Some indices of graphs with more than one cut-edge*

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Abstract A bridge graph is special one of those graphs with more than one cut-edge. In this paper we compute Wiener, hyper-Wiener, PI and vertex PI indices of graphs with more than one cut-edge, which generalize results in [12, 13, 14].
Keywords: Hyper-Wiener index; PI index; Cut-edge.
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1 Introduction

A topological index is a real number related to a molecular graph, which does not depend on the labeling or the pictorial representation of a graph. Several indices have been defined and have found applications as means to model chemical, pharmaceutical and other properties of molecules. The Wiener index, introduced in 1947 by H. Wiener as the path number for characterization of alkanes, is the first topological index used in chemistry [15, 16, 17]. The hyper-Wiener index of acyclic graphs was introduced by M. Randić, and then Klein et al. [11] generalized Randić's definition for all connected graphs as a generalization of the Wiener index. I. Gutman et al. [2, 3, 4, 10, 19] studied the mathematical properties of hyper-Wiener index and its applications in chemistry. P. V. Khadikar and S. Karmarkar introduced the PI index in [5] and showed in [6, 7] that the PI index correlates highly with Wiener index as well as with the physicochemical properties and biological activities of a large number of diverse and complex compounds. M.H. Khalifeh et al. [8] defined the vertex PI index and computed vertex PI indices of some graphs [1, 9, 18].

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Recall the definition of a bridge graph in [12]. Let $\{B_i\}_{i=1}^{h}$ be a set of finite pairwise disjoint graphs with $v_i \in V(B_i)$. The bridge graph $H = B\{B_1, B_2, \cdots B_h\}$ is the graph obtained from $B_1, \cdots, B_h$ by connecting the vertices v_i and v_{i+1} by an edge for all $i = 1, 2, \cdots, h-1$, as in Fig. 1.

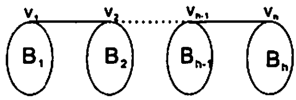

Fig. 1. A bridge graph $H = B\{B_1, B_2, \cdots B_h\}$

Wiener, hyper-Wiener, PI and vertex PI indices of a bridge graph have been computed in [12, 13, 14], respectively. A bridge graph is clearly one of these graphs with more than one cut-edge. In this paper we obtain formulae of Wiener, hyper-Wiener, PI and vertex PI indices of a graph with more than one cut-edge, which generalize the results in [12, 13, 14].

2 Some notations

Suppose that $G = (V(G), E(G))$ is a graph with the cut-edge set $\mathscr{C}$. Let $\mathscr{B} = G \setminus \mathscr{C}$. A component of $\mathscr{B}$ is a *block* of G. We denote by $T(G)$ the *characteristic tree* of G obtained from G by contracting each block of G into a vertex, as illustrated in Fig. 2.

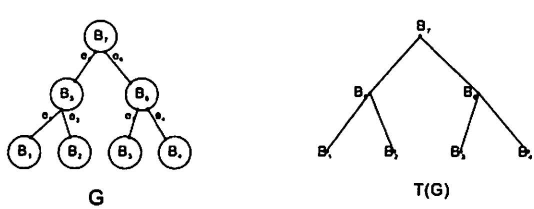

Fig. 2. The graph G and its characteristic graph $T(G)$

If one end of cut-edge e belongs to block B, then e and B are *adjacent*, denoted by $e \sim B$. If $e \sim B$ then we write e_B for e. Let $P = \{(e_B, B) | B \in \mathscr{B}\}$. Given $(e_B, B) \in P$, we let $\mathscr{B}_{(e_B,B)}$ be the set of such blocks B' that the path from B to B' in $T(G)$ contains e_B, and let $N(e_B, B) = \sum_{B' \in \mathscr{B}_{(e_B,B)}} |B'|$ and $N'(e_B, B) = \sum_{B' \in \mathscr{B}_{(e_B,B)}} d_{T(G)}(B', B)|B'|$, where $d_{T(G)}(B, B')$ denotes the distance between vertices B and B' in $T(G)$ and $|B'|$ is the cardinality of the vertex set $V(B')$.

3 Indices of graphs with more than one cut edge

If e and f are two edges of a graph G, then we denote by $d_G(e,f)$ the shortest one of distances between ends of e and ones of f; if v is a vertex of G, then we denote by $d_G(v,e)$ the shorter one of distances between v and ends of e. Set $d^+(e,G) = \sum_{v\in V(G)} d_G(v,e)$ and $d^{++}(e,G) = \sum_{v\in V(G)} d_G^2(v,e)$. Recall that the Wiener index $W(G)$ of a graph G is defined as $W(G) = \sum_{\{u,v\}\subset V(G)} d_G(u,v)$, where $d_G(u,v)$ denotes the distance between vertices u and v in G.

Theorem 1. *Suppose that G is a graph with cut-edge set $\mathscr{C}$. Let $\mathscr{B} = G\backslash\mathscr{C}$ and $Q = \{(e_B, B, f_B)|\ B \in \mathscr{B}, e_B, f_B \in \mathscr{C}\}$. Then*

$$
\begin{aligned}
W(G) = \sum_{B\in\mathscr{B}} W(B) + &\sum_{(e_B,B)\in P} N(e_B,B)d^+(e_B,B)\\
+ &\sum_{(e_B,B,f_B)\in Q} N(e_B,B)N(f_B,B)d_G(e_B,f_B)\\
+ &\sum_{\{B',B''\}\subset\mathscr{B}} |B^{'}||B^{''}|d_{T(G)}(B^{'},B^{''}).
\end{aligned}
$$

Proof. Suppose that x and y are two vertices of G, and that B' and B'' are two blocks of G containing x and y, respectively. Let $B'PB''$ be the path in $T(G)$ between B' and B'', and $e_{B'}$ and $e_{B''}$ be two cut-edges on $B'PB''$. Write $B \in B'PB''$ if cut-edges e_B and f_B are on $B'PB''$. Thus we have

$$
d_G(x,y) = d_G(x,e_{B'}) + d_G(y,e_{B''}) + d_{T(G)}(B',B'') + \sum_{B\in B'PB''} d_G(e_B,f_B) \tag{1}
$$

Using the formula (1) we obtain

$$
\begin{aligned}
\Lambda_1(B^{'},B^{''}) =& \sum_{x\in B'}\sum_{y\in B''} d_G(x,y)\\
=&|B^{''}|d^+(e_{B'},B^{'}) + |B^{'}|d^+(e_{B''},B^{''})\\
&+|B^{'}||B^{''}|d_{T(G)}(B',B'') + |B^{'}||B^{''}| \sum_{B\in B'PB''} d_G(e_B,f_B)
\end{aligned}
$$

Therefore, by the definition of $W(G)$ and the equation above, we have

$$\begin{aligned} W(G) &= \sum_{B\in\mathscr{B}} W(B) + \sum_{\{B',B''\}\subset\mathscr{B}} \Lambda_1(B',B'') \\ &= \sum_{B\in\mathscr{B}} W(B) + \sum_{(e_B,B)\in P} N(e_B,B)d^+(e_B,B) \\ &+ \sum_{(e_B,B,f_B)\in Q} N(e_B,B)N(f_B,B)d_G(e_B,f_B) \\ &+ \sum_{\{B',B''\}\subset\mathscr{B}} |B'||B''|d_{T(G)}(B',B''). \end{aligned}$$

□

Recall that the hyper-Wiener index $WW(G)$ of a graph G is defined as follows

$$WW(G) = \frac{1}{2}\sum_{\{u,v\}\subset V(G)} \Big(d_G(u,v) + d_G(u,v)^2\Big).$$

Theorem 2. *Suppose that G is a graph with cut-edge set $\mathscr{C}$, and let $\mathscr{B} = G\setminus\mathscr{C}$. Then*

$$\begin{aligned} WW(G) &= \sum_{B\in\mathscr{B}} WW(B) \\ &+ \frac{1}{2}\sum_{\{B',B''\}\subset\mathscr{B}} |B'||B''|\Big(\big(d_{T(G)}(B',B'')\big)^2 + d_{T(G)}(B',B'')\Big) \\ &+ \frac{1}{2}\sum_{(e_B,B)\in P} \Big(N(e_B,B)d^{++}(e_B,B) + N(e_B,B)d^+(e_B,B) \\ &\qquad + 2N'(e_B,B)d^+(e_B,B)\Big) \\ &+ \frac{1}{2}\sum_{(e_B,B,f_B)\in Q} \Big(2N'(e_B,B)N(f_B,B) + 2N'(f_B,B)N(e_B,B) \\ &\qquad + N(f_B,B)N(e_B,B)\Big)d_G(e_B,f_B) \\ &+ \sum_{\{B',B''\}\subset\mathscr{B}} d_{T(G)}(B',B'')|B'||B''| \sum_{B\in B'PB''} d_G(e_B,f_B) \\ &+ \sum_{\{B',B''\}\subset\mathscr{B}} \Big(d^+(e_{B'},B')d^+(e_{B''},B'') + \frac{1}{2}|B'||B''|\big(\sum_{B\in B'PB''} d_G(e_B,f_B)\big)^2\Big). \end{aligned}$$

Proof. Suppose that x and y are two vertices of G, and that B' and B'' are two blocks of G containing x and y, respectively. Let $B'PB''$ be the path in $T(G)$ between B' and B'', and $e_{B'}$ and $e_{B''}$ be two cut-edges on $B'PB''$. Write $B \in B'PB''$ if cut-edges e_B and f_B are on $B'PB''$. According to the formula (1), we have

$$\begin{aligned}
\Lambda_2(B', B'') &= \sum_{x\in B'}\sum_{y\in B''} d_G(x,y)^2 \\
=& \sum_{x\in B'}\sum_{y\in B''} \Big(d_G(x, e_{B'}) + d_G(y, e_{B''}) + d_{T(G)}(B', B'') \\
&\qquad + \sum_{B\in B'PB''} d_G(e_B, f_B)\Big)^2 \\
=& 2d^+(e_{B'}, B')d^+(e_{B''}, B'') + |B'|d^{++}(e_{B''}, B'') \\
&+ 2|B''|d^+(e_{B'}, B') \sum_{B\in B'PB''} d_G(e_B, f_B) \\
&+ 2|B'|d_{T(G)}(B', B'')d^+(e_{B''}, B'') \\
&+ |B'||B''|\Big(\sum_{B\in B'PB''} d_G(e_B, f_B)\Big)^2 + |B''|d^{++}(e_{B'}, B') \\
&+ 2|B''|d_{T(G)}(B', B'')d^+(e_{B'}, B') \\
&+ 2d_{T(G)}(B', B'')|B'||B''| \sum_{B\in B'PB''} d_G(e_B, f_B) \\
&+ 2|B'|d^+(e_{B''}, B'') \sum_{B\in B'PB''} d_G(e_B, f_B) + |B'||B''|\Big(d_{T(G)}(B', B'')\Big)^2.
\end{aligned}$$

Therefore, by the equation above, we further have

$$\begin{aligned}
&\sum_{\{B',B''\}\subset\mathscr{B}} \Lambda_2(B', B'') \\
&= 2\sum_{\{B',B''\}\subset\mathscr{B}} \Big(|B''|d^+(e_{B'}, B') + |B'|d^+(e_{B''}, B'')\Big) \sum_{B\in B'PB''} d_G(e_B, f_B) \\
&+ \sum_{\{B',B''\}\subset\mathscr{B}} |B'||B''|\Big(d_{T(G)}(B', B'')\Big)^2 \\
&+ 2\sum_{\{B',B''\}\subset\mathscr{B}} d^+(e_{B'}, B')d^+(e_{B''}, B'') \\
&+ \sum_{\{B',B''\}\subset\mathscr{B}} \Big(|B''|d^{++}(e_{B'}, B') + |B'|d^{++}(e_{B''}, B'')\Big)
\end{aligned}$$

$$+\sum_{\{B',B''\}\subset\mathscr{B}}|B'||B''|\Big(\sum_{B\in B'PB''}d_G(e_B,f_B)\Big)^2$$

$$+2\sum_{\{B',B''\}\subset\mathscr{B}}\Big(|B''|d_{T(G)}(B',B'')d^+(e_{B'},B')$$

$$+|B'|d_{T(G)}(B',B'')d^+(e_{B''},B'')\Big)$$

$$+2\sum_{\{B',B''\}\subset\mathscr{B}}d_{T(G)}(B',B'')|B'||B''|\sum_{B\in B'PB''}d_G(e_B,f_B)$$

$$=2\sum_{(e_B,B,f_B)\in Q}\Big(N'(e_B,B)N(f_B,B)+N'(f_B,B)N(e_B,B)\Big)d_G(e_B,f_B)$$

$$+2\sum_{\{B',B''\}\subset\mathscr{B}}d_{T(G)}(B',B'')|B'||B''|\sum_{B\in B'PB''}d_G(e_B,f_B)$$

$$+\sum_{\{B',B''\}\subset\mathscr{B}}|B'||B''|\Big(\sum_{B\in B'PB''}d_G(e_B,f_B)\Big)^2$$

$$+\sum_{\{B',B''\}\subset\mathscr{B}}|B'||B''|\Big(d_{T(G)}(B',B'')\Big)^2$$

$$+\sum_{(e_B,B)\in P}N(e_B,B)d^{++}(e_B,B)+2\sum_{(e_B,B)\in P}N'(e_B,B)d^+(e_B,B)$$

$$+2\sum_{\{B',B''\}\subset\mathscr{B}}d^+(e_{B'},B')d^+(e_{B''},B'').$$

Therefore, by the definition of the Hyper-Wiener index and Theorem 1, we obtain

$$WW(G)=\frac{1}{2}\sum_{\{u,v\}\subset V(G)}\Big(d_G(u,v)+d_G(u,v)^2\Big)$$

$$=\frac{1}{2}W(G)+\frac{1}{2}\sum_{\{u,v\}\in V(G)}d_G(u,v)^2$$

$$=\frac{1}{2}W(G)+\frac{1}{2}\sum_{B\in\mathscr{B}}\sum_{\{u,v\}\subset B}d_B(u,v)^2+\frac{1}{2}\sum_{\{B',B''\}\in\mathscr{B}}\Lambda_2(B',B'')$$

$$=\frac{1}{2}W(G)+\sum_{\{B',B''\}\in\mathscr{B}}d^+(e_{B'},B')d^+(e_{B''},B'')$$

$$+\frac{1}{2}\sum_{\{B',B''\}\in\mathscr{B}}|B'||B''|\Big(d_{T(G)}(B',B'')\Big)^2+\frac{1}{2}\sum_{(e_B,B)\in P}N(e_B,B)d^{++}(e_B,B)$$

$$
\begin{aligned}
&+ \sum_{(e_B,B)\in P} N'(e_B,B)d^+(e_B,B) + \frac{1}{2}\sum_{B\in\mathscr{B}}\sum_{\{u,v\}\subset B} d_B(u,v)^2 \\
&+ \sum_{(e_B,B,f_B)\in Q} \Big(N'(e_B,B)N(f_B,B) + N'(f_B,B)N(e_B,B)\Big)d_G(e_B,f_B) \\
&+ \sum_{\{B',B''\}\subset\mathscr{B}} \Big(d_{T(G)}(B',B'')|B'||B''| \sum_{B\in B'PB''} d_G(e_B,f_B) \\
&\qquad + \frac{1}{2}|B'||B''|\Big(\sum_{B\in B'PB''} d_G(e_B,f_B)\Big)^2\Big) \\
&= \sum_{B\in\mathscr{B}} WW(B) + \frac{1}{2}\sum_{\{B',B''\}\in\mathscr{B}} |B'||B''|\Big(\big(d_{T(G)}(B',B'')\big)^2 + d_{T(G)}(B',B'')\Big) \\
&+\frac{1}{2}\sum_{(e_B,B)\in P} \Big(N(e_B,B)d^{++}(e_B,B) + N(e_B,B)d^+(e_B,B) \\
&\qquad + 2N'(e_B,B)d^+(e_B,B)\Big) \\
&+\frac{1}{2}\sum_{(e_B,B,f_B)\in Q} \Big(2N'(e_B,B)N(f_B,B) + 2N'(f_B,B)N(e_B,B) \\
&\qquad + N(f_B,B)N(e_B,B)\Big)d_G(e_B,f_B) \\
&+ \sum_{\{B',B''\}\subset\mathscr{B}} d_{T(G)}(B',B'')|B'||B''| \sum_{B\in B'PB''} d_G(e_B,f_B) \\
&+ \sum_{\{B',B''\}\in\mathscr{B}} d^+(e_{B'},B')d^+(e_{B''},B'') \\
&+ \frac{1}{2}\sum_{\{B',B''\}\in\mathscr{B}} |B'||B''|\Big(\sum_{B\in B'PB''} d_G(e_B,f_B)\Big)^2. \qquad \square
\end{aligned}
$$

Suppose that $e = uv$ is an edge of a graph $G = \big(V(G), E(G)\big)$. Then we denote by $N_u(e|G)$ and $N'_u(e|G)$ the set of vertices and edges lying closer to u than to v, respectively. Let $n_u(e|G) = |N_u(e|G)|$ and $n'_u(e|G) = |N'_u(e|G)|$. Then the vertex PI index $PI_v(G)$ and PI index $PI(G)$ of G are defined as following.$PI_v(G) = \sum_{e\in E(G)} \big[n_u(e|G) + n_v(e|G)\big]$, $PI(G) = \sum_{e\in E(G)} \big[n'_u(e|G) + n'_v(e|G)\big]$.

If we denote by $n_{uv}(e|G)$ and $n'_{uv}(e|G)$ the number of vertices and edges in G having the same distance from ends u and v of e, respectively, then

$$PI_v(G) = |E(G)||G| - \sum_{e\in E(G)} n_{uv}(e|G), PI(G) = |E(G)|^2 - \sum_{e\in E(G)} n'_{uv}(e|G).$$

Theorem 3. *Suppose that G is a graph with cut-edge set $\mathscr{C}$, and let $\mathscr{B} = G \setminus \mathscr{C}$. For $(f_B, B) \in P$ and $e = uv \in E(B)$, define*

$$F(e,(f_B,B)) = \begin{cases} 1, & \text{if } d_G(f_B,u) = d_G(f_B,v); \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

$$PI_v(G) = \sum_{B\in\mathscr{B}} PI_v(B) + |E(G)||G| - \sum_{B\in\mathscr{B}} |E(B)||B| - \sum_{(f_B,B)\in P} \sum_{e\in E(B)} F(e,(f_B,B))N(f_B,B).$$

Proof. We first have

$$\sum_{e\in E(G)} n_{uv}(e|G) = \sum_{B\in\mathscr{B}} \sum_{e\in E(B)} n_{uv}(e|G) + \sum_{e\in\mathscr{C}} n_{uv}(e|G)$$

Note that $\sum_{e\in\mathscr{C}} n_{uv}(e|G) = 0$. We further have

$$\sum_{e\in E(G)} n_{uv}(e|G) = \sum_{B\in\mathscr{B}} \sum_{e\in E(B)} \Big(n_{uv}(e|B) + \sum_{\overline{B}\in\mathscr{B}\setminus B} n_{uv}(e|\overline{B})\Big)$$
$$= \sum_{B\in\mathscr{B}} |E(B)||B| - \sum_{B\in\mathscr{B}} PI_v(B) + \sum_{(f_B,B)\in P} \sum_{e\in E(B)} F(e,(f_B,B))N(f_B,B)$$

Thus

$$\begin{aligned} PI_v(G) =& |E(G)||G| - \sum_{e\in E(G)} n_{uv}(e|G) \\ =& \sum_{B\in\mathscr{B}} PI_v(B) + |E(G)||G| - \sum_{B\in\mathscr{B}} |E(B)||B| \\ &- \sum_{(f_B,B)\in P} \sum_{e\in E(B)} F(e,(f_B,B))N(f_B,B). \end{aligned}$$ □

Similarly, we can obtain the following

Theorem 4. *Suppose that G is a graph with cut-edge set $\mathscr{C}$, and let $\mathscr{B} = G \setminus \mathscr{C}$. For $(f_B, B) \in P$ and $e = uv \in E(B)$, define*

$$F(e,(f_B,B)) = \begin{cases} 1, & \text{if } d_G(f_B,u) = d_G(f_B,v); \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

$$\begin{aligned} PI(G) = \sum_{B\in\mathscr{B}} PI(B) + |E(G)|^2 - \sum_{B\in\mathscr{B}} |E(B)|^2 \\ - \sum_{(f_B,B)\in P} \sum_{e\in E(B)} F(e,(f_B,B))\Big(\sum_{B'\in\mathscr{B}_{(e_B,B)}} |E(B')| \\ + \frac{1}{2} \sum_{B'\in\mathscr{B}_{(e_B,B)}} d_{T(G)}(B') - 1\Big). \end{aligned}$$

Remarks. As an application we use these formulae above to compute those corresponding indices of bridge graph $H = B\{B_1, B_2 \cdots B_h\}$. In the bridge graph H, if we let $e_i = v_i v_{i+1}$, then we easily see that

$$P = \{(e_i, B_i), (e_i, B_{i+1}) | i = 1, 2, \cdots, h-1\}.$$

$$Q = \{(e_i, B_i, e_{i+1}) | i = 1, 2, \cdots, h-1\}.$$

$$\mathscr{B}_{(e_i,B_i)} = \{B_j | j = i, i+1, \cdots, h\}, N(e_i, B_i) = \sum_{j=i+1}^{h} |B_j|,$$

$$N'(e_i, B_i) = \sum_{j=i+1}^{h} (j-i)|B_j|.$$

$$\mathscr{B}_{(e_i,B_{i+1})} = \{B_j | j = 1, 2, \cdots i\}, N(e_i, B_{i+1}) = \sum_{j=1}^{i} |B_j|,$$

$$N'(e_i, B_{i+1}) = \sum_{j=1}^{i} (i-j+1)|B_j|.$$

$$d^+(e_i, B_i) = \sum_{v\in V(B_i)} d_H(v, v_i) = d^+(v_i, B_i).$$

$$d^+(e_i, B_{i+1}) = \sum_{v\in V(B_{i+1})} d_H(v, v_{i+1}) = d^+(v_{i+1}, B_{i+1}).$$

$$d^{++}(e_i, B_i) = \sum_{v\in V(B_i)} d_H(v, v_i)^2 = d^{++}(v_i, B_i).$$

$$d^{++}(e_i, B_{i+1}) = \sum_{v\in V(B_{i+1})} d_H(v, v_{i+1})^2 = d^{++}(v_{i+1}, B_{i+1}).$$

Thus, according to corresponding formulae above, we obtain the following results.

$$\begin{aligned}
W(H) &= \sum_{i=1}^{h} W(B_i) + N(e_1, B_1)d^+(e_1, B_1) + N(e_{h-1}, B_h)d^+(e_{h-1}, B_h) \\
&+ \sum_{i=2}^{h-1}\Big(N(e_{i-1}, B_i)d^+(e_{i-1}, B_i) + N(e_i, B_i)d^+(e_i, B_i)\Big) \\
&+ \sum_{1<i<j<h} |B_i||B_j|d_{T(H)}(B_i, B_j) \\
&= \sum_{i=1}^{h} W(B_i) + \sum_{j=2}^{h} |B_j| \sum_{v\in V(B_1)} d_H(v, v_1) + \sum_{j=1}^{h-1} |B_j| \sum_{v\in V(B_h)} d_H(v, v_h) \\
&+ \sum_{i=2}^{h-1}\Big(\sum_{j=1}^{i-1} |B_j| \sum_{v\in V(B_i)} d(v, v_i) + \sum_{j=i+1}^{h} |B_j| \sum_{v\in V(B_i)} d(v, v_i)\Big) \\
&+ \sum_{1<i<j<h} |B_i||B_j|(j-i) \\
&= \sum_{i=1}^{h} W(B_i) + \sum_{i=1}^{h}\Big((|H| - |B_i|)d^+(v_i, B_i)\Big) + \sum_{1<i<j<h} |B_i||B_j|(j-i).
\end{aligned}$$

$$\begin{aligned}
WW(H) &= \sum_{i=1}^{h} WW(B_i) + \sum_{1<i<j<h} |B_i||B_j|\binom{j-i+1}{2} \\
&+ \frac{1}{2}N(e_1, B_1)d^+(e_1, B_1) \\
&+ \frac{1}{2}N(e_{h-1}, B_h)d^+(e_{h-1}, B_h) \\
&+ \frac{1}{2}\sum_{i=2}^{h-1}\Big(N(e_{i-1}, B_i)d^+(e_i, B_i) + N(e_i, B_i)d^+(e_i, B_i)\Big) \\
&+ \frac{1}{2}N(e_1, B_1)d^{++}(e_1, B_1) + \frac{1}{2}N(e_{h-1}, B_h)d^{++}(e_{h-1}, B_h)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\sum_{i=2}^{h-1}\Big(N(e_{i-1},B_i)d^{++}(e_i,B_i)+N(e_i,B_i)d^{++}(e_i,B_i)\Big)\\
&+N'(e_1,B_1)d^{+}(e_1B_1)+N'(e_{h-1},B_h)d^{+}(e_{h-1},B_h)\\
&+\sum_{i=2}^{h-1}\Big(N'(e_{i-1},B_i)d^{+}(e_{i-1},B_i)+N'(e_i,B_i)d^{+}(e_i,B_i)\Big)\\
&+\sum_{1<i<j<h}\Big(\sum_{v\in V(B_i)}d(v,v_i)\sum_{v\in V(B_j)}d(v,v_j)\Big)\\
=&\sum_{i=1}^{h}WW(B_i)+\sum_{1<i<j<h}|B_i||B_j|\binom{j-i+1}{2}+\frac{1}{2}\sum_{i=2}^{h}|B_i|\sum_{v\in V(B_1)}d(v,v_1)\\
&+\frac{1}{2}\sum_{i=1}^{h-1}|B_i|\sum_{v\in V(B_h)}d(v,v_h)\\
&+\frac{1}{2}\sum_{i=2}^{h-1}\Big(\sum_{j=1}^{i-1}|B_j|\sum_{v\in V(B_i)}d(v,v_i)+\sum_{j=i+1}^{h}|B_j|\sum_{v\in V(B_i)}d(v,v_i)\Big)\\
&+\frac{1}{2}\sum_{i=2}^{h}|B_i|\sum_{v\in V(B_1)}d(v,v_1)^2+\frac{1}{2}\sum_{i=1}^{h-1}|B_i|\sum_{v\in V(B_h)}d(v,v_h)^2\\
&+\frac{1}{2}\sum_{i=2}^{h-1}\Big(\sum_{j=1}^{i-1}|B_j|\sum_{v\in V(B_i)}d(v,v_i)^2+\sum_{j=i+1}^{h}|B_j|\sum_{v\in V(B_i)}d(v,v_i)^2\Big)\\
&+\sum_{i=1}^{h}\Big(\sum_{j=1}^{i-1}(i-j)|B_j|\sum_{v\in V(B_i)}d(v,v_i)+\sum_{j=i+1}^{h}(j-i)|B_j|\sum_{v\in V(B_i)}d(v,v_i)\Big)\\
&+\sum_{1<i<j<h}\Big(\sum_{v\in V(B_i)}d(v,v_i)\sum_{v\in V(B_j)}d(v,v_j)\Big)\\
=&\sum_{i=1}^{h}WW(B_i)+\sum_{1<i<j<h}|B_i||B_j|\binom{j-i+1}{2}\\
&+\sum_{i=1}^{h}\Big(|H|-|B_i|\Big)\Big(d^{++}(v_i,B_i)+d^{+}(v_i,B_i)\Big)\\
&+2\sum_{1<i<j<h}(j-i)|B_j|d^{+}(v_i,B_i)+\sum_{1<i<j<h}d^{+}(v_i,B_i)d^{+}(v_j,B_j).
\end{aligned}
$$

If $e=uv$ is an edge of B_i, then $F\big(e,(e_i,B_i)\big)=\begin{cases}1, & \text{if } d_G(u,v_i)=d_G(v,v_i);\\ 0, & \text{otherwise.}\end{cases}$

It is obvious that $F\big(e,(e_{i-1},B_i)\big) = F\big(e,(e_i,B_i)\big)$.

$$
\begin{aligned}
PI_v(H) &= \sum_{i=1}^{h} PI_v(B_i) + |E(H)||H| - \sum_{i=1}^{h} |E(B_i)||B_i| \\
&- \sum_{e\in E(B_h)} F\big(e,(e_{h-1},B_h)\big)N(e_{h-1},B_h) \\
&-\sum_{i=2}^{h-1}\sum_{e\in E(B_i)} \Big(F\big(e,(e_{i-1},B_i)\big)N(e_{i-1},B_i) + F\big(e,(e_i,B_i)\big)N(e_i,B_i)\Big) \\
&- \sum_{e\in E(B_1)} F\big(e,(e_1,B_1)\big)N(e_1,B_1) \\
&= \sum_{i=1}^{h} PI_v(B_i) + |E(H)||H| \\
&- \sum_{i=1}^{h} |E(B_i)||B_i| - \sum_{e\in E(B_h)} F\big(e,(e_{h-1},B_h)\big)\sum_{i=1}^{h-1}|B_i| \\
&-\sum_{i=1}^{h-1}\sum_{e\in E(B_i)} F\big(e,(e_i,B_i)\big)\Big(\sum_{j=1}^{i-1}|B_j| + \sum_{j=i+1}^{h}|B_j|\Big) \\
&= \sum_{i=1}^{h} PI_v(B_i) + |E(H)||H| - \sum_{i=1}^{h} |E(B_i)||B_i| \\
&- \sum_{e\in E(B_h)} F\big(e,(e_{h-1},B_h)\big)\Big(|H| - |B_h|\Big) \\
&-\sum_{i=1}^{h-1}\sum_{e\in E(B_i)} F\big(e,(e_i,B_i)\big)\Big(|H| - |B_i|\Big).
\end{aligned}
$$

If we denote by $M_v(G)$ the set of all edges ww' in G such that $d(w,v) = d(w',v)$ and by $m_v(G)$ the cardinality of $M_v(G)$, then

$$
\begin{aligned}
PI_v(H) =& \sum_{i=1}^{h} PI_v(B_i) + |E(H)||H| - \sum_{i=1}^{h} |E(B_i)||B_i| \\
&- \sum_{i=1}^{h} m_{v_i}(B_i)\Big(|H| - V(B_i)\Big).
\end{aligned}
$$

These results identify with those corresponding ones in [12, 13, 14].

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