Full Cycle Extendability of Nearly Claw-Free Graphs

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Abstract

We say that G is nearly claw-free if for every $v \in A$, the set of centers of claws of G, there exist two vertices $x, y \in N_G(v)$ such that $x, y \notin A$ and $N_G(v) \subseteq N_G(x) \cup N_G(y) \cup \{x, y\}$. A graph G is triangularly connected if for every pair of edges $e_1, e_2 \in E(G)$, G has a sequence of 3-cycles $C_1, C_2, \cdots, C_l$ such that $e_1 \in C_1, e_2 \in C_l$ and $E(C_i) \cap E(C_{i+1}) \neq \emptyset$ for $1 \leq i \leq l-1$. In this paper, we will show that (i) every triangularly connected $K_{1,4}$-free nearly claw-free graph on at least three vertices is fully cycle extendable if the clique number of the subgraph induced by the set of centers of claws of G is at most 2, and (ii) every 4-connected line graph of a nearly claw-free graph is hamiltonian connected.

Keywords: nearly claw-free graphs, triangularly connected graphs, fully cycle extendability, line graph, hamiltonian connected

1 Introduction

For terms not defined in this paper, we will use the notation and definitions of [1]. In addition, we will only consider finite graphs. The neighborhood of vertex v in G is denoted by $N_G(v)$ and the subgraph induced by $A \subseteq V(G)$ is denoted by $\langle A \rangle$. Denote $d_G(v) = |N_G(v)|$. A clique in a graph G is a set of pairwise adjacent vertices. The clique number $\omega(G)$ of a graph G is the order of a largest clique in G. A graph G is locally connected if for each $v \in V(G)$, the subgraph induced by $N_G(v)$ is connected. For an integer $k \geq 2$, a k-cycle C_k is a 2-regular connected graph with k edges.

If F is a graph, then we say that G is F-free if it does not contain an induced subgraph isomorphic to F. A $K_{1,3}$ is called a claw, and a $K_{1,3}$-

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free graph is called a claw-free graph. The vertex whose degree is r in $K_{1,r}(r \geq 3)$ is called the center of $K_{1,r}$.

Claw-free graphs have been a subject of interest of many authors in the recent years. It is also interesting to investigate classes of graphs containing claw-free graphs, and to generalize results on claw-free graphs to these superclasses. The classes of almost claw-free graphs and nearly claw-free graphs are two of these superclasses.

Definition 1.1 *(Ryjáček [9]) Let G be a graph and let A be the set of centers of claws of G. The graph G is called almost claw-free if A is independent, and for every vertex $v \in A$, there are two vertices $x, y \in N_G(v)$ such that $N_G(v) \subseteq N_G(x) \cup N_G(y) \cup \{x, y\}$.*

Definition 1.2 *Let G be a graph and let A be the set of centers of claws of G. The graph G is called nearly claw-free if for every vertex $v \in A$, there are two vertices $x, y \in N_G(v)$ such that $x, y \notin A$ and $N_G(v) \subseteq N_G(x) \cup N_G(y) \cup \{x, y\}$.*

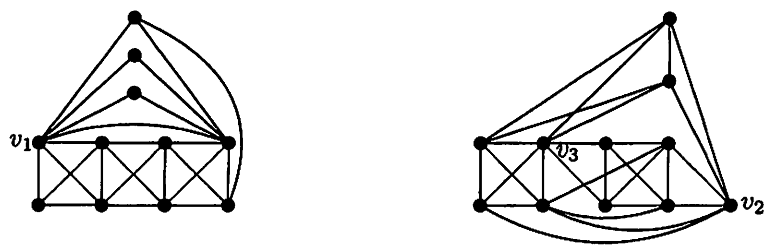
Such vertices x and y in Definitions 1.1 and 1.2 are called the dominating vertices on $N_G(v)$. Obviously, an almost claw-free graph is nearly claw-free, and a nearly claw-free graph is almost claw-free if A is independent, i.e., the clique number of the subgraph induced by A is 1. Our main goal in this paper is to extend some of the results for almost claw-free graphs to nearly claw-free graphs. In section 2, we will consider the fully cycle extendability of nearly claw-free graphs. The hamiltonicity of 4-connected line graphs of nearly claw-free graphs will be discussed in Section 3.

2 Fully cycle extendability of nearly claw-free graphs

The graphs considered are without isolated vertices. A graph G is pancyclic if for every integer k such that $3 \leq k \leq |V(G)|$, G has a k-cycle. G is vertex pancyclic if for each vertex $v \in V(G)$, and for each integer k with $3 \leq k \leq |V(G)|$, G has a k-cycle C_k such that $v \in V(C_k)$. G is said to be fully cycle extendable if every vertex of G lies on a triangle and for every nonhamiltonian cycle C there is a cycle C' in G such that $V(C) \subseteq V(C')$ and $|V(C')| = |V(C)| + 1$. In [8], Oberly and Summer proved that every connected, locally connected claw-free graph on at least three vertices is hamiltonian. Clark [2] proved that, under these conditions, G is vertex pancyclic. Later, Hendry observed that Clark essentially proved the following stronger result.

Theorem 2.1 *(Hendry [4]) If G is a connected, locally connected claw-free graph on at least three vertices, then G is fully cycle extendable.*

Theorem 2.2 *(Ryjáček [9]) Every connected, locally connected $K_{1,4}$-free almost claw-free graph on at least three vertices is fully cycle extendable.*



(A) (B)

Figure 1. Triangularly connected graphs

As a generalization of the concept of locally connected graphs, triangularly connected graphs were introduced in [10]. A graph G is triangularly connected if for every pair of edges $e_1, e_2 \in E(G)$, G has a sequence of 3-cycles $C_1, C_2, \cdots, C_l$ such that $e_1 \in C_1, e_2 \in C_l$ and $E(C_i) \cap E(C_{i+1}) \neq \emptyset$ for $1 \leq i \leq l-1$. Clearly, every connected, locally connected graph is triangularly connected. But not every triangularly connected graph is locally connected. The graphs in Figure 1 are triangularly connected graphs which are not locally connected since the subgraphs induced by the neighborhoods of v_1, v_2 and v_3 are not connected.

Theorem 2.3 *(Zhan [12]) Every triangularly connected $K_{1,4}$-free almost claw-free graph on at least three vertices is fully cycle extendable.*

Our goal here is to extend Theorems 2.1, 2.2, and 2.3 to triangularly connected nearly claw-free graphs.

Theorem 2.4 *Let G be a triangularly connected, $K_{1,4}$-free nearly claw-free graph on at least three vertices. If the clique number of the subgraph induced by the set of centers of claws of G is at most 2, then G is fully cycle extendable.*

Proof of Theorem 2.4. Since every vertex of G lies on a triangle, it is sufficient to prove that for every cycle C of length $3 \leq r \leq |V(G)|-1$ there is a cycle C' of length $r+1$ such that $V(C) \subset V(C')$. We will prove the theorem by contradiction. For every cycle $C \subset G$, one of its orientations is chosen. For every $v \in V(C)$, we denote by u^- and u^+ the predecessor and successor of u on C, respectively. Denote $u^{++} = (u^+)^+$ and $u^{--} = (u^-)^-$. For $u, v \in V(C)$, $C[u,v]$ and $\overleftarrow{C}[v,u]$ denote the path between u and v with the same or opposite orientation as the designated orientation of C. If $u = v$, then $C[u,v]$ and $\overleftarrow{C}[v,u]$ are denoted to be a single vertex. When vertices of a claw or $K_{1,4}$ are listed, the center is always listed as the first vertex. Recall that A is the set of all centers of claws in G.

Let $C = v_1 v_2 \cdots v_r v_1$, where $3 \leq r \leq |V(G)| - 1$, and $\mathcal{B}(C) = \{B | B \text{ is a 3-cycle, and } E(B) \cap E(C) \neq \emptyset\}$. Then $E(C) \subseteq \bigcup_{B \in \mathcal{B}(C)} E(B)$. If there is some $B \subseteq \mathcal{B}(C)$ such that $|V(B) \cap V(C)| = 2$, it is clear that the subgraph of G induced by the edge set $E(C) \cup E(B) - (E(C) \cap E(B))$ extends C. So we assume that for each $B \in \mathcal{B}(C)$, $V(B) \subseteq V(C)$.

Let $e \in E(G)$ such that e is incident with exactly one vertex in $V(C)$ and C_e be a 3-cycle with $e \in C_e$. Clearly, $C_e \notin \mathcal{B}(C)$. As G is triangularly connected, there is a sequence of 3-cycles $Z_0, Z_1, \cdots, Z_k$ such that $Z_0 = C_e$ and $Z_k \in \mathcal{B}(C)$. Let C, e, and C_e be chosen in such a way that,

(1) among all cycles with vertex set $V(C)$, the number, k, of 3-cycles in this sequence is smallest.

Therefore, $k \geq 1$ is the consequence of the definition of the edge e. Also, $|V(Z_0) \cap V(C)| = 2$ and $V(Z_i) \subseteq V(C)$ for $i \geq 1$. Assume that $Z_0 = \Delta u v_i v_j$ and $Z_1 = \Delta v_i v_j v_h$, where $v_h \in C[v_j^+, v_i^-]$ (see Figure 2). By Condition (1), $v_i v_j \notin E(Z_2)$ if $k \geq 2$. We choose C, e, and C_e so that

(2) subject to Condition (1), $|\{v_i, v_j\} \cap A|$ is as small as possible.

(3) subject to Conditions (1) and (2), $|\{v_j^+ v_j^-, v_i^+ v_i^-\} \cap E(G)|$ is as large as possible.

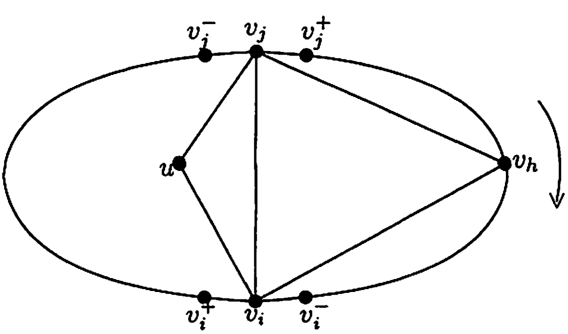


Figure 2.

(1.1) (i) If $k \geq 2$, then $v_i^+ v_h^+, v_i^- v_h^- \notin E(G)$.
(ii) If $k \geq 2$ and $v_i^+ v_i^- \in E(G)$, then $v_i v_h^+, v_i v_h^- \notin E(G)$. Therefore, if $k \geq 2$ and $\{v_h v_i^+, v_h v_i^-\} \cap E(G) \neq \emptyset$, then $v_h^+ v_h^- \notin E(G)$.
(iii) Assume $k \geq 2$. If $v_h \notin A$, then $v_h v_j^-, v_h v_i^+ \notin E(G)$.

Proof. (i) If $v_i^+ v_h^+ \in E(G)$, let $C' = v_i \overleftarrow{C}[v_h, v_i^+]\ C[v_h^+, v_i]$; if $v_i^- v_h^- \in E(G)$, let $C' = v_i C[v_h, v_i^-] \overleftarrow{C}[v_h^-, v_i]$. Then v_i and v_h are adjacent in C', and so the number k is one since $\Delta v_j v_i v_h \in \mathcal{B}(C')$. This contradicts Condition (1).

(ii) If $v_i v_h^- \in E(G)$, let $C' = v_i C[v_h, v_i^-] C[v_i^+, v_h^-] v_i$; if $v_i v_h^+ \in E(G)$, let $C' = v_i C[v_h^+, v_i^-]\ C[v_i^+, v_h] v_i$. Then v_i and v_h are adjacent in C', and so the number k is one since $\Delta v_j v_i v_h \in \mathcal{B}(C')$. This contradicts Condition (1) again.

(iii) Assume that $v_h v_j^- \in E(G)$. The proof for $v_h v_i^+ \notin E(G)$ is similar. By (1.1)(i), $v_j^- v_h^- \notin E(G)$. Since $k \geq 2$, $v_j^- v_i \notin E(G)$. As $\langle v_h, v_h^-, v_j^-, v_i \rangle \neq K_{1,3}$, we have $v_i v_h^- \in E(G)$. Thus $u v_h^- \notin E(G)$ (otherwise, let $Z_0 = \Delta u v_i v_h^-$ and $Z_1 = \Delta v_i v_h^- v_h$. Then $k = 1$). By (1.1)(ii), $v_i^+ v_i^- \notin E(G)$. By (1.1)(i), $v_h^- v_i^- \notin E(G)$. Since $\langle v_i, v_i^+, v_i^-, u, v_h^- \rangle \neq K_{1,4}$, we have $v_h^- v_i^+ \in E(G)$. Thus the cycle $C' = v_i u C[v_j, v_h^-] C[v_i^+, v_j^-] C[v_h, v_i]$ extends C, a contradiction. ∎

(1.2) Let $v_j \in A$. If $d \in N_G(v_j)$ dominates v_j^+, then the following statements hold.
(i) $d \in V(C)$.
(ii) $du \notin E(G)$.
(iii) $dv_j^- \in E(G)$. Therefore, $d \neq v_h$.
(iv) If $k \geq 2$, then $dv_h \in E(G)$.

Proof. (i) Obviously, $d \in V(C)$.

(ii) Assume $du \in E(G)$. Since $d \notin A$ and $ud^+, ud^- \notin E(G)$, we have $d^+ d^- \in E(G)$. Then the cycle C can be extended by $v_j u d C[v_j^+, d^-]\ C[d^+, v_j]$, a contradiction.

(iii) If $dv_j^- \notin E(G)$, then v_j^- and u would be dominated by d'. If $d' \notin V(C)$, then C would be extended by $v_j d' \overleftarrow{C}[v_j^-, v_j]$. If $d' \in V(C)$, then the same contradiction as (ii) occurs. Thus $dv_j^- \in E(G)$.

(iv) By contradiction, assume $dv_h \notin E(G)$. Then $d \notin \{v_h^+, v_h^-\}$ since v_h^+ and v_h^- are neighbors of v_h. As $k \geq 2$, $v_i v_j^+, v_i v_j^-, v_j v_i^+, v_j v_i^- \notin E(G)$. Thus $d \notin \{v_i^+, v_i, v_i^-\}$. Since $du \notin E(G)$, there exists $d_2 \in N_G(v_j)$ that dominates u and v_h. We consider two cases.

Case 1. $d_2 \notin V(C)$

Clearly, $v_j^- d_2, v_j^+ d_2 \notin E(G)$. Also, $d_2 v_i \notin E(G)$ (otherwise, let $Z_0' = \begin{cases} \Delta v_j d_2 v_h, & \text{if } v_j v_h \in E(Z_2) \\ \Delta v_i d_2 v_h, & \text{if } v_i v_h \in E(Z_2) \end{cases}$. Then the new chain $Z_0', Z_2, \cdots, Z_k$ would have one fewer triangles, a contradiction). Thus we have $dv_i \in E(G)$. Since $\langle d, v_i, v_j^+, v_j^- \rangle \neq K_{1,3}$, we have $v_j^+ v_j^- \in E(G)$. Consider the following cases:

Case	Cycle C'
$d^+d^- \in E(G)$	$v_j dC[v_j^+, d^-]C[d^+, v_j]$
$d^+v_j \in E(G)$	$v_j \overleftarrow{C}[d, v_j^+]\overleftarrow{C}[v_j^-, d^+]v_j$
$d^-v_j \in E(G)$	$v_j C[d, v_j^-]C[v_j^+, d^-]v_j$

In each case, $v_j d$ is an edge of the cycle C'. So we can replace the original chain with the new chain Z_0' and Z_1', where $Z_0' = Z_0, Z_1' = \Delta v_i v_j d$. Thus the number of triangles corresponding to C' is one, a contradiction. So $d^+d^-, d^+v_j, d^-v_j \notin E(G)$, and $\langle d, d^+, d^-, v_j \rangle = K_{1,3}$, a contradiction.

Case 2. $d_2 \in V(C)$

Since $d_2 \notin A$, then $d_2^+ d_2^- \in E(G)$.

Claim 1. $d_2 = v_i$.

By contradiction, we assume that $d_2 \neq v_i$. If $d_2 v_i \in E(G)$, then $v_i \notin A$(otherwise, let $Z_0' Z_1' = \begin{cases} \Delta u d_2 v_i \Delta v_i d_2 v_h, & \text{if } v_i v_h \in E(Z_2) \\ \Delta u d_2 v_j \Delta v_j d_2 v_h & \text{if } v_j v_h \in E(Z_2) \end{cases}$. Then the chain of $Z_0', Z_1', Z_2 \cdots, Z_k$ have same number of triangles, but $d_2 \notin A$. This contradicts Condition (2)). By considering $\langle v_i, v_i^+, u, v_h \rangle$, we can see that $v_i^+ v_h \in E(G)$. This means $k = 2$. Consider the new chain with $Z_0' = \Delta u d_2 v_i, Z_1' = \Delta d_2 v_h v_i, Z_2' = \Delta v_h v_i v_i^+$. This new chain still has three triangles but $d_2, v_i \notin A$, contrary to Condition (2). This implies $d_2 v_i \notin E(G)$. Therefore, $dv_i \in E(G)$. As $\langle d, v_i, v_j^+, v_j^- \rangle \neq K_{1,3}$ and $v_i v_j^+, v_i v_j^- \notin E(G)$, we have $v_j^+ v_j^- \in E(G)$. Consider the following cases:

Case	Cycle C'
$d^+d^- \in E(G)$	$v_j dC[v_j^+, d^-]C[d^+, v_j]$
$d^+v_j \in E(G)$	$v_j \overleftarrow{C}[d, v_j^+]\overleftarrow{C}[v_j^-, d^+]v_j$
$d^-v_j \in E(G)$	$v_j C[d, v_j^-]C[v_j^+, d^-]v_j$

In each case, $v_j d$ is an edge of the cycle C'. So we can replace the original chain with the new chain Z_0' and Z_1', where $Z_0' = Z_0, Z_1' = \Delta v_i v_j d$. Thus the number of triangles corresponding to C' is one, a contradiction. So $d^+d^-, d^+v_j, d^-v_j \notin E(G)$, and $\langle d, d^+, d^-, v_j \rangle = K_{1,3}$. This contradiction implies that $d_2 = v_i$.

By Claim 1, $d_2 = v_i$ and by the definition of a nearly claw-free graph, $v_i \notin A$. Thus $v_i^+ v_i^-, v_h v_i^+, v_h v_i^- \in E(G)$. By (1.1)(iii), $v_h \in A$. By (1.1)(ii), we have the following claim.

Claim 2. $v_h^+v_h^-, v_iv_h^-, v_iv_h^+ \notin E(G)$.

Claim 3. (i) $v_jv_h^+, v_jv_h^- \notin E(G)$.
(ii) $v_i^+v_h^- \in E(G), v_i^-v_h^+ \in E(G)$.
(iii) $v_hv_j^+, v_hv_j^- \notin E(G)$. Therefore, $v_j^+v_j^- \in E(G)$.

(i). We use contradiction to prove $v_jv_h^- \notin E(G)$. The discussion for $v_jv_h^+ \notin E(G)$ would be similar. Assume $v_jv_h^- \in E(G)$. By (1.1)(ii), $v_j^+v_j^- \notin E(G)$. Thus, $d \notin \{v_j^+, v_j^-\}$. Since d and v_i are the dominating vertices in $N_G(v_j)$ and $v_iv_h^- \notin E(G)$, $dv_h^- \in E(G)$. If $d \in C[v_i^+, v_j^-]$, then consider the following cases:

Case	Cycle C'
$d^+d^- \in E(G)$	$v_ju\overleftarrow{C}[v_i, v_h]C[v_i^+, d^-]C[d^+, v_j^-]d\overleftarrow{C}[v_h^-, v_j]$
$d^+v_h^- \in E(G)$	$v_ju\overleftarrow{C}[v_i, v_h]C[v_i^+, d]\overleftarrow{C}[v_j^-, d^+]\overleftarrow{C}[v_h^-, v_j]$
$d^-v_h^- \in E(G)$	$v_ju\overleftarrow{C}[v_i, v_h]C[v_i^+, d^-]\overleftarrow{C}[v_h^-, v_j^+]C[d, v_j]$

If $d \in C[v_h^+, v_i^-]$, then consider the following cases:

Case	Cycle C'
$d^+d^- \in E(G)$	$v_ju\overleftarrow{C}[v_i, d^+]\overleftarrow{C}[d^-, v_h]C[v_i^+, v_j^-]d\overleftarrow{C}[v_h^-, v_j]$
$d^+v_h^- \in E(G)$	$v_juv_iC[v_h, d]C[v_j^+, v_h^-]C[d^+, v_i^-]C[v_i^+, v_j]$
$d^-v_h^- \in E(G)$	$v_juv_iC[v_h, d^-]\overleftarrow{C}[v_h^-, v_j^+]C[d, v_i^-]C[v_i^+, v_j]$

If $d \in C[v_j^+, v_h^-]$, then consider the following cases:

Case	Cycle C'
$d^+d^- \in E(G)$	$v_ju\overleftarrow{C}[v_i, v_h]C[v_i^+, v_j^-]d\overleftarrow{C}[v_h^-, d^+]\overleftarrow{C}[d^-, v_j]$
$d^+v_j^- \in E(G)$	$v_ju\overleftarrow{C}[v_i, v_h]C[v_i^+, v_j^-]C[d^+, v_h^-]\overleftarrow{C}[d, v_j]$
$d^-v_j^- \in E(G)$	$v_ju\overleftarrow{C}[v_i, v_h]C[v_i^+, v_j^-]\overleftarrow{C}[d^-, v_j^+]C[d, v_h^-]v_j$

For these nine cases, C' extends C, a contradiction. Thus $d \in A$, a contradiction. So Claim 3(i) holds.

(ii). By (1.1)(i), $v_i^+v_h^+ \notin E(G)$. Since $\langle v_h, v_h^+, v_h^-, v_j, v_i^+\rangle \neq K_{1,4}$, $v_i^+v_h^- \in E(G)$. Similarly, $v_i^-v_h^- \notin E(G)$ and $v_i^-v_h^+ \in E(G)$. So Claim 3(ii) holds.

(iii). If $v_hv_j^+ \in E(G)$, then the cycle $C' = v_ju\overleftarrow{C}[v_i, v_h]C[v_j^+, v_h^-]C[v_i^+, v_j]$ extends C, a contradiction. So $v_hv_j^+ \notin E(G)$. Similarly, $v_hv_j^- \notin E(G)$. As $\langle v_j, v_j^+, v_j^-, u, v_h\rangle \neq K_{1,4}$, $v_j^+v_j^- \in E(G)$. So Claim 3(iii) holds.

Since $v_h \in A$, there is a vertex $d_3 \in N_G(v_h)$ dominating two of v_j, v_h^+, v_h^-.

Claim 4. d_3 does not dominate the vertices v_h^+ and v_h^-.

By contradiction. Assume d_3 dominates the vertices v_h^+ and v_h^-. Then $d_3 \in V(C)$. By Claims 2,3, and (1.1)(i), $d_3 \notin \{v_i, v_i^+, v_i^-, v_j, v_j^+, v_j^-, v_h^+, v_h^-\}$. If $d_3 \in C[v_h^+, v_i^-]$, the cycle

$$C' = \begin{cases} v_j u C[v_i, v_j^-] C[v_j^+, v_h^-] d_3 C[v_h^+, d_3^-] C[d_3^+, v_i^-] v_h v_j, & \text{if } d_3^+ d_3^- \in E(G) \\ v_j u C[v_i, v_j^-] C[v_j^+, v_h^-] C[d_3^+, v_i^-] C[v_h^+, d_3] v_h v_j & \text{if } v_h^- d_3^+ \in E(G) \\ v_j u C[v_i, v_j^-] C[v_j^+, v_h^-] \overleftarrow{C}[d_3^-, v_h^+] C[d_3, v_i^-] v_h v_j, & \text{if } v_h^- d_3^- \in E(G) \end{cases}$$

would extend C. So $\langle d_3, d_3^+, d_3^-, v_h^- \rangle = K_{1,3}$, a contradiction. If $d_3 \in C[v_j^+, v_h^-]$, the cycle

$$C' = \begin{cases} v_j u C[v_i, v_j^-] C[v_j^+, d_3^-] C[d_3^+, v_h^-] d_3 C[v_h^+, v_i^-] v_h v_j, & \text{if } d_3^+ d_3^- \in E(G) \\ v_j u \overleftarrow{C}[v_i, v_h^+] C[d_3^+, v_h^-] C[v_i^+, v_j^-] C[v_j^+, d_3] v_h v_j & \text{if } v_h^+ d_3^+ \in E(G) \\ v_j u \overleftarrow{C}[v_i, v_h^+] \overleftarrow{C}[d_3^-, v_j^+] \overleftarrow{C}[v_j^-, v_i^+] \overleftarrow{C}[v_h^-, d_3] v_h v_j, & \text{if } v_h^+ d_3^- \in E(G) \end{cases}$$

would extend C. So $\langle d_3, d_3^+, d_3^-, v_h^+ \rangle = K_{1,3}$, a contradiction. If $d_3 \in C[v_i^+, v_j^-]$, the cycle

$$C' = \begin{cases} v_j u C[v_i, d_3^-] C[d_3^+, v_j^-] C[v_j^+, v_h^-] d_3 C[v_h^+, v_i^-] v_h v_j, & \text{if } d_3^+ d_3^- \in E(G) \\ v_j u \overleftarrow{C}[v_i, v_h^+] \overleftarrow{C}[d_3, v_i^+] \overleftarrow{C}[v_h^-, v_j^+] \overleftarrow{C}[v_j^-, d_3^+] v_h v_j & \text{if } v_h d_3^+ \in E(G) \\ v_j u \overleftarrow{C}[v_i, v_h^+] C[d_3, v_j^-] C[v_j^+, v_h^-] C[v_i^+, d_3^-] v_h v_j, & \text{if } v_h d_3^- \in E(G) \end{cases}$$

would extend C. Thus $\langle d_3, d_3^+, d_3^-, v_h \rangle = K_{1,3}$, a contradiction.

Claim 5. d_3 does not dominate the vertices v_j and v_h^+.

Assume that d_3 dominates the vertices v_j and v_h^+. Then $d_3 \in V(C)$. By Claims 2,3, $d_3 \notin \{v_i, v_j, v_j^+, v_j^-, v_h^+, v_h^-\}$. Since $k \geq 2$, $d_3 \notin \{v_i^+, v_i^-\}$. If $d_3 \in C[v_h^+, v_i^-]$, then the cycle

$$C' = \begin{cases} v_j u C[v_i, v_j^-] C[v_j^+, v_h] \overleftarrow{C}[d_3, v_h^+] \overleftarrow{C}[v_i^-, d_3^+] v_j, & \text{if } v_j d_3^+ \in E(G) \\ v_j u \overleftarrow{C}[v_h, v_j^+] \overleftarrow{C}[v_j^-, v_i^+] \overleftarrow{C}[v_i^-, d_3^+] C[v_h^+, d_3] v_j, & \text{if } v_h^+ d_3^+ \in E(G) \end{cases}$$

would extend C. Thus $\langle d_3, v_h^+, v_j, d_3^+ \rangle = K_{1,3}$, a contradiction. If $d_3 \in C[v_j^+, v_h^-]$, then the cycle

$$C' = \begin{cases} v_j u \overleftarrow{C}[v_h, d_3^+] C[v_h^+, v_i^-] C[v_i^+, v_j^-] C[v_j^+, d_3] v_j, & \text{if } v_h^+ d_3^+ \in E(G) \\ v_j u \overleftarrow{C}[v_i, v_h^+] \overleftarrow{C}[d_3, v_j^+] \overleftarrow{C}[v_j^-, v_i^+] \overleftarrow{C}[v_h, d_3^+] v_j, & \text{if } v_j d_3^+ \in E(G) \end{cases}$$

would extend C. Thus $\langle d_3, v_h^+, v_j, d_3^+ \rangle = K_{1,3}$, a contradiction. If $d_3 \in C[v_i^+, v_j^-]$, then the cycle

$$C' = \begin{cases} v_j u \overleftarrow{C}[v_i, v_h^+] C[d_3^+, v_j^-] C[v_j^+, v_h] C[v_i^+, d_3] v_j, & \text{if } v_h^+ d_3^+ \in E(G) \\ v_j u \overleftarrow{C}[v_i, v_h^+] \overleftarrow{C}[d_3, v_i^+] \overleftarrow{C}[v_h, v_j^+] \overleftarrow{C}[v_j^-, d_3^+] v_j, & \text{if } v_j d_3^+ \in E(G) \end{cases}$$

would extend C. Thus $\langle d_3, v_h^+, v_j, d_3^+ \rangle = K_{1,3}$, a contradiction.

By Claims 4 and 5, d_3 dominates the vertices v_j and v_h^-. Then $d_3 \in V(C)$. By Claims 2,3, $d_3 \notin \{v_h^+, v_h^-, v_i, v_j, v_j^+, v_j^-\}$. Since $k \geq 2$, $d_3 \notin \{v_i^+, v_i^-\}$. If $d_3 \in C[v_h^+, v_i^-]$, then the cycle

$$C' = \begin{cases} v_j u \overleftarrow{C}[v_i, d_3] \overleftarrow{C}[v_h^-, v_j^+] \overleftarrow{C}[v_j^-, v_i^+] C[v_h, d_3^-] v_j, & \text{if } v_j d_3^- \in E(G) \\ v_j u v_i C[v_h, d_3^-] \overleftarrow{C}[v_h^-, v_j^+] \overleftarrow{C}[v_j^-, v_i^+] \overleftarrow{C}[v_i^-, d_3] v_j, & \text{if } v_h^- d_3^- \in E(G) \end{cases}$$

would extend C. Thus $\langle d_3, v_h^-, v_j, d_3^- \rangle = K_{1,3}$, a contradiction. If $d_3 \in C[v_j^+, v_h^-]$, then the cycle

$$C' = \begin{cases} v_j u \overleftarrow{C}[v_i, v_h] C[v_i^+, v_j^-] C[v_j^+, d_3^-] \overleftarrow{C}[v_h^-, d_3] v_j, & \text{if } v_h^- d_3^- \in E(G) \\ v_j u \overleftarrow{C}[v_i, v_h] C[d_3, v_h^-] C[v_i^+, v_j^-] C[v_j^+, d_3^-] v_j, & \text{if } v_j d_3^- \in E(G) \end{cases}$$

would extend C. Thus $\langle d_3, v_h^-, v_j, d_3^- \rangle = K_{1,3}$, a contradiction. If $d_3 \in C[v_i^+, v_j^-]$, then the cycle

$$C' = \begin{cases} v_j u \overleftarrow{C}[v_i, v_h] C[v_i^+, d_3^-] \overleftarrow{C}[v_h^-, v_j^+] \overleftarrow{C}[v_j^-, d_3] v_j, & \text{if } v_h^- d_3^- \in E(G) \\ v_j u \overleftarrow{C}[v_i, v_h] C[d_3, v_j^-] C[v_j^+, v_h^-] C[v_i^+, d_3^-] v_j, & \text{if } v_j d_3^- \in E(G) \end{cases}$$

would extend C. Thus $\langle d_3, v_h^-, v_j, d_3^- \rangle = K_{1,3}$, a contradiction. This finishes the proof of (1.2). ∎

(1.3) $k \geq 2$. Therefore, $v_i v_j^+, v_i v_j^-, v_j v_i^+, v_j v_i^- \notin E(G)$.

Proof. By contradiction, assume that $k = 1$. Then we have $v_h \in \{v_j^+, v_j^-, v_i^+, v_i^-\}$. Without loss of generality, let's assume that $v_i v_j^+ \in E(G)$. Then $v_i^+ v_i^- \notin E(G)$ because otherwise, the cycle $v_j u v_i C[v_j^+, v_i^-] C[v_i^+, v_j]$ would extend C, a contradiction. Thus, we have $\langle v_i, u, v_i^+, v_i^- \rangle = K_{1,3}$, and $v_i \in A$. Let $d_1 \in N_G(v_i)$ dominate v_i^-. Then, by (1.2), $d_1 \in V(C), d_1 v_i^+ \in E(G)$, and $d_1 u \notin E(G)$. Furthermore, we have $d_1 \neq \{v_i^+, v_i^-\}$ because $v_i^+ v_i^- \notin E(G)$.

Suppose $d_1 = v_j$. Then $v_j v_i^+, v_j v_i^- \in E(G)$. Since $v_j \notin A$, $v_j^+ v_j^- \in E(G)$. Then the cycle $v_j u \overleftarrow{C}[v_i, v_j^+] \overleftarrow{C}[v_j^-, v_i^+] v_j$ extends C, a contradiction. So $d_1 \neq v_j$. Also, we have $v_i^+ v_j^+ \notin E(G)$ otherwise cycle $v_j u \overleftarrow{C}[v_i, v_j^+] C[v_i^+, v_j]$ extends C. Similarly, $v_i^- v_j^- \notin E(G)$. So $d_1 \notin \{v_j^-, v_j^+\}$.

We consider the following two cases.

Case 1. d_1 dominates v_j^+

If $d_1 \in C[v_i^+, v_j^-]$, then consider the following cases:

Case	Cycle C'
$d_1^+v_j^+ \in E(G)$	$v_juC[v_i, d_1]\overleftarrow{C}[v_i^-, v_j^+]C[d_1^+, v_j]$
$d_1^+v_i^+ \in E(G)$	$v_ju\overleftarrow{C}[v_i, v_j^+]\overleftarrow{C}[d_1, v_i^+]C[d_1^+, v_j]$

If $d_1 \in C[v_j^+, v_i^-]$, then consider the following cases:

Case	Cycle C'
$d_1^+v_j^+ \in E(G)$	$v_ju\overleftarrow{C}[v_i, d_1^+]C[v_j^+, d_1]C[v_i^+, v_j]$
$d_1^+v_i^+ \in E(G)$	$v_juv_iC[v_j^+, d_1]\overleftarrow{C}[v_i^-, d_1^+]C[v_i^+, v_j]$

In each of four cases, C' extends C. Thus, $d_1^+v_j^+, d_1^+v_i^+ \notin E(G)$. Since $v_i^+v_j^+ \notin E(G)$, we have $\langle d_1, v_j^+, v_i^+, d_1^+\rangle = K_{1,3}$, a contradiction.

Case 2. d_1 does not dominate v_j^+.

Then there exists a $d_2 \in N_G(v_i)$ that dominates v_j^+ and u. We consider two subcases.

Case 2.1. $d_2 \notin V(C)$.

Obviously, $v_jd_2 \notin E(G)$, so we have $v_jd_1 \in E(G)$. If $d_1 \in C[v_i^{++}, v_j^-]$, then consider the following cases:

Case	Cycle C'
$d_1^+v_i^+ \in E(G)$	$v_juv_iC[v_j^+, v_i^-]\overleftarrow{C}[d_1, v_i^+]C[d_1^+, v_j]$
$d_1^+v_i^- \in E(G)$	$\overleftarrow{C}[v_j, d_1^+]\overleftarrow{C}[v_i^-, v_j^+]d_2C[v_i, d_1]v_j$

In each of case, C' extends C. Thus $d_1^+v_i^+, d_1^+v_i^- \notin E(G)$. Since it was previously proven that $v_i^+v_i^- \notin E(G)$, we have $\langle d_1, d_1^+, v_i^+, v_i^-\rangle = K_{1,3}$, a contradiction.

If $d_1 \in C[v_j^{++}, v_i^{--}]$, then consider the following cases:

Case	Cycle C'
$d_1^+d_1^- \in E(G)$	$v_jd_1\overleftarrow{C}[v_i^-, d_1^+]\overleftarrow{C}[d_1^-, v_j^+]d_2C[v_i, v_j]$
$v_jd_1^- \in E(G)$	$\overleftarrow{C}[v_j, v_i^+]C[d_1, v_i]d_2C[v_j^+, d_1^-]v_j$
$v_jd_1^+ \in E(G)$	$\overleftarrow{C}[v_j, v_i]d_2C[v_j^+, d_1]\overleftarrow{C}[v_i^-, d_1^+]v_j$

In all cases, the cycles C' extend C, so $d_1^+d_1^-, v_jd_1^-, v_jd_1^+ \notin E(G)$ and $\langle d_1, d_1^+, d_1^-, v_j\rangle = K_{1,3}$, a contradiction.

Case 2.2. $d_2 \in V(C)$.

If $d_2 \neq v_j$, then $d_2^+d_2^- \in E(G)$. Thus the cycle $v_jud_2C[v_j^+, d_2^-]C[d_2^+, v_j]$ extends C, a contradiction. So, $d_2 = v_j$ and $v_j^+v_j^- \in E(G)$. Consider $\langle v_i, u, v_j^+, v_i^+, v_i^-\rangle$. Obviously, $uv_i^-, uv_i^+, uv_j^+ \notin E(G)$. Since $v_i^+v_i^-, v_i^+v_j^+ \notin E(G)$, we have $v_i^-v_j^+ \in E(G)$.

If $d_1 \in C[v_j^{++}, v_i^{--}]$, then we consider the following cases.

Case	Cycle C'
$d_1^+d_1^- \in E(G)$	$v_juv_iC[v_j^+, d_1^-]C[d_1^+, v_i^-]d_1C[v_i^+, v_j]$
$v_id_1^- \in E(G)$	$v_juv_i\overleftarrow{C}[d_1^-, v_j^+]\overleftarrow{C}[v_i^-, d_1]C[v_i^+, v_j]$
$v_id_1^+ \in E(G)$	$v_juv_iC[d_1^+, v_i^-]C[v_j^+, d_1]C[v_i^+, v_j]$

In each case, the cycle C' extends C. So $d_1^+d_1^-, v_id_1^+, v_id_1^- \notin E(G)$. Thus $\langle d_1, d_1^+, d_1^-, v_i\rangle = K_{1,3}$, a contradiction. So $d_1 \in C[v_i^{++}, v_j^{--}]$. If $v_iv_j^- \in E(G)$, then $v_i^+v_j^- \in E(G)$ and $d_1^+d_1^-, v_id_1^+, v_id_1^- \notin E(G)$ using the arguments above. Thus $\langle d_1, d_1^+, d_1^-, v_i\rangle = K_{1,3}$, a contradiction. So $v_iv_j^- \notin E(G)$. Consider the following cases:

Case	Cycle C'
$d_1^+d_1^- \in E(G)$	$v_juv_iC[v_j^+, v_i^-]d_1C[v_i^+, d_1^-]C[d_1^+, v_j]$
$d_1^-v_i^- \in E(G)$	$v_juv_iC[v_j^+, v_i^-]\overleftarrow{C}[d_1^-, v_i^+]C[d_1, v_j]$

In each of the two cases above, the cycle C' extends C, a contradiction, so $d_1^+d_1^-, d_1^-v_i^- \notin E(G)$. Consider $\langle d_1, d_1^+, d_1^-, v_i\rangle$. Since $d_1 \notin A$, we have $d_1^+v_i^- \in E(G)$.

We claim that $v_iv_j^{++} \notin E(G)$. By contradiction, assume that $v_iv_j^{++} \in E(G)$. Then either d_1 or $v_j(= d_2)$ dominates v_j^{++}. For each of the following cases,

Case	Cycle C'
$v_jv_j^{++} \in E(G)$	$v_juC[v_i, v_j^-]v_j^+\overleftarrow{C}[v_i^-, v_j^{++}]v_j$
$d_1v_j^{++} \in E(G)$	$v_juC[v_i, d_1]C[v_j^{++}, v_i^-]C[d_1^+, v_j^-]\overleftarrow{C}[v_j^+, v_j]$

C' extends C in each case, a contradiction, so we have $v_iv_j^{++} \notin E(G)$. We also have $v_j^-v_j^{++} \notin E(G)$, otherwise the cycle $v_juC[v_i, v_j^-]C[v_j^{++}, v_i^-]v_j^+v_j$

would extend C. Since $v_iv_j^{++}, v_j^-v_j^{++}, v_iv_j^- \notin E(G)$, we have $\langle v_j^+, v_j^{++}, v_j^-, v_i\rangle = K_{1,3}$. Thus $v_j^+ \in A$. By the hypothesis that the clique number of the subgraph induced by the set of centers of claws of G is at most 2, $v_i^- \notin A$.

We claim that $v_iv_i^{--} \notin E(G)$. By contradiction. Suppose $v_iv_i^{--} \in E(G)$. Then either d_1 or $v_j(= d_2)$ would dominate v_i^{--}. Consider the following cases:

Case	Cycle C'
$v_jv_i^{--} \in E(G)$	$v_juC[v_i, d_1]v_i^-C[d_1^+, v_j^-]C[v_j^+, v_i^{--}]v_j$
$d_1v_i^{--} \in E(G)$	$\overleftarrow{C}[v_j, d_1^+]v_i^-C[v_j^+, v_i^{--}]\overleftarrow{C}[d_1, v_i]uv_j$

In both cases, the cycles C' extend C, a contradiction. Thus $v_iv_i^{--} \notin E(G)$. We also have $d_1^+v_i^{--} \notin E(G)$ otherwise the cycle $v_juC[v_i, d_1]v_i^-C[v_j^+, v_i^{--}]$ $C[d_1^+, v_j]$ would extend C, a contradiction. Thus by considering $\langle v_i^-, v_i, v_i^{--}, d_1^+\rangle$, we see that $v_id_1^+ \in E(G)$.

Consider $\langle v_i, v_i^+, u, d_1^+, v_j^+\rangle$. It is obvious that $uv_i^+, uv_j^+ \notin E(G)$ and it was previously proven that $v_i^+v_j^+ \notin E(G)$. Consider the following cases:

Case	Cycle C'
$d_1^+u \in E(G)$	$d_1^+uC[v_i, d_1]\overleftarrow{C}[v_i, d_1^+]$
$d_1^+v_j^+ \in E(G)$	$v_juC[v_i, d_1]\overleftarrow{C}[v_i^-, v_j^+]C[d_1^+, v_j]$
$d_1^+v_i^+ \in E(G)$	$v_juv_iC[v_j^+, v_i^-]\overleftarrow{C}[d_1, v_i^+]C[d_1^+, v_j]$

In each case, C' extends C, a contradiction. Thus, we have $d_1^+u, d_1^+v_j^+, d_1^+v_i^+ \notin E(G)$ and so $\langle v_i, v_i^+, u, d_1^+, v_j^+\rangle = K_{1,4}$, a contradiction. ∎

(1.4) $k = 2$.

Proof. By contradiction, assume $k \geq 3$. Then, $v_hv_j^-, v_hv_j^+, v_hv_i^-, v_hv_i^+ \notin E(G)$. Thus, we have $\langle v_j, v_j^+, u, v_h\rangle = K_{1,3}$ and $\langle v_i, v_i^-, u, v_h\rangle = K_{1,3}$, which means $v_i, v_j \in A$. Without loss of generality, in the chain of 3-cycles, $Z_0, Z_1, Z_2 \cdots Z_k$, we assume $v_hv_j \in E(Z_2)$. Let $d \in N_G(v_j)$ dominate v_j^+. By (1.2), $d \in V(C)$, and $dv_j^-, dv_h \in E(G)$. Since $v_hv_j^+, v_hv_j^- \notin E(G)$ and $d \notin A$, we have $v_j^+v_j^- \in E(G)$. Consider the following cases:

Case	Cycle C'
$d^+d^- \in E(G)$	$v_jdC[v_j^+, d^-]C[d^+, v_j]$
$v_jd^- \in E(G)$	$v_jC[d, v_j^-]C[v_j^+, d^-]v_j$
$v_jd^+ \in E(G)$	$v_j\overleftarrow{C}[d, v_j^+]\overleftarrow{C}[v_j^-, d^+]v_j$

In each case, v_jd is an edge of the cycle C'. Then, we can replace the original chain with the new chain Z_0, Z_1, Z_2' where $Z_2' = \Delta v_j v_h d$. Thus $k = 2$, contrary to Condition (1). So, we have $d^+d^-, v_jd^-, v_jd^+ \notin E(G)$. Resultantly, $\langle d, d^+, d^-, v_j \rangle = K_{1,3}$, a contradiction. ∎

(1.5) If $v_j^+v_j^- \notin E(G)$, and d dominates v_j^+, then the following statements hold.
(i) $d^+d^-, v_j^+d^+, v_j^-d^- \notin E(G)$. Therefore, $v_j^-d^+, v_j^+d^- \in E(G)$.
(ii) $d \neq v_h^+, v_h^-$.
(iii) $dv_h^+, dv_h^- \notin E(G)$.

Proof. If $v_j^+v_j^- \notin E(G)$, then $v_j \in A$. By (1.2), we have $d \in V(C)$ and $dv_j^-, dv_h \in E(G)$.

(i) Consider the following cases:

Case	Cycle C'
$d^+d^- \in E(G)$	$v_jdC[v_j^+, d^-]C[d^+, v_j]$
$v_j^+d^+ \in E(G)$	$v_j\overleftarrow{C}[d, v_j^+]C[d^+, v_j]$
$v_j^-d^- \in E(G)$	$v_jC[d, v_j^-]\overleftarrow{C}[d^-, v_j]$

In all of the cases, v_jd is an edge of the cycle C'. So in all of the three cycles above, we can replace the original chain by Z_0, Z_1, Z_2' where $Z_2' = \Delta v_j v_h d$, so Conditions (1) and (2) are not violated, but in C', either d and v_j^+ are adjacent or d and v_j^- are adjacent, contrary to Condition (3). Thus $d^+d^-, v_j^+d^+, v_j^-d^- \notin E(G)$. As $\langle d, d^-, d^+, v_j^+ \rangle \neq K_{1,3}$ and $\langle d, d^-, d^+, v_j^- \rangle \neq K_{1,3}$, we have $v_j^-d^+, v_j^+d^- \in E(G)$.

(ii) It follows by (1.1)(i).

(iii) By contradiction. If $dv_h^+ \in E(G)$, then consider the following cases:

Case	Cycle C_1
$d \in C[v_j, v_h]$	$v_j\overleftarrow{C}[v_h, d^+]\overleftarrow{C}[v_j^-, v_h^+]\overleftarrow{C}[d, v_j]$
$d \in C[v_h, v_j]$	$v_j\overleftarrow{C}[v_h, v_j^+]\overleftarrow{C}[d^-, v_h^+]C[d, v_j]$

If $dv_h^- \in E(G)$, then consider the following cases:

Case	Cycle C_1
$d \in C[v_j, v_h]$	$v_jC[v_h, v_j^-]C[d^+, v_h^-]\overleftarrow{C}[d, v_j]$
$d \in C[v_h, v_j]$	$v_jC[v_h, d^-]C[v_j^+, v_h^-]C[d, v_j]$

For all four cases, v_j and v_h are adjacent, so we can shorten the original chain by Z_0, Z_1, contrary to Condition (1). Thus we have $dv_h^+, dv_h^- \notin E(G)$ ∎

(1.6) Either $v_j^+v_j^- \in E(G)$ or $v_i^+v_i^- \in E(G)$.

Proof. By contradiction, assume $v_i^+v_i^-, v_j^+v_j^- \notin E(G)$. Then, $\langle v_i, v_i^+, v_i^-, u\rangle = K_{1,3}$, $\langle v_j, v_j^+, v_j^-, u\rangle = K_{1,3}$, and $v_i, v_j \in A$. Let $d_1 \in N_G(v_j)$ dominate v_j^+. Since the clique number of the subgraph induced by the set of centers of claws of G is at most 2, we have $v_h \notin A$. By (1.1)(iii), $v_hv_j^- \notin E(G)$. As $\langle v_i, v_j^+, v_j^-, u, v_h\rangle \neq K_{1,4}$, $v_hv_j^+ \in E(G)$. Considering $\langle v_h, v_h^+, v_h^-, d_1\rangle$, by (1.5)(iii), we have $v_h^+v_h^- \in E(G)$. Thus v_j and v_h are adjacent in the new cycle $C' = v_jv_hC[v_j^+, v_h^-]C[v_h^+, v_j]$, contrary to Condition 1. ∎

By (1.6), we assume that $v_j^+v_j^- \in E(G)$. By (1.2)(ii), we have $v_jv_h^+$, $v_jv_h^- \notin E(G)$.

(1.7) $v_i^+v_i^- \in E(G)$.

Proof. Assume that $v_i^+v_i^- \notin E(G)$. Thus $v_i \in A$ and there exists a $d_2 \in N_G(v_i)$ that dominates v_i^+. By (1.2), $d_2v_i^-, d_2v_h \in E(G)$.

We claim that $d_2v_j \notin E(G)$. Suppose that $d_2v_j \in E(G)$. By symmetry, we assume that $d_2 \in C[v_i, v_j]$. We consider three cases:

Case	Cycle C'
$d_2^-d_2^+ \in E(G)$	$v_ju\overleftarrow{C}[v_i, v_j^+]\overleftarrow{C}[v_j^-, d_2^+]\overleftarrow{C}[d_2^-, v_i^+]d_2v_j$
$v_jd_2^+ \in E(G)$	$v_juC[v_i, d_2]\overleftarrow{C}[v_i^-, v_j^+]\overleftarrow{C}[v_j^-, d_2^+]v_j$
$v_jd_2^- \in E(G)$	$v_ju\overleftarrow{C}[v_i, v_j^+]\overleftarrow{C}[v_j^-, d_2]C[v_i^+, d_2^-]v_j$

In each case, the cycles C' extend C. Thus $d_2^-d_2^+, v_jd_2^+, v_jd_2^- \notin E(G)$. So, $\langle d_2, d_2^+, d_2^-, v_j\rangle = K_{1,3}$, a contradiction. This means $d_2v_j \notin E(G)$.

As $\langle v_h, v_h^+, v_h^-, d_2, v_j\rangle \neq K_{1,4}$, by (1.6)(iii), $v_h^+v_h^- \in E(G)$. By (1.2)(ii), $v_hv_i^+, v_hv_i^- \notin E(G)$. Thus $\langle v_i, v_i^+, v_i^-, u, v_h\rangle = K_{1,4}$, a contradiction. ∎

(1.8) (i). $\{v_iv_h^+, v_iv_h^-, v_jv_h^+, v_jv_h^-\} \cap E(G) = \emptyset$.
(ii). $\{v_hv_i^+, v_hv_i^-, v_hv_j^-, v_hv_j^+\} \cap E(G) \neq \emptyset$, $v_h^+v_h^- \notin E(G)$, and $v_h \in A$.

Proof. (i) It follows by (1.6),(1.7), and (1.2)(ii).
(ii) By (1.4), $\{v_hv_j^+, v_hv_j^-, v_hv_i^+, v_hv_i^-\} \cap E(G) \neq \emptyset$. By (1.2)(ii), $v_h^+v_h^- \notin E(G)$. Thus $\langle v_h, v_h^+, v_h^-, v_i\rangle = K_{1,3}$, and so $v_h \in A$. ∎

(1.9) Either $v_i \in A$ or $v_j \in A$.

Proof. Assume that both v_i and $v_j \notin A$. By considering $\langle v_j, v_j^+, u, v_h\rangle$, we have $v_j^+v_h \in E(G)$. Similarly, $v_j^-v_h, v_i^+v_h, v_i^-v_h \in E(G)$.

We claim that $v_j^-v_h^+$ and $v_j^-v_h^- \notin E(G)$. If $v_j^-v_h^- \in E(G)$, let $C' = v_ju\overleftarrow{C}[v_i, v_h]C[v_i^+, v_j^-]\ \overleftarrow{C}[v_h^-, v_j]$; if $v_j^-v_h^+ \in E(G)$, let $C' = v_ju\overleftarrow{C}[v_i, v_h^+]$

$\overleftarrow{C}[v_j^-, v_i^+]\overleftarrow{C}[v_h, v_j]$. Then C' extends C, a contradiction. So $v_j^- v_h^+, v_j^- v_h^- \notin E(G)$.

By (1.4), $v_i v_j^- \notin E(G)$. By (1.8), $\langle v_h, v_h^+, v_h^-, v_j^-, v_i\rangle = K_{1,4}$, a contradiction. ▮

By (1.9), we may assume that $v_j \in A$.

(1.10). The following statements hold.
(i) If $v_i^+ v_h \in E(G)$, then , $v_i^+ v_h^- \in E(G)$.
(ii) If $v_i^- v_h \in E(G)$, then $v_i^- v_h^+ \in E(G)$.

Proof. Assume $v_i^+ v_h \in E(G)$. By (1.1)(i), $v_i^+ v_h^+ \notin E(G)$. Consider $\langle v_h, v_h^-, v_h^+, v_i^+, v_j\rangle$. By (1.4), $v_j v_i^+ \notin E(G)$. By (1.5), $v_h^+ v_h^- \notin E(G)$. By (1.8), $v_j v_h^+, v_j v_h^- \notin E(G)$. Thus $v_i^+ v_h^- \in E(G)$. So (i) holds. Similarly, (ii) also holds. ▮

(1.11) $v_i \in A$.

Proof. Suppose $v_i \notin A$. Then $v_i^+ v_h, v_i^- v_h \in E(G)$. By (1.10), $v_i^+ v_h^-, v_i^- v_h^+ \in E(G)$. Since $v_j \in A$, there exists a $d \in N_G(v_j)$ that dominates v_j^+ and thus dominates v_j^- and v_h by (1.2). Consider the following cases.

Case 1. $d \in C[v_i^+, v_j^-]$.

By (1.4), $d \neq v_i^+$. Also, $d \neq v_j^-$, otherwise the cycle $v_j u\overleftarrow{C}[v_i, v_h]\overleftarrow{C}[d, v_i^+]$ $\overleftarrow{C}[v_h^-, v_j]$ extends C, a contradiction. Consider the following subcases:

Case	Cycle C'
$d^- v_h \in E(G)$	$v_j u\overleftarrow{C}[v_i, v_h]\overleftarrow{C}[d^-, v_i^+]\overleftarrow{C}[v_h^-, v_j^+]C[d, v_j]$
$d^+ v_h \in E(G)$	$v_j u\overleftarrow{C}[v_i, v_h]C[d^+, v_j^-]C[v_j^+, v_h^-]C[v_i^+, d]v_j$
$d^- d^+ \in E(G)$	$v_j u\overleftarrow{C}[v_i, v_h]d\overleftarrow{C}[v_j^-, d^+]\overleftarrow{C}[d^-, v_i^+]\overleftarrow{C}[v_h^-, v_j]$

C' extends C in each case. Thus $d^- v_h, d^+ v_h, d^- d^+ \notin E(G)$. So, $\langle d, v_h, d^-, d^+\rangle$ $= K_{1,3}$, a contradiction.

Case 2. $d \in C[v_j^+, v_h^-]$.

By (1.8)(i), $v_j v_h^- \notin E(G)$, so $d \neq v_h^-$. Also, $d \neq v_j^+$. Otherwise, the cycle $C = v_j u\overleftarrow{C}[v_i, v_h]C[d, v_h^-]C[v_i^+, v_i]$ extends C. Consider the following subcases:

Case	Cycle C'
$d^- v_h \in E(G)$	$v_j u\overleftarrow{C}[v_i, v_h]\overleftarrow{C}[d^-, v_j^+]C[d, v_h^-]C[v_i^+, v_j]$
$d^+ v_h \in E(G)$	$v_j u\overleftarrow{C}[v_i, v_h]C[d^+, v_h^-]C[v_i^+, v_j^-]\overleftarrow{C}[d, v_j]$
$d^- d^+ \in E(G)$	$v_j u\overleftarrow{C}[v_i, v_h]dC[v_j^+, d^-]C[d^+, v_h^-]C[v_i^+, v_j]$

C' extends C in each case. Thus $d^-v_h, d^+v_h, d^-d^+ \notin E(G)$. So, $\langle d, v_h, d^-, d^+ \rangle = K_{1,3}$, a contradiction.

Case 3. $d \in C[v_h^+, v_i^-]$.

By (1.4), $d \neq v_i^-$. By (1.8)(i), $v_j v_h^+ \notin E(G)$, so $d \neq v_h^+$. Consider the following subcases:

Case	Cycle C'
$d^-d^+ \in E(G)$	$v_j u \overleftarrow{C}[v_i, d^+] \overleftarrow{C}[d^-, v_h] d \overleftarrow{C}[v_j^-, v_i^+] \overleftarrow{C}[v_h^-, v_j]$
$d^+v_j \in E(G)$	$v_j u C[v_i, v_j^-] C[v_j^+, v_h] \overleftarrow{C}[d, v_h^+] \overleftarrow{C}[v_i^-, d^+] v_j$
$d^-v_j \in E(G)$	$v_j u C[v_i, v_j^-] C[v_j^+, v_h] C[d, v_i^-] C[v_h^+, d^-] v_j$

C' extends C in each case. Thus $d^-d^+, d^+v_j, d^-v_j \notin E(G)$. So, $\langle d, d^-, d^+, v_j \rangle = K_{1,3}$, a contradiction. ∎

By (1.8)(ii) and (1.11), we have $v_i, v_j, v_h \in A$, contrary to the hypothesis that the clique number of the subgraph induced by the set of centers of claws of G is at most 2. This contradiction concludes our proof of Theorem 2.4. ∎

Conjecture 2.5 *Every triangularly connected, $K_{1,4}$-free nearly claw-free graph on at least three vertices is fully cycle extendable.*

3 Line graph of a nearly claw-free graph

The line graph of a graph G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G have at least one vertex in common.

Conjecture 3.1 *(Thomassen [11]) Every 4-connected line graph is hamiltonian.*

A graph G is hamiltonian connected if every two vertices of G are connected by a hamiltonian path. So far it is known that every 7-connected line graph is hamiltonian connected [13], and that every 4-connected line graph of a claw-free graph is hamiltonian connected [5], and that every 4-connected line graph of an almost claw-free graph is hamiltonian connected [7]. Thomassen's conjecture has also been proved to be true for 4-connected line graphs of planar simple graphs [6]. Here we consider the hamiltonicity of the line graph of a nearly claw-free graph and have the following.

Theorem 3.2 *Every 4-connected line graph of a nearly claw-free graph is hamiltonian connected.*

To prove our finding, we need one more concept. Let G be a graph such that $L(G)$ is 3 connected and $L(G)$ is not complete. The core of a graph G, denoted by G_0, is obtained by deleting the vertices of degree 1 and replacing each path xyz in G with $d_G(y) = 2$ by an edge xz. The core of a graph was first introduced by Dulmage and Mendelsohn [3], but the definition they have given is different from ours.

Theorem 3.3 *([7]) Let G be a graph in which every 3-edge-cut of G_0 has at least one edge lying in a cycle of length at most 3 in G_0. Then the following are equivalent.*
(i) $L(G)$ is hamiltonian connected;
(ii) $L(G)$ is 3-connected.

Proof of Theorem 3.2. Let G be a nearly claw-free graph such that $L(G)$ is 4-connected. Let G_0 be the core of G, and let $X = \{e_1, e_2, e_3\}$ be a 3-edge cut in G_0 and let H_1 and H_2 be components of $G_0 - X$. By Theorem 3.3, it suffices to prove that X has at least one edge lying in a cycle of length at most 3 in G_0. By contradiction, assume that X has no edge lying in a cycle of length at most 3 in G_0. Since a cycle of length at most 3 is either a C_2 or C_3, there is no parallel edges in X. Since $L(G)$ is 4-connected, X must be incident to a common vertex, say v, in G_0 (otherwise, X is a vertex cut in $L(G)$). Let $X = \{vu_1, vu_2, vu_3\}$, where $e_i = vu_i (i = 1, 2, 3)$, u_1, u_2, u_3 are different vertices. Without loss of generality, we assume that $v \in V(H_1)$ and $u_1, u_2, u_3 \in V(H_2)$. Since $L(G)$ is 4-connected, $V(H_1) = \{v\}$. Thus $N_{G_0}(v) = \{u_1, u_2, u_3\}$. Since X has no edge lying in a cycle of length at most 3 in G_0, $\{u_1u_2, u_1u_3, u_2u_3\} \cap E(G_0) = \emptyset$.

Case 1. $N_G(v) = N_{G_0}(v)$.

Then v is the center of a claw in G. Thus there are two vertices $d_1, d_2 \in N_G(v)$ such that $N_G(v) \subseteq N_G(d_1) \cup N_G(d_2) \cup \{d_1, d_2\}$. Thus $d_{G_0}(v) \geq 5$. It contradicts the hypothesis that $d_G(v) = 3$.

Case 2. $N_G(v) \neq N_{G_0}(v)$.

If some e_i, say $e_1 = vu_1$, is not in $E(G)$, then, by the definition of G_0, we assume that $\{w_1, w_2, w_3\} \subseteq N_G(v)$, where, for $i = 1, 2, 3$, $vw_i, w_iu_i \in E(G)$ (possibly $w_2 = u_2, w_3 = u_3$) and $d_G(w_i) = 2$ (if $w_i \neq u_i$). Thus $\{w_1u_1, vw_2, vw_3\}$ is a 3-cut in $L(G)$. This contrary implies that $X \subseteq E(G)$. As $N_{G_0}(v) \neq N_G(v)$, we have $N_G(v) = \{u_1, u_2, u_3, p_1, \cdots, p_k\} (k \geq 1)$, where $d_G(p_i) = 1$ for $i = 1, \cdots, k$. Then X is a 3-cut in $L(G)$, contrary to the hypothesis that $L(G)$ is 4-connected again. ∎

References

[1] J. A. Bondy and U. S. R. Murty, Graph theory with applications, Macmillan, London and Elsevier, New York, 1976.

[2] L. Clark, Hamiltonain properties of connected, locally connected graphs, Congr. Numer. 32 (1981), 199-204.

[3] A. L. Dulmage and N. S. Mendelsohn, Coverings of bipartite graphs, Canadian J. Math. 10 (1958) 517-34.

[4] G. R. T. Hendry, Extending cycles in graphs, Discrete Math. 85 (1990), 59-72.

[5] M. Kriesell, All 4-connected line graphs of claw-free graphs are hamiltonian-connected, J. Combin. Theory, Ser. B 82(2001), 306-315.

[6] H.-J. Lai, Every 4-connected line graph of a planar graph is hamiltonian, Graphs Combin. 10(1994), 249-253.

[7] H.-J. Lai, G. Yu, Y. Shao, and M. Zhan, Hamilton cycles in 3-connected claw-free graphs, submitted.

[8] D. Oberly and D. Summer, Every connected, locally connected nontrivial graph with no induced claw is hamiltonian, J. Graph Theory 3 (1979), 351-356.

[9] Z. Ryjáček, Almost claw-free graphs, J. Graph Theory 18(1994), 469-477.

[10] Y. Shao, Claw-free graphs and line graphs, Ph. D. Dissertation, West Virginia Univeristy, 2005.

[11] C. Thomassen, Reflections on graph theory, J. Graph Theory, 10 (1986), 309-324.

[12] M. Zhan, Full cycle extendability of triangularly connected almost claw-free graphs, ARS Combinatoria, Vol. 96, 2010.

[13] S. Zhan, On Hamiltonian line graphs and connectivity, Discrete Math. 89 (1991) 89-95.