigma_2^{2r} - 1}{\sigma_2^{2r} - 1} \\
&= \frac{\sigma_1^{2rn} - \sigma_1^{2r(n+1)} - \sigma_2^{2r} + 1 + \sigma_2^{2rn} - \sigma_2^{2r(n+1)} - \sigma_1^{2r} + 1}{(\sigma_1\sigma_2)^{2r} - \sigma_1^{2r} - \sigma_2^{2r} + 1} \\
&= \frac{\sigma_1^{2r(n+1)} + \sigma_2^{2r(n+1)} - (\sigma_1^{2rn} + \sigma_2^{2rn}) + (\sigma_1^{2r} + \sigma_2^{2r}) - 2}{\sigma_1^{2r} + \sigma_2^{2r} - 2} \\
&= \frac{L_{k,2r(n+1)} - L_{k,2rn} + L_{k,2r} - 2}{L_{k,2r} - 2}
\end{aligned}$$

Theorem 3 **Sum of alternated even k–Lucas numbers**
For $r \geq 1$:

$$\sum_{j=0}^{n}(-1)^j L_{k,2rj} = (-1)^n \frac{L_{k,2r(n+1)} + L_{k,2rn}}{L_{k,2r} + 2} + 1 \tag{4}$$

Proof.

$$\begin{aligned}
&\sum_{j=0}^{n}(-1)^j L_{k,2rj} = \sum_{j=0}^{n}(-1)^j\left(\sigma_1^{2rj} + \sigma_2^{2rj}\right) \\
&= \frac{(-1)^n\sigma_1^{2rn}(-\sigma_1^{2r}) - 1}{-\sigma_1^{2r} - 1} + \frac{(-1)^n\sigma_2^{2rn}(-\sigma_2^{2r}) - 1}{-\sigma_2^{2r} - 1} \\
&= \frac{(-1)^n\sigma_1^{2rn} + (-1)^n\sigma_1^{2r(n+1)} + \sigma_2^{2r} + 1 + (-1)^n\sigma_2^{2rn} + (-1)^n\sigma_2^{2r(n+1)} + \sigma_1^{2r} + 1}{(\sigma_1\sigma_2)^{2r} + \sigma_1^{2r} + \sigma_2^{2r} + 1} \\
&= \frac{(-1)^n\left(\sigma_1^{2r(n+1)} + \sigma_2^{2r(n+1)}\right) + (-1)^n\left(\sigma_1^{2rn} + \sigma_2^{2rn}\right) + (\sigma_1^{2r} + \sigma_2^{2r}) + 2}{\sigma_1^{2r} + \sigma_2^{2r} + 2} \\
&= \frac{(-1)^n L_{k,2r(n+1)} + (-1)^n L_{k,2rn} + L_{k,2r} + 2}{L_{k,2r} + 2}
\end{aligned}$$

because $\sigma_1^{2r}\sigma_2^{2r} = (\sigma_1\sigma_2)^{2r} = (-1)^{2r} = 1$

2 On the odd powers of the k–Fibonacci numbers

In this section we will study the powers $F_{k,n}^{2r+1}$. We will prove a formula that relates these powers to the k–Fibonacci numbers of odd index. This section ends with a formula for the sum of these numbers.

Theorem 4 **Odd power of a k–Fibonacci number**
An odd power of a k–Fibonacci number is related to the k- Fibonacci numbers of odd index by mean of the following formula:

$$F_{k,n}^{2r+1} = \frac{1}{(k^2+4)^r}\sum_{j=0}^{r}(-1)^{(n+1)j}\binom{2r+1}{j}F_{k,(2r+1-2j)n}, \quad r \geq 0 \qquad (5)$$

Proof. From the Binet formula, $F_{k,n}^{2r+1} = \left(\frac{\sigma_1^n - \sigma_2^n}{\sigma_1 - \sigma_2}\right)^{2r+1}$.
Now we will expand this formula taking into account $\sigma_1 - \sigma_2 = \sqrt{k^2+4}$, $\sigma_1\sigma_2 = -1$ *and that this expansion has* $2r+2$ *terms:*

$$\begin{aligned}
F_{k,n}^{2r+1} &= \frac{1}{\sqrt{(k^2+4)^{2r+1}}}\sum_{j=0}^{2r+1}(-1)^j\binom{2r+1}{j}(\sigma_1^n)^{2r+1-j}(\sigma_2^n)^j \\
&= \frac{1}{(k^2+4)^r}\frac{1}{\sqrt{k^2+4}} \\
&\quad \left[(\sigma_1^n)^{2r+1} - \binom{2r+1}{1}(\sigma_1^n)^{2r}(\sigma_2^n) + \binom{2r+1}{2}(\sigma_1^n)^{2r-1}(\sigma_2^n)^2 - \cdots\right] \\
&= \frac{1}{(k^2+4)^r}\left[\frac{(\sigma_1^n)^{2r+1} - (\sigma_2^n)^{2r+1}}{\sqrt{k^2+4}}\right. \\
&\quad - \frac{\binom{2r+1}{1}(\sigma_1^n)^{2r-1}(\sigma_1\sigma_2)^n - \binom{2r+1}{2r}(\sigma_2^n)^{2r-1}(\sigma_1\sigma_2)^n}{\sqrt{k^2+4}} \\
&\quad \left. + \frac{\binom{2r+1}{2}(\sigma_1^n)^{2r-3}(\sigma_1\sigma_2)^{2n} - \binom{2r+1}{2r-1}(\sigma_2^n)^{2r-3}(\sigma_1\sigma_2)^{2n}}{\sqrt{k^2+4}} - \cdots\right] \\
&= \frac{1}{(k^2+4)^r}\left[F_{k,(2r+1)n} - \binom{2r+1}{1}(-1)^n F_{k,(2r-1)n}\right. \\
&\quad \left. + \binom{2r+1}{2}F_{k,(2r-3)n} - \binom{2r+1}{3}(-1)^n F_{k,(2r-5)n} + \cdots\right] \\
&= \frac{1}{(k^2+4)^r}\sum_{j=0}^{r}(-1)^{(n+1)j}\binom{2r+1}{j}F_{k,(2r+1-2j)n}
\end{aligned}$$

We have used the property of symmetry $\binom{n}{m} = \binom{n}{n-m}$ *[5, p. 174].*

From the Formula (5), we can express $F_{k,(2r+1)n}$ *as a combination of powers of* $F_{k,n}$ *and see below. For* $r = 1, 2, 3$:

$$
\begin{aligned}
F_{k,n}^3 &= \frac{1}{k^2+4}[F_{k,3n} - 3(-1)^n F_{k,n}] \\
F_{k,n}^5 &= \frac{1}{(k^2+4)^2}[F_{k,5n} - 5(-1)^n F_{k,3n} + 10F_{k,n}] \\
F_{k,n}^7 &= \frac{1}{(k^2+4)^3}[F_{k,7n} - 7(-1)^n F_{k,5n} + 21F_{k,3n} - 35(-1)^n F_{k,n}]
\end{aligned}
$$

From these formulas we obtain

$$
\begin{aligned}
F_{k,3n} &= (k^2+4)F_{k,n}^3 + 3(-1)^n F_{k,n} \\
F_{k,5n} &= (k^2+4)^2 F_{k,n}^5 + 5(-1)^n (k^2+4) F_{k,n}^3 + 5F_{k,n} \\
F_{k,7n} &= (k^2+4)^3 F_{k,n}^7 + 7(-1)^n (k^2+4)^2 F_{k,n}^5 + 14(k^2+4)F_{k,n}^3 + 7(-1)^n F_{k,n} \\
F_{k,9n} &= (k^2+4)^4 F_{k,n}^9 + 9(-1)^n (k^2+4)^3 F_{k,n}^7 + 27(k^2+4)^2 F_{k,n}^5 \\
&\quad +30(-1)^n (k^2+4)F_{k,n}^3 + 9F_{k,n}
\end{aligned}
$$

With the coefficients of these expressions we generate the Table 1, where the column Σ is the sum by rows and the last column is $\sum_{j=0}^{r}(-1)^{r+c}a_{r,c}$*:*

Table 1: Coefficients of $F_{k,(2r+1)n}$

r\ c	0	1	2	3	4	5	6	7	Σ	Σ′
0	1								1	1
1	1	3							4	2
2	1	5	5						11	1
3	1	7	14	7					29	-1
4	1	9	27	30	9				76	-2
5	1	11	44	77	55	11			199	-1
6	1	13	65	156	182	91	13		521	1
7	1	15	90	275	450	378	140	15	1364	2

This table corresponds to the odd coefficients of the Lucas Polynomials (OEIS, A034807).

It is easy to build this table if we follow the following instructions: if $a_{r,c}$ *is the entry of the r–th row (for* $r = 0,1,2,\ldots$*) and the c–th column (for* $c = 0,1,2,\ldots,r$*), then*

1. $a_{r,0} = 1$ *for all* r
2. $a_{1,1} = 3$

3. $a_{r,c} = 2a_{r-1,c-1} + a_{r-1,c} - a_{r-2,c-2}$

For instance: $a_{6,4} = 2a_{5,3} + a_{5,4} - a_{4,2} = 2 \cdot 77 + 55 - 27 = 182$

All the diagonal sequences of this table 1 as well as the first four columns are listed in OEIS.

The sequence of the penultimate column Σ : $\{1, 4, 11, 29, 76, 199, 521, 1364, \ldots\}$ *is the bisection of the classical Lucas sequence,* L_{2n+1}. *These numbers verify the recurrence relation* $b_{n+1} = 3b_n - b_{n-1}$ *for* $n \geq 1$ *with the initial conditions* $b_0 = 1,\ b_1 = 4$.

The sequence of the last column Σ' : $\{1, 2, 1, -1, -2, -1, 1, 2, \ldots\}$ *corresponds to the sum* $d_r = \sum_{c=0}^{r} (-1)^{r+c} a_{r,c}$. *Each term can be calculate by the formula* $d_r = \frac{3(-1)^{\lfloor \frac{r}{3} \rfloor} - (-1)^r}{2}$. *It is the sequence A057079 in OEIS.*

We can find these numbers by appling the formula

$\forall r,\ a_{r,0} = 1$ *and* $a_{r,c} = \frac{2r+1}{c}\binom{2r-c}{c-1}$ *for* $r \geq 1$. *Consequently,*

$$F_{k,(2r+1)n} = (k^2+4)^r F_{k,n}^{2r+1} + \sum_{c=1}^{r} (-1)^{nc} \frac{2r+1}{c}\binom{2r-c}{c-1}(k^2+4)^{r-c} F_{k,n}^{2(r-c)+1} \qquad (6)$$

For the classical Fibonacci numbers (k = 1), it is:

$$\begin{aligned}
F_{3n} &= 5F_n^3 + 3(-1)^n F_n \\
F_{5n} &= 25F_n^5 + 25(-1)^n F_n^3 + 5F_n \\
F_{7n} &= 125F_n^7 + 175(-1)^n F_n^5 + 70F_n^3 + 7(-1)^n F_n \\
F_{9n} &= 625F_n^9 + 1125(-1)^n F_n^7 + 675F_n^5 + 150(-1)^n F_n^3 + 9F_n
\end{aligned}$$

2.1 Sum of the odd consecutive powers of the k–Fibonacci numbers

From the Equation (5), it is $F_{k,n}^3 = \frac{1}{k^2+4}[F_{k,3n} - (-1)^n 3F_{k,n}]$ *and so*

$\sum_{j=0}^{n} F_{k,j}^3 = \frac{1}{k^2+4}\left[\sum_{j=0}^{n} F_{k,3j} - \sum_{j=0}^{n} (-1)^j 3F_{k,j}\right]$ *and applying the equations (1) and (2), it is*

$$\sum_{j=0}^{n} F_{k,j}^3 = \frac{1}{k^2+4}\left[\frac{F_{k,3(n+1)} + F_{k,3n} - F_{k,3}}{L_{k,3}} - \frac{3}{L_{k,1}}\left[(-1)^n(F_{k,n+1} - F_{k,n}) - 1\right]\right]$$

Following the same process, we can find

$$\sum_{j=0}^{n} F_{k,n}^5 = \frac{1}{(k^2+4)^2}\left[\frac{F_{k,5(n+1)} + F_{k,5n} - F_{k,5}}{L_{k,5}} - 5\frac{(-1)^n F_{k,3(n+1)} + (-1)^n F_{k,3n} - F_{k,3}}{L_{k,3}} + 10\frac{F_{k,n+1} - F_{k,n} - F_{k,1}}{L_{k,1}}\right]$$

$$\sum_{j=0}^{n} F_{k,n}^7 = \frac{1}{(k^2+4)^3}\left[\frac{F_{k,7(n+1)} + F_{k,7n} - F_{k,7}}{L_{k,7}} - 7\frac{(-1)^n F_{k,5(n+1)} - (-1)^n F_{k,5n} - F_{k,5}}{L_{k,5}} + 21\frac{F_{k,3(n+1)} + F_{k,3n} - F_{k,3}}{L_{k,3}} - 35\frac{(-1)^n F_{k,n+1} - (-1)^n F_{k,n} - F_{k,1}}{L_{k,1}}\right]$$

And, in general,

$$\sum_{j=0}^{n} F_{k,j}^{2r+1} = \frac{1}{(k^2+4)^r}\sum_{j=0}^{r}\binom{2r+1}{j}$$

$$\frac{(-1)^{j(n+1)}F_{k,(2r+1-2j)(n+1)} - (-1)^{j(n+1)}F_{k,(2r+1-2j)n} - F_{k,2r+1-2j}}{L_{k,2r+1-2j}}$$

3 Even powers of the k–Fibonacci numbers

We will follow in this section a similar process that in the section 2: finding a combinatorial formula for $F_{k,n}^{2r}$ and then a formula for its sum.

Theorem 5 **Combinatorial formula for the even k–Fibonacci numbers**
For an even power of a k–Fibonacci number, it is

$$F_{k,n}^{2r} = \frac{1}{(k^2+4)^r}\left[\sum_{j=0}\binom{2r}{j}(-1)^{(n+1)j}L_{k,(2r-2j)n} + (-1)^{(n+1)r}\binom{2r}{r}\right] \tag{7}$$

Proof. Developing the Binnet formula

$$F_{k,n}^{2r} = \left(\frac{\sigma_1^n-\sigma_2^n}{\sigma_1-\sigma_2}\right)^{2r} = \frac{1}{\sqrt{(k^2+4)^{2r}}}\sum_{j=0}^{2r}(-1)^j\binom{2r}{j}(\sigma_1^n)^{2r-j}(\sigma_2^n)^j$$

$$= \frac{1}{(k^2+4)^r}\left[(\sigma_1^n)^{2r} - \binom{2r}{1}(\sigma_1^n)^{2r-1}(\sigma_2^n) + \binom{2r}{2}(\sigma_1^n)^{2r-2}(\sigma_2^n)^2 - \cdots\right]$$

$$= \frac{1}{(k^2+4)^r}\left[(\sigma_1^n)^{2r} - \binom{2r}{1}(\sigma_1^n)^{2r-2}(\sigma_1\sigma_2)^n + \binom{2r}{2}(\sigma_1^n)^{2r-4}(\sigma_1\sigma_2)^{2n} - \right.$$

$$\cdots + \binom{2r}{r}(\sigma_1\sigma_2)^{rn} + \cdots + \binom{2r}{2r-2}(\sigma_2^n)^{2r-4}(\sigma_1\sigma_2)^{2n} -$$

$$\left.\binom{2r}{2r-1}(\sigma_2^n)^{2r-2}(\sigma_1\sigma_2)^n + (\sigma_2^n)^{2r}\right]$$

$$= \frac{1}{(k^2+4)^r}\left[\left(\sigma_1^{2rn}+\sigma_2^{2rn}\right) - \binom{2r}{1}(-1)^n\left(\sigma_1^{(2r-2)n}+\sigma_2^{(2r-2)n}\right)\right.$$

$$+\binom{2r}{2}\left(\sigma_1^{(2r-4)n}+\sigma_2^{(2r-4)n}\right) - \binom{2r}{3}(-1)^n\left(\sigma_1^{(2r-6)n}+\sigma_2^{(2r-6)n}\right)$$

$$\left.+\cdots+(-1)^{r+n}\binom{2r}{r}\right]$$

$$= \frac{1}{(k^2+4)^r}\left[L_{k,2rn} - \binom{2r}{1}(-1)^nL_{k,2(r-1)n} + \binom{2r}{2}L_{k,2(r-2)n}\right.$$

$$\left.-\binom{2r}{3}(-1)^nL_{k,2(r-3)n} + \cdots + (-1)^{r+n}\binom{2r}{r}\right]$$

$$= \frac{1}{(k^2+4)^r}\left[\sum_{j=0}^{r-1}\binom{2r}{j}(-1)^{(n+1)j}L_{k,(2r-2j)n} + (-1)^{(n+1)r}\binom{2r}{r}\right]$$

For the powers 2, 4, 6 and 8, it is

$$F_{k,n}^2 = \frac{1}{k^2+4}\left(L_{k,2n} - 2(-1)^n\right)$$

$$F_{k,n}^4 = \frac{1}{(k^2+4)^2}\left(L_{k,4n} - 4(-1)^nL_{k,2n} + 6\right)$$

$$F_{k,n}^6 = \frac{1}{(k^2+4)^3}\left(L_{k,6n} - 6(-1)^nL_{k,4n} + 15L_{k,2n} - (-1)^n20\right)$$

$$F_{k,n}^8 = \frac{1}{(k^2+4)^4}\left(L_{k,8n} - 8(-1)^nL_{k,6n} + 28L_{k,4n} - 56(-1)^nL_{k,2n} + 70\right)$$

As particular cases, and following the same process when r is odd, for $r = 1, 2, 3, 4$ *we obtain* $(k^2+4)F_{k,n}^2 + 2(-1)^n$

$$\begin{aligned}
L_{k,4n} &= (k^2+4)^2F_{k,n}^4 + 4(-1)^n(k^2+4)F_{k,n}^2 + 2\\
L_{k,6n} &= (k^2+4)^3F_{k,n}^6 + 6(-1)^n(k^2+4)^2F_{k,n}^4 + 9(k^2+4)F_{k,n}^2 + 2(-1)^n\\
L_{k,8n} &= (k^2+4)^4F_{k,n}^8 + 8(-1)^n(k^2+4)^3F_{k,n}^6 + 20(k^2+4)^2F_{k,n}^4\\
&\quad +16(-1)^n(k^2+4)F_{k,n}^2 + 2
\end{aligned}$$

With the coefficients of these expressions we generate the Table 2, where the column Σ *is the sum by rows and the last column is* $\sum_{j=0}^{r}(-1)^{r+c}a_{r,c}$*: This table corresponds to the even*

Table 2: Coefficients of $F_{k,2rn}$

r\ c	0	1	2	3	4	5	6	7	Σ	Σ'
0	2								2	2
1	1	2							3	1
2	1	4	2						7	-1
3	1	6	9	2					18	-2
4	1	8	20	16	2				47	-1
5	1	10	35	50	25	2			123	1
6	1	12	54	112	105	36	2		322	2
7	1	14	77	210	294	196	49	2	843	1

coefficients of the Lucas Polynomials (OEIS, A034807).

We can build this table following the same rule as in the case of odd power: if $a_{r,c}$ *is the entry of the r–th row (for* $r = 0, 1, 2, \ldots$*) and the c–th column (for* $c = 0, 1, 2, \ldots, r$*), then*

1. $a_{0,0} = a_{1,1} = 2$
2. $a_{r,0} = 1$ *for all* $r \geq 1$
3. $a_{r,c} = 2a_{r-1,c-1} + a_{r-1,c} - a_{r-2,c-2}$ *: the same relation that for the odd k–Fibonacci numbers.*

All the diagonal sequences and the first four columns of this table are listed in OEIS.

The sequence $\{2, 3, 7, 18, 47, 123, 322, 843, \ldots\}$ *is the bisection of the classical Lucas sequence,* L_{2n}*. The terms of this sequence verify the same recurrence relation that in the case of the Table 1:* $b_{n+1} = 3b_n - b_{n-1}$ *but with the initial conditions* $b_0 = 2$ *and* $b_1 = 3$*.*

We can find these numbers by appling the following formula: $a_{0,0} = 2$*, for all* $r \geq 1$*,* $a_{r,0} = 1$*, and* $a_{r,c} = \frac{2r}{c}\binom{2r-c-1}{c-1}$*. Consequently,*

$$L_{k,2rn} = (k^2+4)^rF_{k,n}^{2r} + \sum_{c=1}^{r}(-1)^{nc}\frac{2r}{c}\binom{2r-c-1}{c-1}(k^2+4)^{r-c}F_{k,n}^{2(r-c)} \tag{8}$$

For the classical Fibonacci and Lucas sequences, it is $L_{2n} = 5F_n^2 + 2(-1)^n$

$$\begin{aligned}
L_{4n} &= 25F_n^4 + 20(-1)^nF_n^2 + 2\\
L_{6n} &= 125F_n^6 + 150(-1)^nF_n^4 + 45F_n^2 + 2(-1)^n\\
L_{8n} &= 625F_n^8 + 1000(-1)^nF_n^6 + 500F_n^4 + 80(-1)^nF_n^2 + 2
\end{aligned}$$

For instance, $L_8 = 625 - 1000 + 500 - 80 + 2 = 47$

As final conclussion of this part, we can merge the formulas 6 and 8, obtaining

$$(k^2+4)^{\lfloor\frac{r}{2}\rfloor}F_{k,n}^r + \sum_{c=1}^{\lfloor\frac{r}{2}\rfloor}(-1)^{nc}\frac{r}{c}\binom{r-c-1}{c-1}(k^2+4)^{\lfloor\frac{r}{2}\rfloor-c}F_{k,n}^{r-2c} =$$

- *$F_{k,rn}$ if r is odd*
- *$L_{k,rn}$ if r is even.*

In such case, the table of coefficients takes the form This table corresponds to the coefficients

Table 3: Coefficients of $F_{k,r\,n}$

r\ c	0	1	2	3	4	Σ
0	2					2
1	1					1
2	1	2				3
3	1	3				4
4	1	4	2			7
5	1	5	5			11
6	1	6	9	2		18
7	1	7	14	7		29
8	1	8	20	16	2	47

of the Lucas Polynomials (OEIS, A034807) and the last column is the sequence of the classical Lucas numbers.

The antidiagonal sums is the sequence $\{2,1,1,3,4,5,8,12,17,\ldots\} =$ A179070 which terms verify the recurrence relation $a_{n+1} = a_n + a_{n-2}$ for $n \geq 2$, with the initial conditions $a_0 = 2,\ a_1 = 1,\ a_2 = 1$

Theorem 6 **Sum of the consecutive even powers of the k–Fibonacci numbers**
The sum of the consecutive even powers of the k–Fibonacci numbers is given by

$$\sum_{j=0}^{n} F_{k,j}^{2r} = \frac{1}{(k^2+4)^r}\left(\sum_{j=0}^{r-1}(-1)^{(n+1)j}\binom{2r}{j}\frac{L_{k,(2r-2j)(n+1)} - (-1)^j L_{k,(2r-2j)n}}{L_{k,2r-2j} - 2(-1)^j}\right.$$
$$\left. + \binom{2r-1}{r-1}((-1)^{(n+1)r} + ((-1)^r + 1)n)\right)$$

It is enough to apply the formulas (3) and (4) to the formula (7).

For $r = 1,2,3$ it is

$$\sum_{j=0}^{n} F_{k,j}^2 = \frac{1}{k^2+4}\left(\frac{L_{k,2(n+1)} - L_{k,2n}}{L_{k,2} - 2} - (-1)^n\right)$$

$$\sum_{j=0}^{n} F_{k,j}^4 = \frac{1}{(k^2+4)^2}\left(\frac{L_{k,4(n+1)} - L_{k,4n}}{L_{k,4} - 2} - 4(-1)^n\frac{L_{k,2(n+1)} + L_{k,2n}}{L_{k,2} + 2} + 6n + 3\right)$$

$$\sum_{j=0}^{n} F_{k,j}^6 = \frac{1}{(k^2+4)^3}\left(\frac{L_{k,6(n+1)} - L_{k,6n}}{L_{k,6} - 2} - 6(-1)^n\frac{L_{k,4(n+1)} + L_{k,4n}}{L_{k,4} + 2} + \right.$$
$$\left. +15\frac{L_{k,2(n+1)} - L_{k,2n}}{L_{k,2} - 2} - 10(-1)^n\right)$$

Finally, for the classical Fibonacci numbers ($k = 1$), it is

$$\sum_{j=0}^{n} F_j^2 = \frac{1}{5}(L_{2n+1} - (-1)^n)$$

$$\sum_{j=0}^{n} F_j^4 = \frac{1}{25}\left(\frac{3L_{4n+1} + L_{4n}}{5} - 4(-1)^n\frac{L_{2n+1} + L_{2n}}{5} + 6n + 3\right)$$

$$\sum_{j=0}^{n} F_j^6 = \frac{1}{125}\left(\frac{2L_{6n+1} + L_{6n}}{4} - 2(-1)^n L_{4n+2} + 15L_{2n+1} - 10(-1)^n\right)$$

3.0.1 Conclusion

We have found a formula to calculate the sum of the powers of the k-Fibonacci numbers by means of a linear combination of k-Fibonacci numbers or of k-Lucas numbers, according to the power is odd or even, respectively.

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