Paired-domination in claw-free graphs with minimum degree at least four*

Xinguo Cao[1], Erfang Shan[1,2†]
[1]School of Management, Shanghai University, Shanghai 200444, China
[2]Department of Mathematics, Shanghai University, Shanghai 200444, China

ABSTRACT. A paired-dominating set of a graph G is a dominating set of vertices whose induced subgraph has a perfect matching. The paired-domination number is the minimum cardinality of a paired-dominating set of G. In this paper we investigate the paired-domination number in claw-free graphs with minimum degree at least four. We show that a connected claw-free graph G with minimum degree at least four has paired-domination number at most four-sevenths its order.

Keywords: Paired-domination; Claw-free; Minimum degree

MSC: 05C69

1 Introduction

In this article, we continue the study of paired-domination in graphs. Paired-domination in graphs is now well studied in graph theory. The literature on this subject has been surveyed and detailed in the two books by Haynes et al. [8, 9].

A *matching* in a graph G is a set of independent edges in G. A *perfect matching* M is a matching such that every vertex of G is incident with an edge of M. A *paired-dominating set*, abbreviated PDS, of G is a set S of vertices of G such that every vertex is adjacent to some vertex in S and the subgraph $G[S]$ induced by S contains a perfect matching. (not necessarily induced). Clearly, every graph without isolated vertices has a PDS, since the end-vertices of any maximal matching form such a PDS.

*Research was partially supported by the National Nature Science Foundation of China (Nos. 11426144, 11571222).

† *Corresponding author. Email address: efshan@shu.edu.cn*

The *paired-domination number* is the minimum cardinality of a paired-dominating set of G, denoted by $\gamma_{pr}(G)$. Paired-domination was introduced by Haynes and Slater [10, 11] as a model for assigning backups to guards for security purposes. Recent progress on this subject can be found in [2, 3, 5, 6, 12, 13, 16] and elsewhere.

The decision problem to determine the paired-domination number of a graph is known to be NP-complete [10]. Hence it is of interest to determine bounds on the paired-domination number of graphs. Upper bounds on the paired-domination number of graphs in terms of their order under the minimum degree condition have been investigated in recent years (see, [1, 7, 10, 12]). In [7] Goddard and Henning posed the following conjecture.

Conjecture 1. (Goddard and Henning [7]) *If $G \neq P$ is a connected graph of order n with minimum degree $\delta(G) \geq 3$, then $\gamma_{pr}(G) \leq \frac{4}{7}n$, where P is the Petersen graph.*

In this paper we show that Conjecture 1 is true for claw-free graphs with minimum degree at least four. Our main result is as follows.

Theorem 2. *If G is a connected claw-free graph of order n with minimum degree $\delta(G) \geq 4$, then $\gamma_{pr}(G) \leq \frac{4}{7}n$.*

For notation and graph theory terminology we in general follow [8]. Specifically, let $G = (V, E)$ be a graph with *vertex set* V and *edge set* E. For a set $S \subseteq V$, the subgraph induced by S is denoted by $G[S]$. We denote the *degree* of a vertex v in G by $d_G(v)$, and the *minimum degree* among the vertices of G is denoted by $\delta(G)$. The *open neighborhood* of a vertex $v \in V$ is denoted by $N(v) = \{u \in V | uv \in E\}$, and the *closed neighborhood* of v is denoted by $N[v] = \{v\} \cup N(v)$. For a set $S \subseteq V$, the open neighborhood of S is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighborhood of S is $N[S] = S \cup N(S)$. For a vertex $x \in V$, we define $N_S(x) = N(x) \cap S$. If X and Y are two subsets of V, we say that X *dominates* Y if $Y \subseteq N[X]$, and denote by $[X, Y]$ the set of edges between X and Y. A graph is called *claw-free* if it does not contain the complete bipartite graph $K_{1,3}$ as an induced subgraph. Let P_n denote a *path* on n vertices. A component of a disconnected graph H is called an *F-component* of H if it is isomorphic to a given graph F.

We shall proceed as follows. In Section 2, we start with some preliminary results that will help to prove our main result, and define a weight function on which our proof is based. In Section 3, we prove our main result.

2 Preliminary results

In this section we first describe the structure of a special minimum PDS S of G. Secondly, we define a weight function and give some properties of this function.

2.1 The paired-dominating set S

From now on, we begin to consider a connected claw-free graph with minimum degree at least four. Let $G = (V, E)$ be a connected claw-free graph of order n with $\delta(G) \geq 4$. For a subset $T \subseteq V$, let $\lambda(T)$ be the number of edges in $G[T]$. Among all the minimum paired-dominating sets of G, let us choose a PDS S of G so that $\lambda(S)$ is minimized.

Fix some perfect matching M in $G[S]$. For each $v \in S$, let $\overline{v}$ denote the vertex such that $v\overline{v} \in M$. We define $\overline{v}$ as the *partner* of v. Let $S_v = \{v, \overline{v}\}$. We will refer S_v as a pair in S. For a vertex $v \in S$, the *S-private neighborhood* of v is the set $pn(v, S) = N[v] \setminus N[S \setminus \{v\}]$. We call a vertex $u \in pn(v, S)$ an *S-private neighbor* of v. Obviously, $pn(v, S) \subseteq V \setminus S$.

We partition S into four subsets as follows:

$A = \{v \in S|$ both v and $\overline{v}$ have an S-private neighbor$\}$,

$B = \{v \in S|$ v has an S-private neighbor and $\overline{v}$ has no S-private neighbor$\}$,

$C = \{v \in S | \overline{v} \in B\}$,

$D = \{v \in S|$ neither v nor $\overline{v}$ has an S-private neighbor$\}$.

Furthermore, we define a pair of vertices in S (that are partners) to be as follows:

- an *A-pair* if both belong to A;
- a *BC-pair* if one belongs to B and the other to C; and
- a *D-pair* if both belong to D.

According to the definition above, we can see that each pair of S is either an A-pair or a BC-pair or a D-pair. By the choice of S and the claw-freeness of G, the following Claims 1-5 are given by Huang and Shan in [15], which are useful in our proof.

Claim 1. *Each vertex in C is a degree-1 vertex in $G[S]$.*

Claim 2. *Each vertex in B is a degree-1 vertex in $G[S]$.*

By Claims 1 and 2, every BC-pair is a P_2-component in $G[S]$.

Claim 3. *At least one vertex from every D-pair is a degree-1 vertex in $G[S]$.*

Claim 4. *If there is an edge joining vertices from two distinct D-pairs, then the set of these four vertices forms a P_4-component in $G[S]$.*

By Claim 4, we see that if a D-pair is joined to another D-pair of S, then it must be joined to such a unique D-pair. We define a D-pair to be a *linked D-pair* if it is joined to some other D-pair by an edge; and a *solo D-pair* if it is not a linked D-pair.

For two disjoint vertex subsets X and Y of V, we call that X is adjacent to Y if $[X, Y] \neq \emptyset$. Clearly, if an A-pair is adjacent to D-pairs, then it is only adjacent to solo D-pairs by Claim 4. More precisely, we have

Claim 5. *Each A-pair is adjacent to at most one solo D-pair.*

2.2 Weight functions

Now by the claw-freeness of G and the Claims above, we can give the following lemma.

Lemma 1. *For each $x \in V \setminus S$, we have*

(1) $|N_C(x)| \leq 2$. If $|N_C(x)| = 2$, then x has no neighbor in D; and if $|N_C(x)| = 1$, then x has at most two neighbors in D, actually, if x has two neighbors in D, the two neighbors must be adjacent vertices.

(2) $|N_{C \cup D}(x)| \leq 4$. If $|N_{C \cup D}(x)| = 4$, then x has no neighbor in C.

(3) If x has at least three neighbors in some P_4-component H in $G[S]$ formed by two linked D-pairs, then $N_S(x) \subseteq V(H)$.

(4) If x is adjacent to some vertex $v \in D$ of degree 1 in $G[S]$ but not to $\overline{v}$, then $|N_{C \cup D}(x)| \leq 3$.

Proof. Let x be a vertex in $V \setminus S$.

(1) Suppose that $|N_C(x)| > 2$. Then x has at least three neighbors in C, say a, b and c. By Claim 1, each one of a, b and c has degree 1 in $G[S]$, thus a claw occurs at x, a contradiction. Similar reasoning shows that if $|N_C(x)| = 2$, then x has no neighbor in D; and if $|N_C(x)| = 1$, then x has a neighbor or two adjacent neighbors in D.

(2) Suppose that $|N_{C \cup D}(x)| > 4$. By Claims 1, 3, 4 and the result in (1), there would be a claw at x, a contradiction. Specially, if $|N_{C \cup D}(x)| = 4$, the four neighbors of x must be two linked D-pairs.

(3) The result follows directly from (2).

(4) The result follows directly from the claw-freeness of G. □

We are going to prove our main result by using the weight function given below. We begin to define a weight function ω on all the edges between S and $V \setminus S$. The weight function is defined so that for each vertex in $V \setminus S$, the total weight of the edges incident with it sums to 1. Hence the total weight of all edges incident with vertices in $V \setminus S$ is equal to $n - |S|$. At the same time, we calculate the weight of edges incident with each pair in S. If we can prove that the sum of the weight of edges incident with each pair in S is at least $\frac{3}{2}$, then the total weight of edges between S and $V \setminus S$ is at least $\frac{3}{4}|S|$. Since the total weight is exactly $n - |S|$, it follows that $|S| \leq \frac{4}{7}n$.

Now we begin to formally define our weight function ω: $[S, V \setminus S] \to [0, 1]$. For each vertex $x \in V \setminus S$, the weights of the edges that from x to S are defined as follows:

(1) If x is an S-private neighbor, then assign the weight 1 to the unique edge from x to S.

(2) Assume that x is not an S-private neighbor. If x has no neighbor in $C \cup D$, share the weight 1 equally among the edges from x to $A \cup B$.

(3) Assume that x has at least one neighbor in $C \cup D$. If $|N_{C\cup D}(x)| = 1$, assign the weight 1 to the unique edge from x to C or D; and assign the weight 0 to each edge from x to $A \cup B$.

(4) Assume that x has at least two neighbors in $C \cup D$. If $|N_C(x)| = 2$, assign the weight $\frac{1}{2}$ to each edge from x to C; if $|N_C(x)| = 1$, then assign the weight $\frac{1}{6}$ to the edge from x to C, and share the weight $\frac{5}{6}$ equally among the edges (one or two, by Lemma 1 (1)) from x to D. If $|N_C(x)| = 0$, then share the weight 1 equally among the edges from x to D. In these three cases, assign the weight 0 to each edge from x to $A \cup B$.

From the definition of the weight function ω, we can straightly get the following claim:

Claim 6. *Let $x \in V \setminus S$ and let e be an edge from x to S. Then the following properties hold:*

(1) The sum of the weights assigned to the edges from x to S is 1.

(2) If e joins x to C, then $\omega(e) \in \{\frac{1}{6}, \frac{1}{2}, 1\}$, and $\omega(e) = \frac{1}{6}$ if and only if $|N_{C\cup D}(x)| \geq 2$ and $|N_C(x)| = 1$.

(3) If e joins x to D, then $\omega(e) \in \{\frac{1}{4}, \frac{1}{3}, \frac{5}{12}, \frac{1}{2}, \frac{5}{6}, 1\}$, and $\omega(e) = \frac{1}{4}$ if and only if $|N_D(x)| = 4$.

We next define a function f that assigns to each subset $S' \subseteq S$ the sum of the weights of the edges from S' to $V \setminus S$; that is,

$$f(S') = \sum_{e \in [S', V\setminus S]} \omega(e).$$

Specially, if $S' = S$, then $f(S)$ is the sum of the weight of all edges in $[S, V \setminus S]$ (namely, $|V \setminus S|$).

Finally, we define another function g that assigns to each pair $S_v = \{v, \overline{v}\}$ in S the weight as follows:

$$g(S_v) = \begin{cases} f(S_v) - \frac{1}{4}|[S_v, D]| & \text{if } v \in A; \\ f(S_v) + \frac{1}{4}|[S_v, A]| & \text{if } v \in D; \\ f(S_v) & \text{if } v \in B \text{ or } v \in C. \end{cases}$$

3 Proof of main result

In this section we are going to prove our main result. Before the proof of Theorem 2, we need to establish a lemma by the definition of weight function g.

Lemma 2. *Let S be a minimum PDS of G such that it is chosen and partitioned into four groups A, B, C and D in the same way as we did in the last section. Then for each pair S_v of vertices in S, we have $g(S_v) \geq \frac{3}{2}$.*

Proof. We consider each type of the pair S_v.

Suppose that the pair S_v is an A-pair. Then by Claim 5, we have $|[S_v, D]| \leq 2$. So $g(S_v) = f(S_v) - \frac{1}{4}|[S_v, D]| \geq 2 - 2 \times \frac{1}{4} = \frac{3}{2}$.

Suppose that the pair S_v is a BC-pair. Without loss of generality, let $v \in B$ and $\overline{v} \in C$. Since $\delta(G) \geq 4$ and $\overline{v}$ is a degree-1 vertex in $G[S]$, $\overline{v}$ has at least three neighbors in $V \backslash S$. By Claim 6(2), $g(S_v) = f(S_v) \geq 1 + 3 \times \frac{1}{6} = \frac{3}{2}$.

Suppose that the pair S_v is a solo D-pair, by Claim 3, we may assume that $\overline{v}$ is a degree-1 vertex in $G[S]$. Thus $\overline{v}$ has at least three neighbors outside S. By Claim 6(3), we have $\omega(e) \geq \frac{1}{4}$ for each edge $e \in [S_v, V \setminus S]$. We know that v is only adjacent to vertices of A-pairs in S. If v is adjacent to at least two A-pairs, then $|[S_v, A]| \geq 4$. So $g(S_v) = f(S_v) + \frac{1}{4}|[S_v, A]| \geq 3 \times \frac{1}{4} + \frac{1}{4} \times 4 > \frac{3}{2}$. If S_v is adjacent to exactly one A-pair, then $|[S_v, A]| = 2$. Since $\delta(G) \geq 4$, v has at least one neighbor outside S. So $g(S_v) = f(S_v) + \frac{1}{4}|[S_v, A]| \geq 4 \times \frac{1}{4} + \frac{1}{4} \times 2 = \frac{3}{2}$. If S_v has no neighbor in $G[S]$, that is, $|[S_v, A]| = 0$, then we have that $g(S_v) = f(S_v)$. Since $\delta(G) \geq 4$, both v and $\overline{v}$ have at least three neighbors in $V \setminus S$. So $g(S_v) = f(S_v) \geq 6 \times \frac{1}{4} = \frac{3}{2}$.

Suppose that S_v and S_u are linked D-pairs such that $uv \in E$. Recall that a linked D-pair forms a P_4-component of $G[S]$. We consider the set $S' = S \setminus \{\overline{v}, \overline{u}\}$. Since $|S'| < |S|$, S' is not a PDS of G, so there exists at least one vertex $x \in V \setminus S$ such that $N_S(x) \subseteq \{\overline{v}, \overline{u}\}$. Furthermore, since neither $\overline{v}$ nor $\overline{u}$ has an S-private neighbor, there exists one vertex $x \in V \setminus S$ such that $N_S(x) = \{\overline{v}, \overline{u}\}$. By the definition of the function f, we have that $\omega(\overline{v}x) = \omega(\overline{u}x) = \frac{1}{2}$. Since $\delta(G) \geq 4$, both v and u have at least two neighbors in $V \setminus S$, and both $\overline{v}$ and $\overline{u}$ have at least two neighbors in $V \setminus S$ other than x. So $g(S_v) = f(S_v) \geq \frac{1}{2} + 4 \times \frac{1}{4} = \frac{3}{2}$. □

Proof of Theorem 2. Let S_P be a subset of S that consists of one vertex from each pair in S. By Lemma 2, we have

$$\sum_{v \in S_P} g(S_v) \geq \frac{3}{2}|S_P| = \frac{3}{2} \times \frac{|S|}{2} = \frac{3}{4}|S|.$$

Since

$$\bigcup_{v \in A} [S_v, D] = [A, D] = \bigcup_{v \in D} [S_v, A],$$

we have that

$$\sum_{v \in S_P} g(S_v) = \sum_{v \in S_P} f(S_v) - \frac{1}{4}|[A, D]| + \frac{1}{4}|[A, D]| = f(S) = n - |S|.$$

Therefore, $n - |S| \geq \frac{3}{4}|S|$, that is, $\gamma_{pr}(G) = |S| \leq \frac{4}{7}n$. □

Acknowledgment

The authors would like to thank the anonymous referee for valuable comments.

References

[1] X.G. Chen, L. Sun and H.M. Xing, Paired-domination numbers of cubic graphs (Chinese), *Acta Math. Sci. Ser. A Chin. Ed.* 27 (2007) 166–170.

[2] X.G. Chen, C.S. Wai, H.C. Wai, Upper bounds on the paired-domination number, *Applied Mathematics Letters* 21 (2008) 1194–1198.

[3] T.C.E. Cheng, L.Y. Kang and C.T. Ng, Paired domination on interval and circular-arc graphs, *Discrete Applied Mathematics* 155 (2007) 2077–2086.

[4] P. Dorbec, M.A. Henning, Upper paired-domination in claw-free graphs, *Journal of Combinatorial Optimization* 22 (2011) 235–251.

[5] P. Dorbec, S. Gravier, M.A. Henning, Paired-domination in generalized claw-free graphs, *Journal of Combinatorial Optimization* 14 (2007) 1–7.

[6] O. Favaron, M.A. Henning, Paired domination in claw-free cubic graphs, *Graph and Combinatorics* 20 (2004) 447–456.

[7] W. Goddard, M.A. Henning, A characterization of cubic graphs with paired-domination number three-fifths their order, *Graph and Combinatorics* 25 (2009) 675–692.

[8] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[9] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York, 1998.

[10] T.W. Haynes, P.J. Slater, Paired-domination in graphs. *Networks* 32 (1998) 199–206.

[11] T.W. Haynes, P.J. Slater, Paired-domination and the paired-domatic number, *Congr. Numer.* 109 (1995) 65–72.

[12] M.A. Henning, Graphs with large paired-domination number, *Journal of Combinatorial Optimization* 13 (2007) 61–78.

[13] M.A. Henning, M.D. Plummer, Vertices contained in all or in no minimum paired-dominating set of a tree, *Journal of Combinatorial Optimization* 10 (2005) 283–294.

[14] S.W. Huang, E.F. Shan, A note on the upper bound for the paired-domination number of a graph with minimum degree at least two, *Networks* 57 (2) (2011) 115–116.

[15] S.W. Huang, E.F. Shan, Paired-domination in claw-free graphs, manuscript.

[16] H. Qiao, L.Y. Kang, M. Cardei and D.Z. Du, Paired-domination of trees, *Journal of Global Optimization* 25 (2003) 43–54.