The partition dimension of corona product graphs

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Abstract

Given a set of vertices $S = \{v_1, v_2, ..., v_k\}$ of a connected graph G, the metric representation of a vertex v of G with respect to S is the vector $r(v|S) = (d(v, v_1), d(v, v_2), ..., d(v, v_k))$, where $d(v, v_i)$, $i \in \{1, ..., k\}$ denotes the distance between v and v_i. S is a resolving set of G if for every pair of distinct vertices u, v of G, $r(u|S) \neq r(v|S)$. The metric dimension $dim(G)$ of G is the minimum cardinality of any resolving set of G. Given an ordered partition $\Pi = \{P_1, P_2, ..., P_t\}$ of vertices of a connected graph G, the partition representation of a vertex v of G, with respect to the partition Π is the vector $r(v|\Pi) = (d(v, P_1), d(v, P_2), ..., d(v, P_t))$, where $d(v, P_i)$, $1 \leq i \leq t$, represents the distance between the vertex v and the set P_i, that is $d(v, P_i) = \min_{u \in P_i}\{d(v, u)\}$. Π is a resolving partition for G if for every pair of distinct vertices u, v of G, $r(u|\Pi) \neq r(v|\Pi)$. The partition dimension $pd(G)$ of G is the minimum number of sets in any resolving partition for G. Let G and H be two graphs of order n_1 and n_2 respectively. The corona product $G \odot H$ is defined as the graph obtained from G and H by taking one copy of G and n_1 copies of H and then joining by an edge, all the vertices from the i^{th}-copy of H with the i^{th}-vertex of G. Here we study the relationship between $pd(G \odot H)$ and several parameters of the graphs $G \odot H$, G and H, including $dim(G \odot H)$, $pd(G)$ and $pd(H)$.

Keywords: Resolving sets; resolving partition; metric dimension; partition dimension; corona graph.

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1 Introduction

The concepts of resolvability and location in graphs were described independently by Harary and Melter [10] and Slater [19], to define the same structure in a graph. After these papers were published several authors developed diverse theoretical works about this topic [3, 4, 5, 6, 7, 8, 9, 16, 18, 20]. Slater described the usefulness of these ideas into long range aids to navigation [19]. Also, these concepts have some applications in chemistry for representing chemical compounds [14, 15] or to problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures [17]. Other applications of this concept to navigation of robots in networks and other areas appear in [6, 12, 16]. Some variations on resolvability or location have been appearing in the literature, like those about conditional resolvability [18], locating domination [11], resolving domination [1] and resolving partitions [5, 8, 9]. In this work we are interested into study the relationship between $pd(G \odot H)$ and several parameters of the graphs $G \odot H$, G and H, including $dim(G \odot H)$, $pd(G)$ and $pd(H)$.

We begin with some basic concepts and notations. Let $G = (V, E)$ be a simple graph. Let $u, v \in V$ be two different vertices in G, the distance $d_G(u, v)$ between two vertices u and v of G is the length of a shortest path between u and v. If there is no ambiguity, we will use the notation $d(u, v)$ instead of $d_G(u, v)$. The diameter of G is defined as $D(G) = \max_{u,v \in V}\{d(u, v)\}$. Given $u, v \in V$, $u \sim v$ means that u and v are adjacent vertices. Given a set of vertices $S = \{v_1, v_2, ..., v_k\}$ of a connected graph G, the *metric representation* of a vertex $v \in V$ with respect to S is the vector $r(v|S) = (d(v, v_1), d(v, v_2), ..., d(v, v_k))$. We say that S is a *resolving set* for G if for every pair of distinct vertices $u, v \in V$, $r(u|S) \neq r(v|S)$. The *metric dimension* of G is the minimum cardinality of any resolving set for G, and it is denoted by $dim(G)$.

Given an ordered partition $\Pi = \{P_1, P_2, ..., P_t\}$ of vertices of a connected graph G, the *partition representation* of a vertex $v \in V$ with respect to the partition Π is the vector $r(v|\Pi) = (d(v, P_1), d(v, P_2), ..., d(v, P_t))$, where $d(v, P_i)$, $1 \leq i \leq t$, represents the distance between the vertex v and the set P_i, that is $d(v, P_i) = \min_{u \in P_i}\{d(v, u)\}$. We say that Π is a *resolving partition* of G if for every pair of distinct vertices $u, v \in V$, $r(u|\Pi) \neq r(v|\Pi)$. The *partition dimension* of G is the minimum number of sets in any resolving partition for G and it is denoted by $pd(G)$. The partition dimension of graphs is studied in [5, 8, 18, 20, 21].

Let G and H be two graphs of order n_1 and n_2, respectively. The corona product $G \odot H$ is defined as the graph obtained from G and H by taking one copy of G and n_1 copies of H and joining by an edge each vertex from the i^{th}-copy of H with the i^{th}-vertex of G. We will denote by $V = \{v_1, v_2, ..., v_n\}$ the set of vertices of G and by $H_i = (V_i, E_i)$ the copy of H such that $v_i \sim v$ for every $v \in V_i$.

2 Majorizing $pd(G \odot H)$

It was shown in [8] that for any nontrivial connected graph G we have $pd(G) \leq dim(G) + 1$. Thus,

$$pd(G \odot H) \leq dim(G \odot H) + 1. \quad (1)$$

In order to give another interesting relationship between $pd(G \odot H)$ and $dim(G \odot H)$ that allow us to derive tight bounds on $pd(G \odot H)$, we present the following lemma.

Lemma 1. [22] *Let $G = (V, E)$ be a connected graph of order $n \geq 2$ and let H be a graph of order at least two. Let $H_i = (V_i, E_i)$ be the subgraph of $G \odot H$ corresponding to the i^{th} copy of H.*

(i) *If $u, v \in V_i$, then $d_{G \odot H}(u, x) = d_{G \odot H}(v, x)$ for every vertex x of $G \odot H$ not belonging to V_i.*

(ii) *If S is a resolving set for $G \odot H$, then $V_i \cap S \neq \emptyset$ for every $i \in \{1, ..., n\}$.*

(iii) *If S is a resolving set for $G \odot H$ of minimum cardinality, then $V \cap S = \emptyset$.*

Theorem 2. *Let G be a connected graph of order $n_1 \geq 2$ and let H be a graph of order n_2. Then*

$$pd(G \odot H) \leq \frac{1}{n_1} dim(G \odot H) + pd(G) + 1.$$

Proof. Let S be a resolving set for $G \odot H$ of minimum cardinality. By Lemma 1 (ii) and (iii) we conclude that $S = \cup_{i=1}^{n_1} S_i$, where $\emptyset \neq S_i \subset V_i$. We note that $|S_i| = \frac{|S|}{n_1} = \frac{1}{n_1} dim(G \odot H)$ for every $i \in \{1, ..., n_1\}$. In order to build a resolving partition for $G \odot H$, we need to introduce some additional notation. Let $\Pi(G) = \{W_1, W_2, ..., W_{pd(G)}\}$ be a resolving partition for G, let $A = \cup_{i=1}^{n_1}(V_i - S_i)$, let $S_i = \{v_{i1}, v_{i2}, ..., v_{it}\}$, and let $B_j = \cup_{i=1}^{n_1}\{v_{ij}\}$, $j = 1, ..., t$. Let us prove that $\Pi = \{A, B_1, ..., B_t, W_1, ..., W_{pd(G)}\}$ is a resolving partition for $G \odot H$. Let x, y be two different vertices of $G \odot H$. We have the following cases.

Case 1. $x, y \in V_i$. If $x \in S_i$ or $y \in S_i$ then x and y belong to different sets of Π, so $r(x|\Pi) \neq r(y|\Pi)$. We suppose $x, y \in V_i - S_i$. Since S is a resolving set for $G \odot H$, we have $r(x|S) \neq r(y|S)$. By Lemma 1 (i), $d_{G\odot H}(x, u) = d_{G\odot H}(y, u)$ for every vertex u of $G \odot H$ not belonging to V_i. So, there exists $v \in S_i$ such that $d_{G\odot H}(x, v) \neq d_{G\odot H}(y, v)$. Thus, either ($v \sim x$ and $v \not\sim y$) or ($v \not\sim x$ and $v \sim y$). In the first case we have $d_{G\odot H}(x, v) = d_{H_i}(x, v) = 1$ and $d_{G\odot H}(y, v) = 2 \leq d_{H_i}(y, v)$. The case $v \not\sim x$ and $v \sim y$ is analogous. Therefore, for every $x, y \in V_i$ there exists $v_{il} \in S_i$ such that $d_{G\odot H}(x, B_l) = d_{G\odot H}(x, v_{il}) \neq d_{G\odot H}(y, v_{il}) = d_{G\odot H}(y, B_l)$.

Case 2. $x \in V_i$ and $y \in V_j$, $j \neq i$. There exists $W_k \in \Pi(G)$ such that $d_G(v_i, W_k) \neq d_G(v_j, W_k)$. Thus, $d_{G\odot H}(x, W_k) = 1 + d_G(v_i, W_k) \neq d_G(v_j, W_k) + 1 = d_{G\odot H}(y, W_k)$.

Case 3. $x, y \in V$. There exists $W_k \in \Pi(G)$ such that $d_G(x, W_k) \neq d_G(y, W_k)$. Thus, $d_{G\odot H}(x, W_k) \neq d_{G\odot H}(y, W_k)$.

Case 4. $x \in V$ and $y \notin V$. In this case x and y belong to different sets of Π, so $r(x|\Pi) \neq r(y|\Pi)$.

Therefore, Π is a resolving partition for $G \odot H$. □

We denote by K_n and P_n the complete graph and the path graph of order n, respectively. The following proposition allows us to conclude that for every connected graphs G and H of order greater than or equal to two such that $G \odot H \not\cong K_{n_1} \odot P_2$ and $G \odot H \not\cong K_{n_1} \odot P_3$, the equation in Theorem 2 is never worse than equation (1).

Proposition 3. *Let G and H be two connected graphs of order greater than or equal to two. Let n_1 denote the order of G. If $G \odot H \not\cong K_{n_1} \odot P_2$ and $G \odot H \not\cong K_{n_1} \odot P_3$, then*

$$dim(G \odot H) \geq \frac{n_1}{n_1 - 1} pd(G).$$

Proof. It was shown in [22] that

$$dim(G \odot H) \geq n_1 dim(H). \qquad (2)$$

So we differentiate two cases. Case 1: $dim(H) \geq 2$. Since $n_1 \geq 2$, we have $2n_1(n_1 - 1) \geq n_1^2$. Thus,

$$dim(H)n_1(n_1 - 1) \geq 2n_1(n_1 - 1) \geq n_1^2 \geq n_1 pd(G).$$

Hence, by equation (2) we obtain $dim(G \odot H)(n_1 - 1) \geq n_1 pd(G)$.

Case 2: $dim(H) = 1$. It was shown in [6] that a connected graph H has dimension 1 if and only if H is a path graph. So we have $H \cong P_{n_2}$. Now we consider two subcases.

Subcase 2.1: $G \ncong K_{n_1}$ and $n_2 \geq 2$. Then by equation (2) we obtain

$$(n_1 - 1)dim(G \odot P_{n_2}) \geq n_1(n_1 - 1) \geq n_1 pd(G)$$

and, as a consequence, $dim(G \odot H) \geq \frac{n_1}{n_1 - 1} pd(G)$.

Subcase 2.2: $G \cong K_{n_1}$ and $n_2 \geq 4$. Let S be a resolving set for $K_{n_1} \odot P_{n_2}$ of minimum cardinality. As above we denote by $\{v_1, ..., v_{n_1}\}$ the set of vertices of K_{n_1} and by $H_i = (V_i, E_i)$, $i \in \{1, ..., n_1\}$ the corresponding copies of P_{n_2} in $K_{n_1} \odot P_{n_2}$. By Lemma 1 (ii) we know that $V_i \cap S \neq \emptyset$, for every $i \in \{1, ..., n_1\}$. We suppose $V_i \cap S = \{x_i\}$. In this case, since $n_2 \geq 4$ and $H_i \cong P_{n_2}$, there exist $a, b \in V_i$ such that either $d_{K_{n_1} \odot P_{n_2}}(a, x_i) = d_{K_{n_1} \odot P_{n_2}}(b, x_i) = 1$ or $d_{K_{n_1} \odot P_{n_2}}(a, x_i) = d_{K_{n_1} \odot P_{n_2}}(b, x_i) = 2$. Thus, By Lemma 1 (i) we conclude that $r(a|S) = r(b|S)$, a contradiction. Hence, $|V_i \cap S| \geq 2$ and, as a consequence, $dim(K_{n_1} \odot P_{n_2}) \geq 2n_1$. Then

$$dim(K_{n_1} \odot P_{n_2})(n_1 - 1) \geq 2n_1(n_1 - 1) \geq n_1^2 = n_1 pd(K_{n_1}).$$

Therefore, the result follows. □

In [22] it was shown that for every connected graph G of order $n_1 \geq 2$ and every graph H of order $n_2 \geq 2$,

$$dim(G \odot H) \leq \begin{cases} n_1(n_2 - \alpha - 1) & \text{for } \alpha \geq 1 \text{ and } \beta \geq 1, \\ n_1(n_2 - \alpha) & \text{for } \alpha \geq 1 \text{ and } \beta = 0, \\ n_1(n_2 - 1) & \text{for } \alpha = 0, \end{cases}$$

where α denotes the number of connected components of H and β denotes the number of isolated vertices of H.

By using the above bound on $dim(G \odot H)$ we obtain the following direct consequence of Theorem 2.

Corollary 4. *Let G be a connected graph of order $n_1 \geq 2$ and let H be a graph of order $n_2 \geq 2$. Let α be the number of connected components of H of order greater than one and let β be the number of isolated vertices of H.*

Then

$$pd(G \odot H) \leq \begin{cases} pd(G) + n_2 - \alpha & \text{for } \alpha \geq 1 \text{ and } \beta \geq 1, \\ pd(G) + n_2 - \alpha + 1 & \text{for } \alpha \geq 1 \text{ and } \beta = 0, \\ pd(G) + n_2 & \text{for } \alpha = 0. \end{cases}$$

The reader is referred to [22] for several upper bounds on $dim(G \odot H)$ which lead to bounds on $pd(G \odot H)$.

Theorem 5. *Let G and H be two connected graphs of order $n_1 \geq 2$ and $n_2 \geq 2$, respectively. If $D(H) \leq 2$, then*

$$pd(G \odot H) \leq pd(G) + pd(H).$$

Proof. Let $P = \{A_1, A_2, ...A_k\}$ be a resolving partition in G and let $Q_i = \{B_{i1}, B_{i2}, ...B_{it}\}$ be a resolving partition in the corresponding copy H_i of H. Let $B_j = \bigcup_{i=1}^{n_1} B_{ij}$, $j \in \{1, ..., t\}$. We will show that

$$\Pi = \{A_1, A_2, ..., A_k, B_1, B_2, ..., B_t\}$$

is a resolving partition for $G \odot H$. Let x, y be two different vertices of $G \odot H$. If $x, y \in A_i$, then there exists $A_j \in P \subset \Pi$, $j \neq i$, such that $d(x, A_j) \neq d(y, A_j)$. On the other hand, if $x, y \in B_j$, then we have the following cases.

Case 1: $x, y \in B_{ij}$. Hence, there exists $B_{ik} \in Q_i$, $k \neq j$, such that $d_{H_i}(x, B_{ik}) \neq d_{H_i}(y, B_{ik})$. Since $D(H) \leq 2$, for every $u \in B_{ij}$ we have $d_{H_i}(u, B_{ik}) = d_{G \odot H}(u, B_k)$ and $d_{H_i}(u, B_{ik}) = d_{G \odot H}(u, B_k)$. So, we obtain $d_{G \odot H}(x, B_k) = d_{H_i}(x, B_{ik}) \neq d_{H_i}(y, B_{ik}) = d_{G \odot H}(y, B_k)$.

Case 2: $x \in B_{ij}$ and $y \in B_{kj}$, $k \neq i$. If $v_i, v_k \in A_l$, then there exists $A_q \in P \subset \Pi$ such that $d_G(v_i, A_q) \neq d_G(v_k, A_q)$. So, we have $d_{G \odot H}(x, A_q) = 1 + d_G(v_i, A_q) \neq 1 + d_G(v_k, A_q) = d_{G \odot H}(y, A_q)$.

On the other hand, if $v_i \in A_p$ and $v_k \in A_q$, $q \neq p$, then we have $d_{G \odot H}(x, A_q) = 1 + d_G(v_i, A_q) > 1 = d_G(y, A_q) = d_{G \odot H}(y, A_q)$.

Thus, for every two different vertices x, y of $G \odot H$ we have $r(x|\Pi) \neq r(y|\Pi)$ and, as a consequence, Π is a resolving partition for $G \odot H$. □

Corollary 6. *Let G and H be two connected graphs of order $n_1 \geq 2$ and $n_2 \geq 2$, respectively. If $D(H) \leq 2$, then*

$$pd(G \odot H) \leq dim(G) + dim(H) + 2.$$

In the next section we will show that all the above inequalities are tight.

3 Minorizing $pd(G \odot H)$

Theorem 7. *Let G and H be two connected graphs. Let Π be a resolving partition of $G \odot H$ of minimum cardinality. Let $H_i = (V_i, E_i)$ be the subgraph of $G \odot H$ corresponding to the i^{th}-copy of H, and let Π_i be the set composed by all non-empty sets of the form $S \cap V_i$, where $S \in \Pi$. Then Π_i is a resolving partition for H_i.*

Proof. If Π_i is composed by sets of cardinality one, then the result immediately follows. Now, let x, y be two different vertices of H_i belonging to the same set of Π. We know that there exists $S \in \Pi$ such that $d_{G\odot H}(x, S) \neq d_{G\odot H}(y, S)$. By Lemma 1 (i) we have that for every vertex v of $G \odot H$ not belonging to V_i, it follows that $d_{G\odot H}(x, v) = d_{G\odot H}(y, v)$. Hence we conclude $S' = S \cap V_i \neq \emptyset$ and we can assume, without loss of generality, that $d_{G\odot H}(x, S) = 1$ and $d_{G\odot H}(y, S) = 2$. As a result, $S' \in \Pi_i$ and $d_{H_i}(x, S') = d_{G\odot H}(x, S) = 1 < 2 = d_{G\odot H}(y, S) \leq d_{H_i}(y, S')$. Therefore, the result follows. □

Corollary 8. *For any connected graphs G and H,*

$$pd(G \odot H) \geq pd(H).$$

It is easy to check that for the star graph $K_{1,n}$, $n \geq 2$, it follows $pd(K_{1,n}) = n$. So the following result shows that the above inequality is tight.

Proposition 9. *Let G denote a connected graph of order n_1 and let n be an integer. If $n \geq 2n_1 \geq 4$ or $n > 2n_1 = 2$, then*

$$pd(G \odot K_{1,n}) = n.$$

Proof. Let us suppose $n \geq 2n_1 \geq 4$. For each $v_i \in V$, let $\{a_i, u_{i1}, u_{i2}, ..., u_{in}\}$ be the set of vertices of the i^{th} copy of $K_{1,n}$ in $G \odot K_{1,n}$, where a_i is the vertex of degree n.

We will show that $\Pi = \{S_1, S_2, ..., S_n\}$ is a resolving partition for

$G \odot K_{1,n}$, where

$$\begin{aligned} S_1 &= \{a_1, u_{11}, u_{21}, ..., u_{n_1 1}\}, \\ S_2 &= \{v_1, u_{12}, u_{22}, ..., u_{n_1 2}\}, \\ S_3 &= \{a_2, u_{13}, u_{23}, ..., u_{n_1 3}\}, \\ S_4 &= \{v_2, u_{14}, u_{24}, ..., u_{n_1 4}\}, \\ &\vdots \\ S_{2n_1} &= \{v_{n_1}, u_{1(2n_1)}, u_{2(2n_1)}, ..., u_{n_1(2n_1)}\}, \\ S_{2n_1+1} &= \{u_{1(2n_1+1)}, u_{2(2n_1+1)}, ..., u_{n_1(2n_1+1)}\}, \\ &\vdots \\ S_n &= \{u_{1n}, u_{2n}, ..., u_{n_1 n}\}. \end{aligned}$$

Let x, y be two different vertices of $G \odot K_{1,n}$. We differentiate three cases. Case 1: $x = u_{il}$ and $y = u_{jl}$, $i \neq j$. If $l \neq 2i - 1$, then

$$d(u_{il}, S_{2i-1}) = d(u_{il}, a_i) = 1 < 2 = d(u_{jl}, u_{j(2i-1)}) = d(u_{jl}, S_{2i-1}).$$

If $l = 2i - 1$, then

$$d(u_{jl}, S_{2j-1}) = d(u_{jl}, a_j) = 1 < 2 = d(u_{il}, u_{i(2j-1)}) = d(u_{il}, S_{2j-1}).$$

Case 2: $x = v_i$ and $y = u_{j(2i)}$. If $j = i$, then

$$d(v_i, S_i) = d(v_i, u_{ii}) = 1 < 2 = d(u_{i(2i)}, u_{ii}) = d(u_{i(2i)}, S_i).$$

If $j \neq i$, then

$$d(v_i, S_i) = d(v_i, u_{ii}) = 1 < 2 = d(u_{j(2i)}, u_{ji}) = d(u_{j(2i)}, S_i).$$

Case 3: $x = a_i$ and $y = u_{j(2i-1)}$. If $j = i$, then

$$d(a_i, S_i) = d(a_i, u_{ii}) = 1 < 2 = d(u_{i(2i-1)}, u_{ii}) = d(u_{i(2i-1)}, S_i).$$

If $j \neq i$, then

$$d(a_i, S_i) = d(a_i, u_{ii}) = 1 < 2 = d(u_{j(2i-1)}, u_{ji}) = d(u_{j(2i-1)}, S_i).$$

Therefore, we conclude that Π is a resolving partition for $G \odot K_{1,n}$.

For $n_1 = 1$ and $n \geq 3$ we denote by v the vertex of G, by a the vertex of $K_{1,n}$ of degree n, and by $\{u_1, u_2, ..., v_n\}$ the set of leaves of $K_{1,n}$. Thus, from $d(v, u_3) = 1 < 2 = d(u_2, u_3)$ and $d(a, u_3) = 1 < 2 = d(u_1, u_3)$, we conclude that $\Pi = \{S_1, S_2, ..., S_n\}$ is a resolving partition for $G \odot K_{1,n}$, where $S_1 = \{a, u_1\}$, $S_2 = \{v, u_2\}$, $S_3 = \{u_3\}$, ..., $S_n = \{u_n\}$. □

Lemma 10. *Let G be a connected graph. If Π is a resolving partition for $G \odot K_n$ of cardinality $n+1$, then for every vertex v of $G \odot K_n$ and every $A \in \Pi$, it follows $d(v, A) \leq 3$.*

Proof. Let v_i, v_j be two adjacent vertices of G and let $H_l = (V_l, E_l)$ ($l \in \{i, j\}$) be the copy of K_n in $G \odot K_n$ such that v_l is adjacent to every vertex of H_l. If there exists a vertex v of the subgraph of $G \odot K_n$ induced by $V_i \cup V_j \cup \{v_i, v_j\}$ such that $d(v, A) > 3$, for some $A \in \Pi$, then, since different vertices of V_i (respectively, V_j) belong to different sets of Π, there exist $B, C \in \Pi$, $u_i \in V_i$ and $u_j \in V_j$ such that $u_i, v_i \in B$ and $u_j, v_j \in C$.

If $B = C$, then $d(u_i, A) = d(v_j, A)$ or $d(v_i, A) = d(u_j, A)$. Hence, $r(u_i|\Pi) = r(v_j|\Pi)$ or $r(v_i|\Pi) = r(u_j|\Pi)$, a contradiction. If $B \neq C$, then there exist two vertices $u'_i \in V_i \cap C$ and $u'_j \in V_j \cap B$ and, as a consequence, then $d(u'_i, A) = d(v_j, A)$ or $d(v_i, A) = d(u'_j, A)$. Thus, $r(u'_i|\Pi) = r(v_j|\Pi)$ or $r(v_i|\Pi) = r(u'_j|\Pi)$, a contradiction. Therefore, $d(v, A) \leq 3$, for every $A \in \Pi$. □

Given a graph H which contains a connected component isomorphic to a complete graph, we denote by $c(H)$ the maximum cardinality of any connected component of H which is isomorphic to a complete graph.

Theorem 11. *Let G be a connected graph of order n. Then for any graph H such that $n > 2c(H) + 1 \geq 5$,*

$$pd(G \odot H) \geq c(H) + 2.$$

Proof. We denote by S_i a connected component of H_i isomorphic to $K_{c(H)}$, $i \in \{1, ..., n\}$. Since different vertices of S_i belong to different sets of any resolving partition for $G \odot H$, we conclude $pd(G \odot H) \geq c(H)$. If $pd(G \odot H) = c(H)$, then there exist two vertices $a, b \in S_i \cup \{v_i\}$ such that they belong to the same set of any resolving partition for $G \odot H$. Thus, a and b have the same partition representation, which is a contradiction. So, $pd(G \odot H) \geq c(H) + 1$. Now, let us suppose $pd(G \odot H) = c(H) + 1$ and let $\Pi(G \odot H) = \{A_1, A_2, ..., A_{c(H)+1}\}$ be a resolving partition for $G \odot H$. Now, let $S = \bigcup_{i=1}^{n} (S_i \cup \{v_i\})$ and let $u \in S$. Suppose $u \in A_j$, $j \in \{1, ..., c(H)+1\}$. So, we have that the partition representation of u is given by

$$r(u|\Pi) = (1, 1, ..., \underbrace{1, 0, 1}_{j}, ..., \underbrace{1, t, 1}_{i}, ..., 1),$$

where $i, j \in \{1, ..., c(H) + 1\}$, $i \neq j$, and, by Lemma 10, $t \in \{1, 2, 3\}$. Since for every different vertices $a, b \in S$, $r(a|\Pi) \neq r(b|\Pi)$, the maximum number of possible different partition representations for vertices of S is given by $(c(H)+1)(2c(H)+1)$, i.e., for $t=1$ there are at most $c(H)+1$ different vectors and for $t \in \{2,3\}$ there are at most $2(c(H)+1)c(H)$. Hence, $n(c(H)+1) = |S| \le (2c(H)+1)(c(H)+1)$ and, as a consequence, $n \le 2c(H)+1$. Therefore, if $n > 2c(H)+1$, then $pd(G \odot H) \ge c(H)+2$. □

Corollary 12. *Let G be a graph of order n_1 and let $n_2 \ge 2$ be an integer. If $n_1 > 2n_2+1$, then*

$$pd(G \odot K_{n_2}) \ge n_2 + 2.$$

From Theorem 5 and Corollary 12 we obtain that if $n_1 > 2n_2+1 \ge 5$, then $pd(G) + n_2 \ge pd(G \odot K_{n_2}) \ge n_2 + 2$. Therefore, since the partition dimension of a path P_n with $n > 1$ vertices is two, we obtain the following result.

Remark 13. *Let n_1 and n_2 be integers such that $n_1 > 2n_2+1 \ge 5$. Then*

$$pd(P_{n_1} \odot K_{n_2}) = n_2 + 2.$$

By Remark 13 we conclude that the inequalities in Theorem 2, Corollary 4, Theorem 5, Corollary 6 and Corollary 12 are tight.

An empty graph of order n, denoted by N_n, consists of n isolated nodes with no edges. In the following result $\beta(H)$ denotes the number of isolated vertices of a graph H.

Theorem 14. *Let G be a connected graph of order $n \ge 2$ and let H be any graph. If $n > \beta(H) \ge 2$, then*

$$pd(G \odot H) \ge \beta(H) + 1.$$

Proof. We will proceed similarly to the proof of Theorem 11. Let S_i denote the set of isolated vertices of H_i, $i \in \{1,...,n\}$.

Since different vertices of S_i belong to different sets of any resolving partition for $G \odot H$, we have $pd(G \odot H) \ge \beta(H)$. Let us suppose $pd(G \odot H) = \beta(H)$ and let $\Pi(G \odot H) = \{A_1, A_2, ..., A_{\beta(H)}\}$ be a resolving partition for $G \odot H$. Now, let $S = \bigcup_{i=1}^{n}(S_i \cup \{v_i\})$ and let $u \in S$. If $u \in A_j \cap S_j$, $j \in \{1,...,n_1\}$, then the partition representation of u is given by

$$r(u|\Pi) = (2,2,..., \underbrace{2,0,2}_{j}, ..., \underbrace{2,t,2}_{i}, ...,2),$$

with $i,j \in \{1,...,\beta(H)\}$, $i \ne j$ and $t \in \{1,2\}$. On the other side, if $u \in A_j \cap V$, then

$$r(u|\Pi) = (1,1,..., \underbrace{1,0,1}_{j}, ...,1),$$

with $j \in \{1, ..., \beta(H)\}$. Thus, the maximum number of possible different partition representations for vertices of S is given by $(\beta(H)+1)\beta(H)$. Hence, $n(\beta(H)+1) = |S| \leq \beta(H)(\beta(H)+1)$. Thus, $n \leq \beta(H)$. Therefore, if $n > \beta(H)$, then $pd(G \odot H) \geq \beta(H)+1$. □

Corollary 15. *Let G be a graph of order n_1 and let $n_2 \geq 2$ be an integer. If $n_1 > n_2$, then*

$$pd(G \odot N_{n_2}) \geq n_2 + 1.$$

Proposition 16. *If $n_1 \geq n_2 \geq 2$, then*

$$pd(P_{n_1} \odot N_{n_2}) = n_2 + 1.$$

Proof. Let $V = \{v_1, ..., v_n\}$ be the set of vertices of P_{n_1} and, for each $v_i \in V$, let $V_i = \{u_{i1}, ..., u_{in_2}\}$ be the set of vertices of the i^{th} copy of N_{n_2} in $P_{n_1} \odot N_{n_2}$. Let $\Pi = \{A_1, ..., A_{n_2+1}\}$, where $A_1 = \{v_1, u_{11}\}$, $A_2 = \{v_i, u_{i1} : i \in \{2, ..., n_1\}\}$ and $A_j = \{u_{i(j-1)} : i \in \{1, ..., n_1\}\}$ for $j \in \{3, .., n_2 + 1\}$. Note that $d_{P_{n_1} \odot N_{n_2}}(v_1, A_2) \neq d_{P_{n_1} \odot N_{n_2}}(u_{11}, A_2)$. Moreover, for two different vertices $x, y \in A_j$, $j \in \{3, ..., n_2 + 1\}$, we have $d_{P_{n_1} \odot N_{n_2}}(x, A_1) \neq d_{P_{n_1} \odot N_{n_2}}(y, A_1)$. Now on we suppose $x, y \in A_2$. If $x, y \in V$ or $x, y \in V_i$, for some i, then $d_{P_{n_1} \odot N_{n_2}}(x, A_1) \neq d_{P_{n_1} \odot N_{n_2}}(y, A_1)$. Finally, if $x \in V$ and $y \notin V$, then $d_{P_{n_1} \odot N_{n_2}}(x, A_3) \neq d_{P_{n_1} \odot N_{n_2}}(y, A_3)$. Therefore, Π is a resolving partition for $P_{n_1} \odot N_{n_2}$ and, as a consequence, $pd(P_{n_1} \odot N_{n_2}) \leq n_2 + 1$. By corollary 15 we conclude the proof. □

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