A New Characterization of Disk Graphs and its Application

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Abstract

An $L(2,1)$-labeling of a graph G is a function f from the vertex set $V(G)$ to the set of all nonnegative integers such that $|f(x)-f(y)| \geq 2$ if $d(x,y) = 1$ and $|f(x) - f(y)| \geq 1$ if $d(x,y) = 2$, where $d(x,y)$ denotes the distance between x and y in G. The $L(2,1)$-labeling number, $\lambda(G)$, of G is the smallest number k such that G has an $L(2,1)$-labeling f with $\max\{f(v) : v \in V(G)\} = k$. In this paper, we present a new characterization on d-disk graphs for $d > 1$. As an application, we give upper bounds on the $L(2,1)$-labeling number for this classes of graphs.

Keywords: frequency assignment, $L(2,1)$-labeling, unit disk graph, disk graph

AMS Subject Classification (2001): 05C15, 05C78

1 Introduction

With the development of information technology, wireless communication devices have become of widely use. Most of these devices are radio transmitters which can emit and receive signals. If any two of these devices are assigned nearby frequencies, they may interfere with one another. To maintain satisfactory communication quality, any pair of devices have to be assigned frequencies chosen so that no interference can take place; this is the frequency assignment problem.

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To describe the frequency assignment problem more precisely, Roberts considered two levels of interference: the first level is when two transmitters are "very close" so they must be assigned frequencies that differ by no less than two units; the second level is when transmitters are farther apart, but they still could interfere with each other so they must be assigned frequencies that differ by at least one unit. This description can be modelled as a graph problem: the devices are the vertices of a graph; two vertices are considered "very close" if they are at distance one and they are just "close" if they are at distance two. Motivated by Roberts' model, Griggs and Yeh [12] defined an $L(2,1)$-*labeling* of a graph G as a function f from the vertex set $V(G)$ to the nonnegative integers such that $|f(x)-f(y)| \geq 2$ if x and y are at distance one from one another and $|f(x) - f(y)| \geq 1$ if x and y are at distance two from each other. A k-$L(2,1)$-*labeling* is an $L(2,1)$-labeling such that all labels are at most k. The $L(2,1)$-*labeling number* of G, denoted by $\lambda(G)$, is the smallest number k such that G has a k-$L(2,1)$-labeling.

The $L(2,1)$-labeling problem has been generalized to the $L(h,k)$-labeling problem (and $L(p_1,p_2,...,p_r)$-labeling problem) and a large amount of research has been devoted to L(2,1)-labelings and their generalizations (see references). The problem of computing the $L(2,1)$-labeling number of a graph is a generalization of the vertex coloring problem, which is known to be NP-hard. There are only a few general bounds known for $\lambda(G)$ and most research has centered on computing $\lambda(G)$ for particular classes of graphs. Griggs and Yeh [12] showed that $\lambda(G) \leq \Delta^2$ for any diameter 2 graph G and they conjectured that $\lambda(G) \leq \Delta^2$ for any graph G with maximum degree $\Delta \geq 2$ [12] ($\Delta = 1$ is an exception, since for example, $\Delta(K_2) = 1$ but $\lambda(K_2) = 2$.) Griggs and Yeh [12] gave an upper bound of $\Delta^2 + 2\Delta$ for the $L(2,1)$-labeling number of any graph with maximum degree Δ. Chang and Kuo [5] improved the bound to $\Delta^2+\Delta$, and then Král' and Škrekovski [17] reduced the bound to $\Delta^2 + \Delta - 1$. Goncalves [11] further reduced the bound to $\Delta^2+\Delta-2$. In 2008, Havet, Reed and Sereni [14] proved that the conjecture of Griggs and Yeh is true for large values of Δ.

Roberts [22] and Sakai [23] pointed out that the class of unit interval graphs and its generalization, the class of unit d-disk graphs, are of particular interest in the frequency assignment problem. When transmitters are located in R^d, for $d = 1, 2$ or 3, interference takes place if two transmitters are within a certain distance from each other, so interfering transmitters can be conveniently represented with a unit d-disk graph.

Disk graphs and unit disk graphs have been extensively studied due to their many applications ([6], [15], [10]). The problem of deciding whether a given graph is a disk graph or a unit disk graph has been proved to be NP-hard ([3], [15]). The $L(1,1)$-labeling problem (also called the conflict-free channel assignment problem) is like the $L(2,1)$-labeling problem, except that any two neighboring vertices x, y just need to receive different labels, i.e. $|f(x) - f(y)| \geq 1$. The $L(1,1)$-labeling problem on unit disk graphs is NP-hard ([24]). Sen and Malesinska [25] provided an approximation algorithm with a performance guarantee of 14 for the $L(1,1)$-labeling of disk graphs, which was improved by Wan et al. [34] to 13 (12 if the radii are quasi-uniform) by using FIRST-FIT algorithms. In [25] and [34], approximation algorithms were given for the $L(1,1)$-labeling of unit disk graphs with performance guarantee of 7. For unit disk graphs whose vertices lie in a horizontal strip with height $\sqrt{3}/2$ it is proved in [34] that the $L(1,1)$-labeling problem can be solved in polynomial time. Fiala, Fishkin and Fomin [8] designed an approximation algorithm with performance ratio bounded by 12 for the $L(2,1)$-labeling problem on disk graphs. For the $L(2,1)$-labeling problem on unit disk graphs, they gave a labeling algorithm with constant performance ratio bounded by 32/3. Their algorithm does not require the disk representation of the graph and it either outputs a feasible labeling, or answers that the input is not a unit disk graph. They also studied the more general $L(h,k)$-labeling problem on disk graphs and gave an approximation algorithm with performance ratio depending on the diameter ratio, the ratio between the maximum and the minimum diameters of disks.

Despite their importance, no useful characterizations of unit d-disk graphs are known except for $d = 1, 2, 3$. Sakai [23] gave upper bounds for the $L(2, 1)$-labeling number of unit 1-disk graphs. Recently, Shao et al. [30] characterized unit d-disk graphs for $d = 2, 3$ and gave upper bounds for the $L(2, 1)$-labeling number for this class of graphs.

In this paper, we present a new characterization of d-disk graphs for $d > 1$ and use it to obtain upper bounds on the $L(2, 1)$-labeling number for this classes of graphs.

1.1 Unit d-Disk Graphs and d-Disk Graphs

A *d-sphere*, $d \geq 2$, is the set of points $(x_1, x_2, \cdots, x_d)$ in R^d such that $(x_1 - c_1)^2 + (x_2 - c_2)^2 + \cdots + (x_d - c_d)^2 = r^2$, where r is the *radius* and $(c_1, c_2, \cdots, c_d) \in R^d$ is the center of the d-sphere. The 2-sphere and 3-sphere are the usual *circle* and *sphere*, respectively. The *diameter* of a sphere is $2r$. A d-sphere with diameter one is called a *unit d-sphere.*

A graph G is called a *unit d-disk graph*, if we can assign a unit d-sphere to each vertex of G so that two vertices are adjacent if and only if the corresponding spheres overlap. The set D of spheres assigned to the vertices of G is called the disk representation of G.

Example. The right side of Figure 1 shows a unit 2-disk graph called *the triangular lattice graph*, Γ, and its disk representation is shown on the left side of Figure 1. Γ is an infinite graph and it is $K_{1,4}$-free (i.e. it does not contain any induced $K_{1,4}$ although it contains $K_{1,4}$ as subgraph).

Let D be the disk representation of a d-disk graph G. Let d_{min} and d_{max} be the minimum and maximum diameters of the d-spheres in D. The value d_{max}/d_{min} is called the *diameter ratio* of D, denoted by $\sigma(D)$. A disk graph G is called a $\sigma(D)$-disk graph if it has a disk representation D of diameter ratio $\sigma(D)$. If $\sigma(D) = 1$, then it is not restrictive to assume G is a unit disk graph.

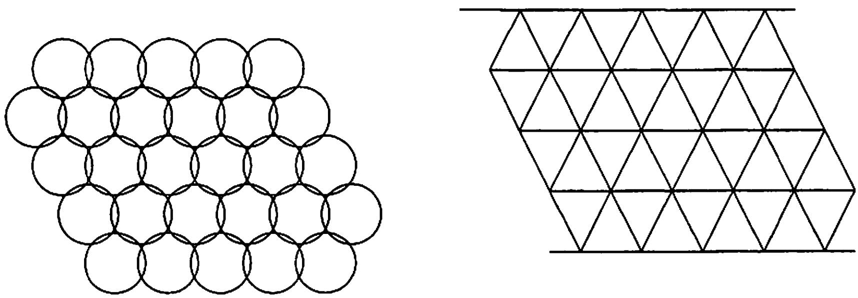

Figure 1: The triangular lattice graph Γ, and its disk representation.

2 A New Characterization of d-Disk Graphs

Shao et al. [30] characterized unit d-disk graphs for $d = 2, 3$. In this section, we characterize the more general class of d-disk graphs for $d \geq 2$.

From the definition of d-disk graphs, we observe that the set of all d-disk graphs is $K_{1,n}$-free if and only if it is not possible to pack n d-spheres around and touching a central d-sphere without the surrounding spheres touching each other. Hence, we consider the problem of bounding the minimum value n such that for any d-disk graph G, G is $K_{1,n}$-free, as this will allow us to bound the $L(2,1)$-labeling number for d-disk graphs.

We first show that the smallest value n for which any 2-disk graph of diameter ratio σ is $K_{1,n}$-free is $n = \lceil \pi / \arcsin(1/(\sigma+1)) \rceil$.

Theorem 2.1 *Any 2-disk graph of diameter ratio σ is $K_{1,n}$-free for any $n \geq \lceil \pi / \arcsin(1/(\sigma+1)) \rceil$.*

Proof. A 2-disk graph of diameter ratio σ is $K_{1,n}$-free if and only if for any collection D' of 2-spheres or circles of diameter ratio σ it is not possible to pack n circles from D' around and touching a central circle $C_0 \in D'$, without the surrounding circles touching each other.

Let $n(\sigma)$ be the smallest value such that any 2-disk graph of diameter ratio σ is $K_{1,n(\sigma)}$-free. Let $D = \{C_0, C_1, \ldots, C_{n(\sigma)-1}\}$ be a collection of $n(\sigma)$ circles of diameter ratio σ such that $C_1, \ldots, C_{n(\sigma)-1}$ can be placed around C_0 in such a way that each $C_i \neq C_0$ touches C_0 and no two $C_i, C_j \neq C_0$ touch each other. Let d_{min}, d_{max} be the minimum and maximum diameters of the circles in D, respectively. Note that by the definition of $n(\sigma)$, D has the property that no additional circle of diameter $d, d_{min} \leq d \leq d_{max}$, can be packed around C_0 along with $C_1, \ldots, C_{n(\sigma)-1}$ without causing the circles surrounding C_0 to touch each other.

Without loss of generality, let C_0 have diameter d_{max}. Consider a packing of $C_1, \ldots, C_{n(\sigma)-1}$ around C_0 as described above. Let us assume that each circle $C_i, i > 0$, is glued to C_0 at the point P_i where they touch. Note that if we reduce the radius of some circle C_i while keeping C_i glued to C_0 at P_i, no intersections among circles can be created by this operation because the distance between C_i and its two adjacent circles does not decrease (see Figure 2).

If we reduce the diameter of each circle $C_j, 1 \leq j \leq n(\sigma)-1$, to d_{max}/σ, we get a new packing where circles $C_i, 1 \leq i \leq n(\sigma)-1$ do not touch and the corresponding disk graph has diameter ratio σ. In this new packing consider two adjacent circles $C_i, C_{(i+1)(mod\ n(\sigma))}$ (see Figure 3). The angle θ between the centers of C_i, C_0, and C_j is $\theta > 2\arcsin((d_{max}/(2\sigma))/(d_{max}/(2\sigma) + d_{max}/2)) = 2\arcsin(1/(\sigma+1))$ and thus $n(\sigma) \leq \lceil 2\pi/\theta \rceil \leq \lceil \pi/\arcsin(1/(\sigma+1)) \rceil = n'$.

It is now easy to show thatanydisk graph of diameter ratio σ is $K_{1,n}$-free for $n \geq n'$. To see this, for the sake of contradiction, let G be a disk graph of diameter ratio σ and let G have $K_{1,n}$ as an induced subgraph. Let $D_{1,n}$ be the disk representation of $K_{1,n}$. Then $D_{1,n}$ consists of n circles $C_1, C_2, \ldots, C_n$ that can be packed around a central circle $C_0 \in D_{1,n}$, but as shown above, this is impossible for $n \geq n'$. ∎

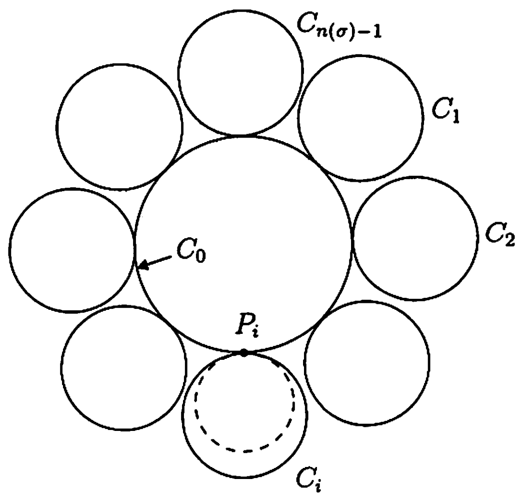

Figure 2: Reducing the radius of a circle C_i does not decrease the distance from C_i to its neighbouring circles.

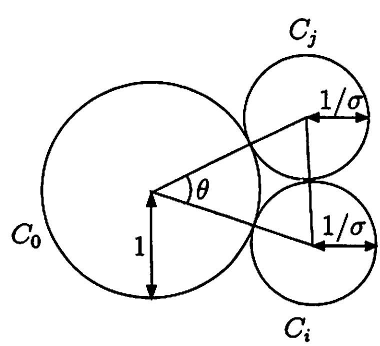

Figure 3: Angle θ between two neighbouring circles

We now turn our attention to d-disk graphs for $d \geq 3$ and compute an upper bound on the minimum value n such that any d-disk graph is $K_{1,n}$-free.

Consider a set D of d-spheres of diameter ratio σ with $|D| = d+1$. Let the d-spheres in D be tangent to each other. Let d of these spheres have radius $1/\sigma$ and the remaining one, S_0, have radius 1. The centers of the spheres in D delimit a d-simplex [1] Δ_d with *edge* lengths $2/\sigma$ and $1/\sigma + 1$. Let the center of S_0 be v. Consider a new d-sphere S with center v and radius $1/\sigma + 1$ (See Figure 4). Observe that all the centers of the spheres in D, except S_0, are on S and, obviously, $\Delta_d \subset S$. Consider the d faces of Δ_d that intersect at the center v of S. Let us extend these faces away from v until they intersect S. The region C delimited by these extended faces and the section of S above them is called a *spherical sector.* An example of Δ_3 and the corresponding spherical sector in 3 dimensions is shown in Figure 4.

Let the volume of S be $V(S)$ and the volume of C be $V(C)$. Let $n_\sigma = \lceil \frac{V(S)}{V(C)} \rceil$.

Lemma 2.2 *Any d-disk graph of diameter ratio σ is K_{1,n_σ}-free, for $d \geq 3$.*

Proof. Consider a d-disk graph G of diameter ratio σ and m vertices. Let $D = \{C_1, \ldots, C_m\}$ be a disk representation for G. Let $C_i \in D$ be a d-sphere for which spheres $C_{i_1}, \ldots, C_{i_{k_i}}$ are packed around and touching C_i, but spheres C_{i_j} do not touch each other. Proceeding as in the proof of Theorem 3.1 we can convert D into a new disk representation for G in which C_i has radius 1 and each sphere $C_{i_j}, j = 1, 2, \ldots, k_i$, has radius $1/\sigma$. Then, by the way in which the spherical sector C and sphere S have been defined, we note that $k_i < \lceil \frac{V(S)}{V(C)} \rceil = n_\sigma$ as it is not possible to pack n_σ spheres around C_i as described above, so G must be K_{1,n_σ}-free. ∎

We now compute the exact values of n_σ for $d = 3$ and then give an estimation of n_σ for $d > 3$.

Lemma 2.3 $n_\sigma = \lceil (4\pi)/(3 \arccos((\sigma^2 + 2\sigma - 1)/(2\sigma^2 + 4\sigma)) - \pi) \rceil$ *if* $d = 3$.

[1] A *d-simplex* is a polytope of dimension d with $d+1$ vertices (cf. [36]).

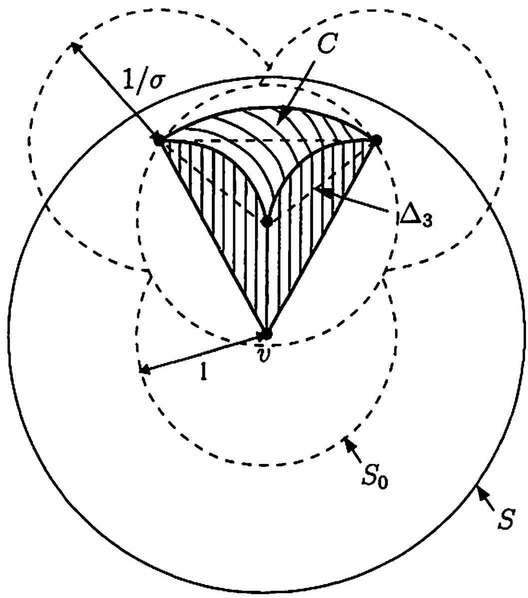

Figure 4: Spherical sector in 3 dimensions.

Proof. First let us recall the definition of a *spherical triangle* [1]: a spherical triangle consists of three vertices on the surface of a sphere S and three sides which are the arcs of the short segments of great circles that join pairs of these vertices. Note that, for a spherical sector C in 3 dimensions, as defined above (see Figure 4), its non-planar face is delimited by a spherical triangle.

Consider 3 spheres S_1, S_2, S_3 of radius $1/\sigma$ and a sphere S_0 of radius 1 with center v, tangent to each other as shown in Figure 5. Let S be a sphere of radius $1 + 1/\sigma$ and center v and let T be the spherical triangle delimiting the spherical sector C defined by these 4 spheres. Let the three angles of T, the area of S and the area of T be α, β, γ, $A(S)$ and $A(T)$, respectively. By Girards formula, $A(T) = R^2(\alpha + \beta + \gamma - \pi)$ [1], where $R = 1 + 1/\sigma$ is the radius of S. Since $\cos(\alpha) = \cos(\beta) = \cos(\gamma) = (\sigma^2 + 2\sigma - 1)/(2\sigma^2 + 4\sigma)$, then $A(T) = R^2(3\arccos((\sigma^2 + 2\sigma - 1)/(2\sigma^2 + 4\sigma)) - \pi)$. Thus, $n_\sigma = \lceil \frac{V(d)}{V(C)} \rceil = \lceil \frac{A(S)}{A(T)} \rceil = \lceil (4\pi R^2)/(R^2(3\arccos((\sigma^2+2\sigma-1)/(2\sigma^2+4\sigma))-\pi)) \rceil =$

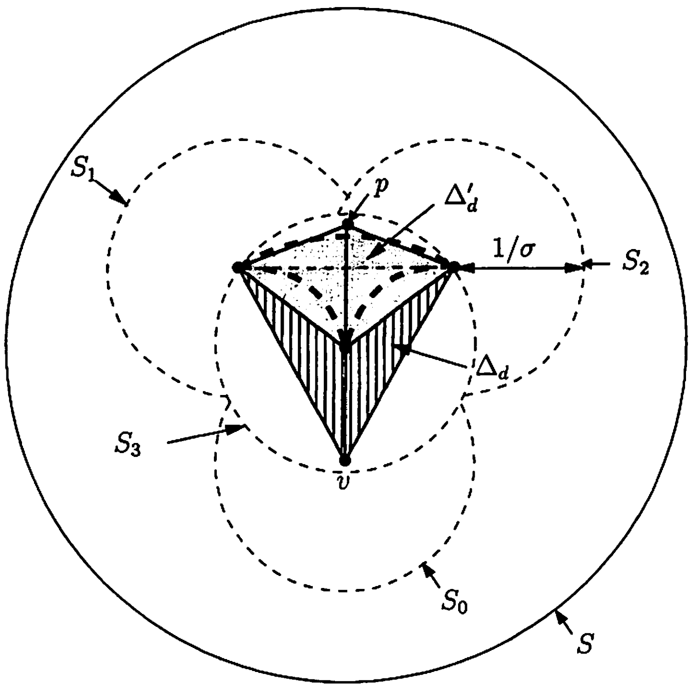

Figure 5: d-Simplexes Δ_d and Δ'_d.

$\lceil (4\pi)/(3\arccos((\sigma^2+2\sigma-1)/(2\sigma^2+4\sigma))-\pi)\rceil$. ■

Lemma 2.4 *$n_\sigma < \lceil d!(1/\sigma+1)^{d-1}\pi^{d/2}/(\Gamma(d/2+1)d^{1/2}(2/\sigma)^{(d-1)/2}\rceil$ (where $\Gamma(x)$ is the gamma function) if $d \geq 4$.*

Proof. Consider a set D of d-spheres tangent to each other in which $|D| = d+1$ and where d of the spheres have radius $1/\sigma$ and a central d-sphere S_0 has radius 1. Let the center of S_0 be v. The centers of the spheres are the vertices of a d-simplex Δ_d. Consider also a sphere S of center v and radius $1/\sigma + 1$. Let us draw a straight line l from v perpendicular[2] to the regular $(d-1)$-simplex Δ_{d-1} formed by all the vertices of Δ_d except v, and let the intersection point of l and S be p.

[2]A line is perpendicular to the simplex if it is orthogonal to each face of the simplex.

By connecting p with all the vertices of Δ_d except v, we get another d-simplex Δ'_d which shares a common $(d-1)$-simplex Δ_{d-1} with Δ_d (See Figure 5). Because Δ_d and Δ'_d share a common $(d-1)$-simplex Δ_{d-1} and both v and p are on a line perpendicular to the regular $(d-1)$-simplex Δ_{d-1}, the combined volume of Δ_d and Δ'_d is $RV(\Delta_{d-1})/d$, where R is the radius of the d-sphere S and $V(\Delta_{d-1})$ is the volume of the regular $(d-1)$-simplex Δ_{d-1}. Since the volume of a d-sphere of radius R is $V(R) = \pi^{d/2}R^d/\Gamma(d/2+1)$ [20] and the volume of a regular d-simplex Δ' with edge length q is $V(\Delta') = (d+1)^{1/2}q^d/((d)!2^{d/2})$ [7], then, the volume of sphere S is $V(S) = \pi^{d/2}(1/\sigma+1)^d/\Gamma(d/2+1)$ and $V(\Delta_d)+V(\Delta'_d) = (1/d)(1/\sigma+1)d^{1/2}(2/\sigma)^{d-1}/((d-1)!2^{(d-1)/2})$ (as the volume of the regular $(d-1)$-simplex Δ_{d-1} is $d^{1/2}(2/\sigma)^{d-1}/((d-1)!2^{(d-1)/2})$). Thus, $n_\sigma = \lceil \frac{V(S)}{V(C)} \rceil < \lceil \frac{V(S)}{V(\Delta_d)+V(\Delta'_d)} \rceil = \lceil (\pi^{d/2}(1/\sigma+1)^d/\Gamma(d/2+1))/((1/d)(1/\sigma+1)d^{1/2}(2/\sigma)^{d-1}/((d-1)!2^{(d-1)/2})) \rceil = \lceil d!(1/\sigma+1)^{d-1}\pi^{d/2}/(\Gamma(d/2+1)d^{1/2}(2/\sigma)^{(d-1)/2} \rceil$.

We note that we can also upper bound n_σ by $\frac{A(S)}{V(\Delta_{d-1})}$, where $A(S)$ is the area of S. Since the area of a d-sphere of radius R is $dV(R)/R = d\pi^{d/2}R^d/(\Gamma(d/2+1)R)$ [20], then, $A(S) = d\pi^{d/2}(1/\sigma+1)^{d-1}/\Gamma(d/2+1)$. So, $n_\sigma < \lceil \frac{A(S)}{V(\Delta_{d-1})} \rceil = \lceil (d\pi^{d/2}(1/\sigma+1)^{d-1}/(\Gamma(d/2+1))/(d^{1/2}(2/\sigma)^{d-1}/((d-1)!2^{(d-1)/2})) \rceil = \lceil d!(1/\sigma+1)^{d-1}\pi^{d/2}/(\Gamma(d/2+1)d^{1/2}(2/\sigma)^{(d-1)/2} \rceil$. Surprisingly, the two approaches yield the same upper bound for n_σ. ■

Note that

$$\Gamma(d/2+1) = \begin{cases} (d/2)! & \text{if } d \text{ is even,} \\ (\pi)^{1/2}((d+1)!/((d+1)/2)!)2^{-(d+1)} & \text{if } d \text{ is odd.} \end{cases}$$

Thus,

$$n_\sigma(d) < \begin{cases} \lceil d!/(d/2)!(\pi/d)^{1/2}((\sigma+1)^2\pi/(2\sigma))^{1/2})^{(d-1)/2} \rceil & \text{if } d \text{ is even,} \\ \lceil ((d+1)/2)!(4/(d+1))d^{-1/2}(2(\sigma+1)^2\pi/\sigma)^{1/2})^{(d-1)/2} \rceil & \text{if } d \text{ is odd.} \end{cases}$$

3 $L(2,1)$-Labelings of d-Disk Graphs

Shao et al. [30] proved that $\lambda(G) \leq \frac{4}{5}\Delta^2 + 2\Delta$ for any unit 2-disk graph G and $\lambda(G) \leq \frac{11}{12}\Delta^2 + 2\Delta$ for any unit 3-disk graph G. In this section, we give upper bounds for the $L(2,1)$-labeling number for the more general class of d-disk graphs for $d \geq 2$.

We cite the following theorem by Shao et al. [30].

Theorem 3.1 *If G is $K_{1,n}$-free then $\lambda(G) \leq \frac{n-2}{n-1}\Delta^2 + 2\Delta$, where Δ is the maximum degree of G.*

By Theorem 3.1, Lemma 3.2 and Theorem 4.1, we have the following Theorem.

Theorem 3.2 *Let G be a d-disk graph of diameter ratio σ and maximum degree Δ, for $d \geq 2$. Then*

$$\lambda(G) \leq \begin{cases} \frac{\lceil \pi/\arcsin(1/(\sigma+1)) \rceil - 2}{\lceil \pi/\arcsin(1/(\sigma+1)) \rceil - 1}\Delta^2 + 2\Delta & \text{if } d = 2, \\ \frac{n_\sigma - 2}{n_\sigma - 1}\Delta^2 + 2\Delta & \text{if } d \geq 3. \end{cases}$$

We do not know whether the above bounds are attainable or not. In the table of Figure 6 we compare the bounds provided by Theorem 3.2 with known labeling numbers for some disk graphs. The triangular lattice graph Γ shown in Figure 1 has been shown to have labeling number $\lambda(\Gamma) = 8$ (see [35]). Let $G_t, t = 3, 4$, denote the hexagonal grid and squared grid, respectively. It has been shown that $\lambda(G_3) = 5$ and $\lambda(G_4) = 6$ (see [4]). All above graphs are unit 2-disk graphs, so for all of them $\sigma = 1$ and $d = 2$. The bounds of Theorem 3.2 for these graphs are listed in the following table. By Theorem 3.2, we can obtain the table in Figure 6.

Graph	Δ	d	σ	λ	Bounds of Theorem 3.2
Γ	6	2	1	8	40
G_3	3	2	1	5	13
G_4	4	2	1	6	20

Figure 6: Bounds of Theorem 3.2 for some graphs.

4 Acknowledgments

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