SELF VERTEX SWITCHINGS OF TREES

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Abstract

A vertex $v \in V(G)$ is said to be a ***self vertex switching*** of G if G is isomorphic to G^v, where G^v is the graph obtained from G by deleting all edges of G incident to v and adding all edges incident to v which are not in G. The set of all self vertex switchings of G is denoted by $SS_1(G)$ and its cardinality by $ss_1(G)$. In [6], the number $ss_1(G)$ is calculated for the graphs, cycle, path, regular graph, wheel, Euler graph, complete graph and complete bipartite graphs. In this paper for a vertex v of a graph G, the graph G^v is characterized for tree, star and forest with a given number of components. Using this, we characterize trees and forests, each with a self vertex switching.

Key words: Switching, Self vertex switching, $SS_1(G)$, $ss_1(G)$.

1. Introduction

For a finite undirected graph $G(V, E)$ with $|V(G)| = p$ and a set $\sigma \subseteq V$, the switching of G by σ is defined as the graph $G^\sigma(V, E')$, which is obtained from G by removing all edges between σ and its complement $V-\sigma$ and adding as edges all non edges between σ and $V-\sigma$. Switching has been defined by Seidel [3, 4] and is also referred to as Seidel switching. When $\sigma = \{v\} \subset V$, we call the corresponding switching $G^{\{v\}}$ as ***vertex switching*** and denoted it as G^v. A subset σ of $V(G)$ to be a ***self switching*** of G if $G \cong G^\sigma$. The set of all self switchings of G with cardinality k is denoted by $SS_k(G)$ and its cardinality by $ss_k(G)$. If $k = 1$, then we call the corresponding self switching as ***self vertex switching*** [1, 6]. A ***branch*** at v in G is a connected subgraph B of G such that $B-v$ is connected and maximal [5]. Two vertices u and v in G are said to be **interchange similar** if there is an automorphism α of G such that $\alpha(u) = v$ and $\alpha(v) = u$ [2].

In [6], the number $ss_1(G)$ for the graphs, cycle, path, regular graph, wheel, Euler graph, complete graph and complete bipartite graphs are calculated. In [5], a connected graph G in which any two self vertex switchings are interchange similar is characterized for $ss_1(G) > 1$. In this paper we find the number $ss_1(G)$ for trees and forests.

Now consider the following results, which are required in the subsequent sections. We consider simple graphs only unless otherwise it is mentioned specifically.

Theorem 1.1.[1] If v is a self vertex switching of a graph G of order p, then $d_G(v) = (p-1)/2$.

Lemma 1.2.[1, 6] In any graph G, vertex adjacent to a vertex of minimum degree is not a self vertex switching.

Theorem 1.3.[5] Let B_i be either a branch at v in G or the union of v and a component of G not containing v, $i = 1, 2, ..., k(G-v)$. Then $G = \bigcup_{i=1}^{k} B_i$ and $G^v = \bigcup_{i=1}^{k} B_i^v$ where $k = k(G-v)$, $k(G)$ is the number of components of G.

Theorem 1.4.[5] Let v be any vertex of a connected graph G such that G^v is connected. Then B is a branch at v in G if and only if B^v is a branch at v in G^v.

2. Characterizing trees each with a self vertex switching

Let G be a graph and v be any vertex of G. Let G^v be the switching of G by v. In this section, we find the number of G^v's to be connected if G is either cycle, path, star, block, tree, complete bipartite graph or complete graph. We characterize a vertex v of G such that G^v is connected. Using this, we characterize the vertex v such that G^v is a tree and in particular a star. Finally we characterize trees, each with a self vertex switching.

First we give a simple lemma which will be used to prove some theorems.

Lemma 2.1. D is a component of G not containing v if and only if $D+v$ is a branch at v in G^v.

Proof. D is a component of G not containing v if and only if v is non-adjacent to all vertices of D in G if and only if v is adjacent to all vertices of D in G^v if and only if $D+v$ is a branch at v in G^v. □

Theorem 2.2. Let $v \in V(G)$ and $|V(G)| \geq 2$. Then G^v is connected if and only if $d_G(v) = 0$ or $d_B(v) < |V(B)|-1$ for every branch B at v in G.

Proof. If $d_G(v) = 0$, then obviously G^v is connected. Let us assume that $d_B(v) < |V(B)|-1$ for every branch B at v in G. Suppose G^v is disconnected. Then let D be a component of G^v not containing v. Using Lemma 2.1, $B = D+v$ is a branch at v in $(G^v)^v = G$. This implies that $d_B(v) = |V(B)|-1$. This is a contradiction to our assumption that $d_B(v) < |V(B)|-1$ and hence G^v is connected.

Conversely, let G^v be connected. If $d_G(v) = 0$, then the proof is over. Now let $d_G(v) \neq 0$. Suppose $d_B(v) = |V(B)|-1$ for at least one branch, say B, at v in G. Then $B-v$ is a component of G^v not containing v and hence G^v is disconnected, which is a contradiction. This implies that $d_B(v) < |V(B)|-1$ for every branch B at v in G. □

Corollary 2.3. Let G be a connected graph of order $p \geq 2$ and $v \in V(G)$. Then G^v is connected if and only if $d_B(v) < |V(B)|-1$ for every branch B at v in G.

Note 2.4. It is interesting to note that corresponding to each $v \in V(G)$, we get G^v, the switching of G by v. In this section, we consider the following notations.

$[G]_1 = \{G^v \mid v \in V(G)\}$ and $[G]_{1c} = \{G^v \mid G^v \text{ is connected}\}$.

From the above notations, the following properties are obvious.

1. $|[\overline{K}_p]_{1c}| = p$ for $p \geq 2$.
2. $|[C_p]_{1c}| = p$ for $p \geq 4$.
3. $|[K_p]_{1c}| = 0$ for $p \geq 2$.
4. $|[K_{1,n}]_{1c}| = n$ for $n \geq 2$.
5. $|[K_{m,n}]_{1c}| = m + n$ for $m, n \geq 2$.
6. $|[P_p]_{1c}| = \begin{cases} 0 & if\ p = 2 \\ 2 & if\ p = 3 \\ p-2 & if\ p \geq 4 \end{cases}$
7. If G is a block of order p, then $|[G]_{1c}| = |\{v \mid d_G(v) < p-1\}|$.

Corollary 2.5. Let G be a nontrivial graph of order p. Then $|[G]_{1c}| = p$ if and only if either $d_G(v) = 0$ or $d_B(v) < |V(B)|-1$ for every branch B at v in G, $v \in V(G)$.

Theorem 2.6. For a tree G of order $p \geq 2$, $|[G]_{1c}| = p-r$ where r is the number of vertices, each of which is adjacent to an end vertex in G.

Proof. Let $v \in V(G)$. Consider the following two cases.

Case 1. v is adjacent to an end vertex.

Let w be an end vertex adjacent to v in G. Then $B = K_2 = vw$ is a branch at v in G and $d_B(v) = |V(B)|-1$. Using Corollary 2.3, G^v is not connected.

Case 2. v is non-adjacent to any end vertex.

Let B be any branch at v in G. This implies that $p \geq 3$ and $B \neq K_2$. Since G is a tree, there exists a vertex, say x, in B such that x is non-adjacent to v and hence $d_B(v) < |V(B)|-1$. Using Corollary 2.3, G^v is connected since B is an arbitrary branch at v in G.

Thus from cases (1) and (2), the result follows. □

Theorem 2.7. Let v be any vertex of a nontrivial connected graph G. Then G^v is a tree if and only if $G-v$ is acyclic and $d_B(v) = |V(B)|-2$ for every branch B at v in G.

Proof. Let G^v be a tree. Then G^v is connected and acyclic. Using Corollary 2.3, $d_B(v) \leq |V(B)|-2$ for every branch B at v in G. Suppose $d_{B^*}(v) < |V(B^*)|-2$ for some branch B^* at v in G. Then there exist at least two vertices, say u and w, in B^* such that they are non-adjacent to v in G. Since B^*-v is connected, there exists a u–w path in B^*-v and hence in G^v also. In this case, the u–w path and the edges wv and vu form a cycle in G^v. This is a contradiction to G^v is acyclic. This implies that $d_B(v) = |V(B)|-2$ for every branch B at v in G.

Conversely, let $G-v$ be acyclic and $d_B(v) = |V(B)|-2$ for every branch B at v in G. Then using Corollary 2.3, G^v is connected. Suppose there exists a cycle, say C, in G^v. Then the cycle C in G^v must contain the vertex v since $G-v$ is acyclic. Let B_1 be the branch at v in G^v, which contains C. Using Theorem 1.4, $B = B_1^v$ is a branch at v in G since G and G^v are connected. Let x and y be adjacent to v in B_1. Clearly x and y are non-adjacent to v in B and hence $d_B(v) < |V(B)|-2$, which is a contradiction to our assumption that $d_B(v) = |V(B)|-2$. This implies that G^v is acyclic and hence is a tree. □

Theorem 2.8. Let v be any vertex of a disconnected graph G. Then G^v is a tree if and only if G is either $\overline{K}_p$ or $D \cup (p-|V(D)|)K_1$ where D is a component of G of order at least 3 containing v such that $D-v$ is acyclic and $d_B(v) = |V(B)|-2$ for every branch B at v in D.

Proof. Let G^v be a tree. Then G^v is connected and acyclic. Using Theorem 2.2, $d_G(v) = 0$ or $d_B(v) \leq |V(B)|-2$ for every branch B at v in G. If $d_G(v) = 0$, then $G = \overline{K}_p$. Suppose $d_B(v) \leq |V(B)|-2$ for every branch B at v in G. This implies that $G \neq \overline{K}_p$ and hence G has at least one nontrivial component. Let v be in a nontrivial component, say D, of G. If G has a nontrivial component, say E, which is different from D, then G^v is not a tree. Thus G has exactly one nontrivial component D and hence $G = D \cup (p-|V(D)|)K_1$. Clearly the branches at v in G are nothing but the branches at v in D and hence $d_B(v) \leq |V(B)|-2$ for every branch B at v in D. If $d_{B^*}(v) < |V(B^*)|-2$ for at least one branch, say B^*, at v in D, then D^v has a cycle and hence G^v also has a cycle since $G^v = D^v \cup (p-|V(D)|)(K_1+v)$. This is a contradiction to our assumption that G^v is acyclic. This implies that $d_B(v) = |V(B)|-2$ for every branch B at v in D. Since G^v is acyclic, $D-v$ is also acyclic.

Conversely, if $G = \overline{K}_p$, then for any vertex v of G, $G^v = K_{1,p-1}$ and hence G^v is a tree. Suppose $G = D \cup (p-|V(D)|)K_1$ where D satisfies the conditions given in the theorem. Then using Theorem 2.2, G^v is connected and using Theorem 2.7, D^v is a tree. This implies that G^v is a tree since $G^v = D^v \cup (p-|V(D)|)(K_1+v)$. □

Theorem 2.9. Let v be any vertex of a graph G of order $p \geq 3$. Then G^v is a star if and only if G is either $\overline{K}_p$ or $K_{2,p-2}$ with $d_G(v) = p-2$.

Proof. Let $V(G^v) = \{u_1, u_2, ..., u_p \mid d_{G^v}(u_1) = p-1 and\, d_{G^v}(u_i) = 1 for\, i = 2, 3, ..., p\}$. If $v = u_1$, then $G = \overline{K}_p$ and if $v = u_i$, $2 \leq i \leq p$, then G is a graph in which u_i and v are non-adjacent but both are adjacent to all other $p-2$ vertices and thereby $G = K_{2,p-2}$.

Conversely, if $G = \overline{K}_p$, then $G^v = K_{1,p-1}$ and if $G = K_{2,p-2}$ with $d_G(v) = p-2$, then $G^v = K_{1,p-1}$. Thus, in both cases, G^v is a star. □

Note 2.10. Let v be a cutvertex of a connected graph G. Let $B_1, B_2, \ldots, B_k$ be the branches with $n_1, n_2, \ldots, n_k$ number of copies at v in G, respectively. In this case, we denote the graph G by $G(v;\ n_1B_1, n_2B_2, \ldots, n_kB_k)$.

As an example, consider the graph G given in figure 2.1. There are four distinct branches B_1, B_2, B_3 and B_4 at v in G and they are given in figure 2.2. Thus $G = G(v;\ 2B_1, B_2, B_3, B_4)$. The graph given in figure 2.3. is $G(v;\ 6P_3)$.

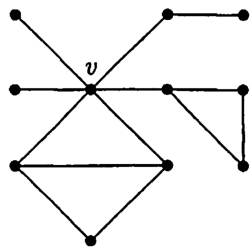

Fig.2.1. $G = G(v;\ 2B_1, B_2, B_3, B_4)$

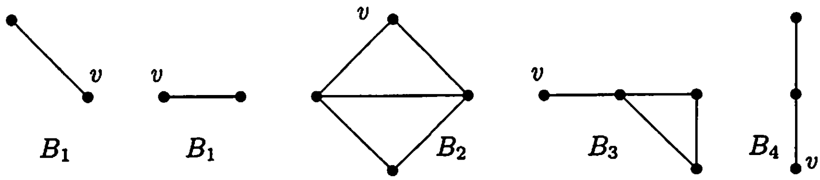

Fig.2.2.

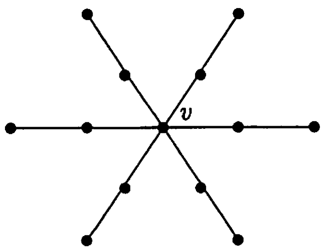

Fig. 2.3.

Theorem 2.11. Let v be a self vertex switching of a connected graph G and B be a branch at v in G. Then $|V(B)| \geq 3$.

Proof. Using Lemma 1.2, v is non-adjacent to a vertex of minimum degree in G. Let B be a branch at v. Then $|V(B)| \geq 2$. If possible, let $|V(B)| = 2$. Let u be the vertex adjacent to v in B. Then $B = vu$. This implies that v is adjacent to the vertex u of minimum degree in G which is a contradiction. Hence $|V(B)| \geq 3$. □

Theorem 2.12. Let G be a tree of order $p = 2n+1$, $n \in N$. Then G has a self vertex switching v if and only if $G = G(v;\ nP_3)$.

Proof. Let v be a self vertex switching of the tree G. Using Theorem 1.1, $d_G(v) = n$. Since G is a tree, there are n branches at v in G. Using Theorem 2.11, $|V(B)| \geq 3$ for any branch B at v in G. If B^* is a branch at v in G such that $|V(B^*)| > 3$, then $p \geq 3(n-1)+4-(n-1) = 2n+2 > p$, which is a contradiction. Hence $|V(B)| = 3$ for every branch B at v in G, which implies that $B = P_3$. Thus $G = G(v;\ nP_3)$.

Conversely, let $G = G(v;\ nP_3)$. If $n = 1$, then $G = P_3$ and v is a self vertex switching of G. For $n \geq 2$, the center v of G is the only self vertex switching of G. □

Corollary 2.13. For any nontrivial tree G,

$$ss_1(G) = \begin{cases} 2 & if\ G = P_3 \\ 1 & if\ G = G(v;\ nP_3), n \geq 2 \\ 0 & otherwise. \end{cases}$$

3. Characterizing forests each with a self vertex switching

In this section we characterize a vertex v of a graph G such that G^v is a forest. We also characterize forests, each with a self vertex switching.

Theorem 3.1. Let v be a vertex of a nontrivial graph G. Then G^v is a disconnected graph with k components if and only if G has at least k–1 branches at v and $d_B(v) = |V(B)|-1$ only for k–1 branches B's at v in G.

Proof. Let G^v be a disconnected graph with k components and v be in a component, say D, of G^v. Let $D_1, D_2, ..., D_{k-1}$ be the remaining k–1 components of G^v. Let $B_i = D_i + v$ for $1 \leq i \leq k$–1. Then using Lemma 2.1, B_i is a branch at v in G with $d_{B_i}(v) = |V(B_i)|-1$, $1 \leq i \leq k$–1. Also $G = D^v \cup (\bigcup_{i=1}^{k-1} B_i)$. In D^v, the vertex v is either a cutvertex or not. Suppose B is a branch at v in G such that $B \neq B_i$, $1 \leq i \leq k$–1. If $d_B(v) = |V(B)|-1$, then B–v is a component of G^v other than D_i and hence the number of components of G^v is greater than k, $1 \leq i \leq k$–1. This is a contradiction, which implies that G has at least k–1 branches B's at v with $d_B(v) = |V(B)|-1$ only for k–1 branches B's.

Conversely, let B_1, B_2, ..., B_{k-1} be the branches at v in G with $d_{B_i}(v) = |V(B_i)|-1$, $1 \leq i \leq k$–1. Using Lemma 2.1, $B_1 - v, B_2 - v$, ..., $B_{k-1} - v$ are components of G^v. Here G may be connected or disconnected and correspondingly we consider the following two cases.

Case 1. G is connected.

Here we consider the following two subcases with respect to the number of branches at v.

Case 1.a. G has only k–1 branches at v.

In this case, $G^v = K_1 \cup (\bigcup_{i=1}^{k-1} (B_i - v))$ where $K_1 = v$ and hence G^v has exactly k components.

Case 1.b. G has at least k branches at v.

Let H be the graph obtained from G by deleting the branches B_1, B_2, ..., B_{k-1} excluding the vertex v. Clearly $G = H \cup (\bigcup_{i=1}^{k-1} B_i)$. By the assumption, we have $d_B(v) < |V(B)|-1$ for any branch $B \neq B_i$ at v in G, $1 \leq i \leq k-1$. This implies that $d_B(v) < |V(B)|-1$ for any branch B at v in H and hence H^v is connected using Corollary 2.3. Now $G^v = H^v \cup (\bigcup_{i=1}^{k-1} (B_i - v))$ implies that G^v has k components.

Case 2. G is disconnected.

Let D, $D_1, D_2, ..., D_r$ be the components of G and v be in D. Then $G^v = D^v \cup (\bigcup_{i=1}^{r} (D_i + v))$. For $1 \leq i \leq r$, $D_i + v$ is a branch at v in G^v. Since D is connected, using case-1, D^v has only k components. Thus there are exactly k components in G^v since $G^v = D^v \cup (\bigcup_{i=1}^{r} (D_i + v))$.

Hence the theorem is proved. □

Theorem 3.2. Let v be a vertex of a nontrivial graph G of order p. Then G^v is a forest with k components if and only if $G = D \cup (p-|V(D)|)K_1$ where D is a nontrivial component of G containing v, $G-v$ is acyclic, $d_B(v) \in \{|V(B)|-1, |V(B)|-2\}$ for any branch B at v in G and $d_B(v) = |V(B)|-1$ only for $k-1$ branches B's.

Proof. Let G^v be a forest with k components. Using Theorem 3.1, G has at least $k-1$ branches at v and $d_B(v) = |V(B)|-1$ only for $k-1$ branches B's at v in G. Let v be in a component D of G and B^* be any branch at v in G with $d_{B^*}(v) \neq |V(B^*)|-1$. If $d_{B^*}(v) < |V(B^*)|-2$, then there exist at least two vertices, say x and y, in B^* which are non-adjacent to v in G. Since B^*-v is connected, there exists a x–y path in B^*-v and hence in G^v also. Now the edges vx and yv and the path x–y form a cycle in G^v, which is a contradiction. This implies, $d_{B^*}(v) = |V(B^*)|-2$ or $|V(B^*)|-1$. Let $E \neq D$ be a component of G. If $E \neq K_1$, then E has an edge uw so that $vuwv$ forms a cycle in G^v, which is a contradiction to the assumption that G^v is a forest. Hence all the components of G, except D, are trivial graphs. Thus $G = D \cup (p-|V(D)|)K_1$, $G-v$ is acyclic, $d_B(v) \in \{|V(B)|-1, |V(B)|-2\}$ for any branch B at v in G and $d_B(v) = |V(B)|-1$ only for $k-1$ branches B's.

On the converse part of the theorem, using Theorem 3.1, G^v is disconnected with k components. If G^v is acyclic, then the proof is over. If not, let us assume that G^v has a cycle, say C. Since $G-v$ is acyclic, each cycle in G^v must contain v. Let B_1 be the branch at v in G^v containing the cycle C. Let $x, y \in V(B_1)$ be such that x and y are adjacent to v in G^v. Let $B^* = B_1^v$ so that $V(B^*) = V(B_1)$. If B^* is not a branch at v in G, then $B_1 - v$ is a nontrivial component of G other than D, which is a contradiction. Therefore B^* is a branch at v in G and so $d_{B^*}(v) < |V(B^*)|-2$ since x and y are non-adjacent to v in G. This is a contradiction. This implies that G^v is acyclic and thereby G^v is a forest with k components. This completes the proof. □

Theorem 3.3. A forest G of order $p = 2n+1$ with k components has a self vertex switching v if and only if $G = D(v;\ (k-1)P_2,\ (n-k+1)P_3) \cup (k-1)K_1$ and $k = p+1-|V(D)|$, $n \in N$.

Proof. Let v be a self vertex switching of the forest G with k components. Using Theorem 1.1, $d_G(v) = n$. G is acyclic and hence there are n branches at v in G. Since G^v is a forest, using Theorem 3.2, $G = D \cup (p-|V(D)|)K_1$ where D is a nontrivial component of G containing v, $G-v$ is acyclic, $d_B(v) \in \{|V(B)|-1,\ |V(B)|-2\}$ for any branch B at v in G and $d_B(v) = |V(B)|-1$ only for $k-1$ branches B's. Using Lemma 2.1, $K_1 + v = K_2$ is a branch at v in G^v. If B is a branch at v in G with $d_B(v) = |V(B)|-2$, then B^v is a branch at v in G^v. If B^* is a branch at v in G with $d_{B^*}(v) = |V(B^*)|-1$, then B^*-v is a component of G^v. Since v is a self vertex switching of G, both G and G^v have k components each and hence $k-1 = p-|V(D)|$. This implies that G has $k-1$ branches at v and each is P_2. Since $G-v$ is acyclic, the remaining $n-(k-1)$ branches at v in G are trees, and each is of order 3 since otherwise G has more than p vertices. Thus $G = D(v;\ (k-1)P_2,\ (n-k+1)P_3) \cup (k-1)K_1$ and $k = p+1-|V(D)|$.

Conversely, if $G = D(v;\ (k-1)P_2,\ (n-k+1)P_3) \cup (k-1)K_1$ and $k = p+1-|V(D)|$, then clearly v is a self vertex switching of G. □

The minimum and maximum values of k are 1 and $n+1$, respectively.

Corollary 3.4. If G is a forest of order p with k components, then $ss_1(G) = 0$ or 1. And $ss_1(G) = 1$ if and only if $G = D(v;\ (k-1)P_2,\ (n-k+1)P_3) \cup (k-1)K_1$ where $p = 2n+1$ and $k = p+1-|V(D)|$, D is a component of G containing v and $n+1 \leq |V(D)| \leq 2n+1$.

Example 3.5. For $n = 5$ and $|\, V(D)\,| = 6, 7, 8, 9, 10, 11$, the six graphs G on $p = 2n{+}1 = 11$ vertices, each of which has v as the self vertex switching are given in figures 3.1 to 3.6, respectively.

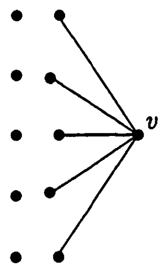

Fig. 3.1. G

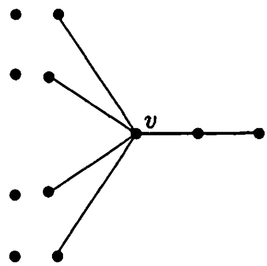

Fig. 3.2. G

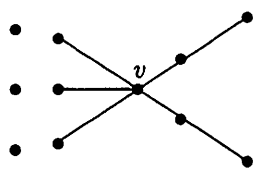

Fig. 3.3. G

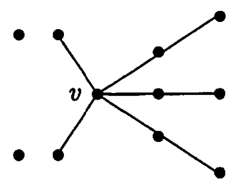

Fig. 3.4. G

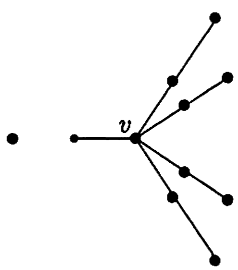

Fig. 3.5. G

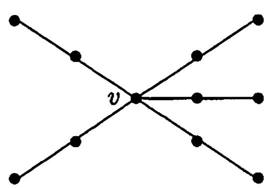

Fig. 3.6. G

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