Some Families of Combination and Permutation Graphs

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Abstract We study: combination and permutation graphs. We introduce some familes to be: combination graphs and permutation graphs.

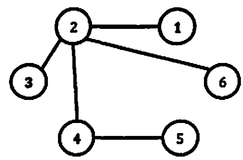
1. ***Introduction*** Hegde and Shetty[2,4] define a graph G with n vertices to be a permutation graph if there exists an injection f from the vertices of G to $\{1,2,3,\dots,n\}$ such that the induced edge function g_f defined as $g_f(uv) = f(u)!/|f(u)-f(v)|!$ is injective. They say a graph G with n vertices to be a combination graph if there exists an injection f from the vertices of G to $\{1,2,3,\dots,n\}$such that the induced edge function g_f defined as $g_f(uv) = f(u)!/|f(u)-f(v)|!\,f(v)!$ is injective. They prove: K_n is a permutation graph if and only if $n \le 5$; K_n is a combination graph if and only if $n \le 2$; C_n is a combination graph for $n > 3$,$k_{n,n}$ is a combination graph if and only if $n \le 2$; W_n is a not a combination graph for $n \le 6$,and a necessary condition for a (p, q)-graph to be a combination graph is that $4q \le p^2$ if p is even and $4q \le p^2 - 1$ if p is odd. They strongly believe that W_n is a combination graph for $n > 6$ and all trees are combination graphs. Baskar Babujee and Vishnupriya [1] prove the following graphs are permutation graphs: P_n ; C_n ; stars; graphs obtained adding a pendent edge to each edge of a star; graphs obtained by joining the centers of two identical stars with an edge or a path of length 2; and complete binary trees with at least three vertices. Throughout this paper, we use the basic notations and conventions in graph theory as in [3].

2. ***Some definitions:***

<u>Definition (2.1):</u>[2,4]A(p, q) graph $G = (V, E)$ is said to be a combination graph if there exists a bijection $f: V(G) \to \{1,2,\dots,p\}$, such that the induced edge function $g_f: E(G) \to \mathbb{N}$ defined as

$$g_f(uv) = \begin{cases} \binom{f(u)}{f(v)} & , \ if\ f(u) > f(v) \\ \binom{f(v)}{f(u)} & , \ if\ f(v) > f(u) \end{cases}$$

is injective, where $\binom{f(u)}{f(v)}$ is the number of combinations of f(u) things taken f(v) at a time. Such a labeling f is called a combination labeling of G . The following example is a combination graph

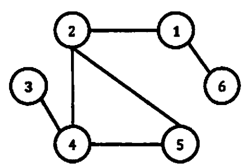


Definition(2.2):[2,4] A(p,q) graph $G=(V,E)$ is said to be a permutation graph if there exists a bijection $f: V(G) \rightarrow \{1,2,\dots,p\}$, such that the induced edge function $g_f: E(G) \rightarrow \mathbb{N}$ defined as

$$g_f(uv) = \begin{cases} {}^{f(u)}P_{f(v)} & , \ if\ f(u) > f(v) \\ {}^{f(v)}P_{f(u)} & , \ if\ f(v) > f(u) \end{cases}$$

is injective, where ${}^{f(u)}P_{f(v)}$ is the number of permutations of f(u) things taken f(v) at a time. Such a labeling f is called a permutation labeling of G.

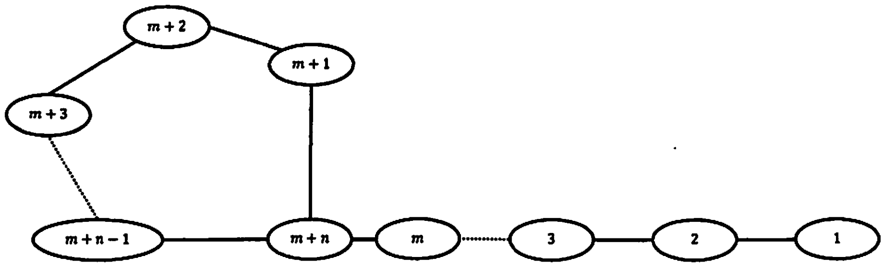
The following example is a permutation graph:



Definition(2.3):[3] For $n \geq 3$, the wheel W_n is defined to be the graph $C_n + K_1$, where the vertex of K_1 is called the center of the wheel.
Definition(2.4): [3] For $n \geq 3$, the fan F_n is defined to be the graph $P_n + K_1$.

3. *Some Combination families*

Theorem (3.1): The maximum minimum degree of all combination graphs of n vertices is $\leq \lfloor n/2 \rfloor$.
Proof: Let $G(n,q)$ be combination graph, and suppose to the contrary that $d_i \geq \lfloor n/2 \rfloor + 1$ for every $i = 1,2,\dots,n$, where d_i is the degree of the vertex labeled by i.It follows that $\sum_{i=1}^{n} d_i \geq n(\lfloor n/2 \rfloor + 1)$, i.e. $2q \geq n(\lfloor n/2 \rfloor + 1)$.
Therefore we have two cases:
If n is even: $2q \geq n(\frac{n}{2}+1)$, hence $4q \geq n^2 + 2n$ which is a contradiction to the assumption that G is a combination graph.
If n is odd:
$2q \geq n(\frac{n-1}{2}+1)$, hence $4q \geq n^2 + n$ which is also a contradiction.
Theorem (3.2): The dragon $D_{n,m}$ is a combination graph for every n, m, where $D_{n,m}$ is the graph obtained from the cycle C_n by joining the end point of a path P_m to one vertex of C_n.
Proof: Consider the following labeling for the dragon (clearly the labels are different).

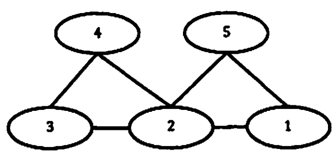


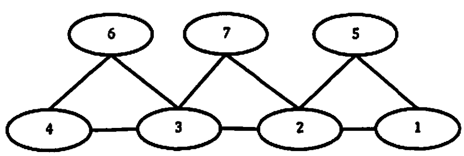
Therefore the dragon is a combination graph for every n, m.

Theorem (3.3): The triangular snake T_n , $n \geq 3$ is a combination graph, where the triangular snake is the graph obtained from the path P_n having the vertices $v_1, v_2, \ldots, v_n$ by adding new vertices $w_1, w_2, \ldots, w_{n-1}$ and connecting w_i to the vertices v_i, v_{i+1} for each i.

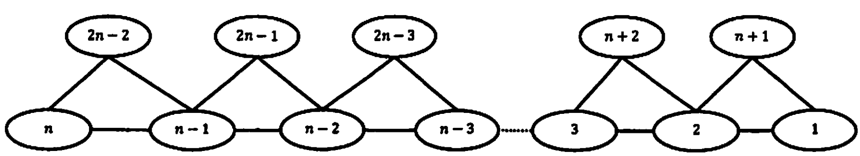
Proof: We will prove the assertion by introducing a labeling for any triangular snake for every $n \geq 3$ to be a combination graph.

Case(1): $n = 3,4$





Case(2): $n \geq 5$



We will divide the set of labels into three sets which are:

$A_1 = \{ {}^{2}C_1, {}^{3}C_2, \cdots, {}^{n}C_{n-1}, {}^{n+1}C_1 \} = \{2, 3, \ldots, n, n+1\}$.

$A_2 = \{ {}^{n+1}C_2, {}^{n+2}C_2, {}^{n+2}C_3, \cdots, {}^{2n-3}C_{n-3}, {}^{2n-3}C_{n-2} \}$.

$A_3 = \{ {}^{2n-2}C_n, {}^{2n-2}C_{n-1}, {}^{2n-1}C_{n-1}, {}^{2n-1}C_{n-2} \}$.

We will prove that the labels in each set of the previous sets are distinct as follows:

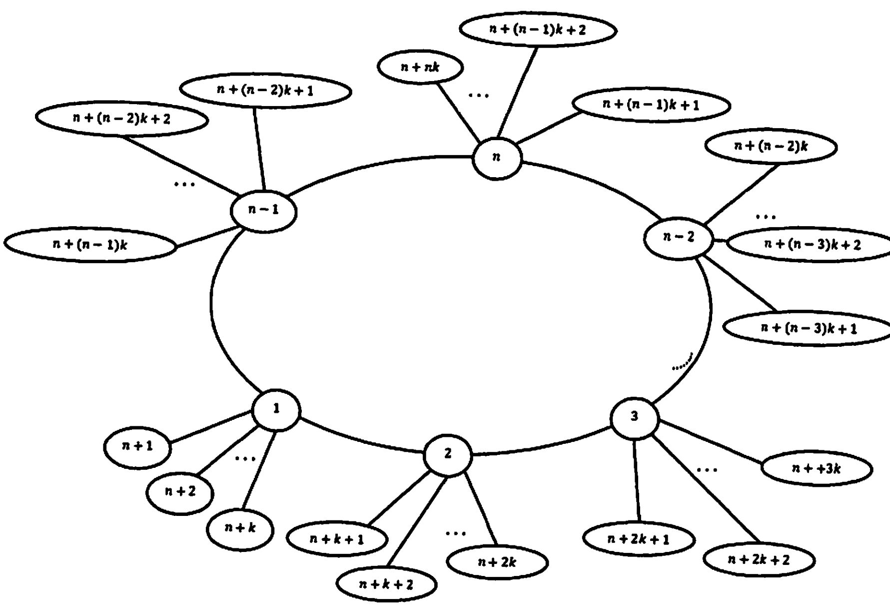
Since we have ${}^{k}C_r < {}^{k+1}C_r$, ${}^{k}C_r < {}^{k+1}C_{r+1}$ and ${}^{n+k}C_k < {}^{n+k}C_{k+1}$ if $k < n -$

1, therefore ${}^{2}C_1 < {}^{3}C_2 < \cdots < {}^{n}C_{n-1} < {}^{n+1}C_1 < {}^{n+1}C_2 < {}^{n+2}C_2 < {}^{n+2}C_3 < \cdots < {}^{2n-3}C_{n-3} < {}^{2n-3}C_{n-2} < {}^{2n-2}C_n = {}^{2n-2}C_{n-2} < {}^{2n-2}C_{n-1} < {}^{2n-1}C_{n-2} < {}^{2n-1}C_{n-1}$.

Therefore the triangular snake is a combination graph for every $n \geq 3$.

Theorem (3.4): The k-crown kC_n $n \geq 3$, the graph obtained by adding k pendant edges to every vertex in the cycle C_n is a combination graph such that $k < {}^{n}C_2 - n$.

Proof: We will prove by introducing a labeling for any kC_n such that $k < {}^{n}C_2 - n$ to be combination graph.



We will divide the set of labels into $n + 1$ subsets as follows:

$A_0 = \{ {}^{2}C_1, {}^{3}C_2, \ldots, {}^{n-2}C_{n-3}, {}^{n}C_{n-2}, {}^{n-1}C_1, {}^{n}C_{n-1}\} = \{2, 3, \ldots, n, {}^{n}C_2\}$.

$A_i = \{ {}^{n+(i-1)k+1}C_i, {}^{n+(i-1)k+2}C_i, \ldots, {}^{n+ik}C_i\}$, $i = 1, 2, \ldots, n$.

Clearly the labels in each set of the previous sets are distinct.

$A_i \cap A_j = \varphi, i \neq j,\ i, j = 1, 2, \ldots, n$, since ${}^{n+ik}C_i < {}^{n+ik+1}C_{i+1}$, $i = 1, 2, \ldots, n$.

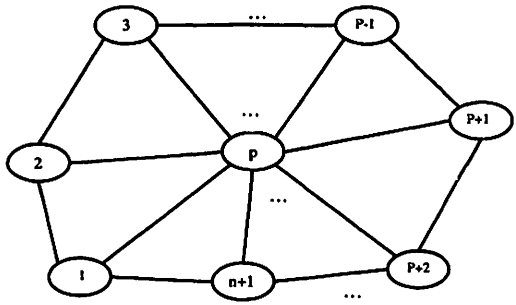
Since $k < {}^{n}C_2 - n$, it follows that $A_1 \cap A_0 = \varphi$.

$A_i \cap A_0 = \varphi, i = 2, \ldots, n$, since every number in A_0 is less than every number in A_i for every $i = 2, \ldots, n$.

4. ***Some Permutation families:***

Theorem (4.1): W_n is a permutation graph for every $n \geq 3$

Proof: We will give a labeling for W_n for every $n \geq 3$.
We have four cases:
Case (1): $n+1$ is odd and is not a prime number. Let p be the greatest prime number less than $n+1$. We label W_n as in the following figure.



We will divide the set of edge labels into five sets which are:
$A_1 = \{ {}^2P_1 , {}^3P_2 , \dots , {}^{p-1}P_{p-2} \} = \{2!, 3!, \dots, (p-1)!\}$.
$A_2 = \{ {}^pP_1 , {}^pP_2 , \dots , {}^pP_{p-1} \}$.
$A_3 = \{ {}^{p+1}P_{p-1} , {}^{p+1}P_p , {}^{p+2}P_p , {}^{p+3}P_p , \dots , {}^{n+1}P_p \}$.
$A_4 = \{ {}^{p+2}P_{p+1} , {}^{p+3}P_{p+2} , \dots , {}^{n+1}P_n \} = \{(p+2)!, (p+3)!, \dots, (n+1)!\}$.
$A_5 = \{ {}^{n+1}P_1 \} = \{n+1\}$.
Clearly the labels in each set of the previous sets are distinct.
$A_1 \cap A_2 = \varphi$, since each label in A_2 is divisible by p while each label in A_1 is not divisible by p.
$A_1 \cap A_3 = \varphi$, since ${}^2P_1 < {}^3P_2 < \cdots < {}^{p-1}P_{p-2} < {}^{p+1}P_{p-1} < {}^{p+1}P_p < {}^{p+2}P_p < {}^{p+3}P_p < \cdots < {}^{n+1}P_p$.
$A_1 \cap A_4 = \varphi$, since ${}^2P_1 < {}^3P_2 < \cdots < {}^{p-1}P_{p-2} < {}^{p+2}P_{p+1} < {}^{p+3}P_{p+2} < \cdots < {}^{n+1}P_n$.
$A_2 \cap A_3 = \varphi$, since ${}^pP_1 < {}^pP_2 < \cdots < {}^pP_{p-1} < {}^{p+1}P_{p-1} < {}^{p+1}P_p < {}^{p+2}P_p < {}^{p+3}P_p < \cdots < {}^{n+1}P_p$.
$A_2 \cap A_4 = \varphi$, since ${}^pP_1 < {}^pP_2 < \cdots < {}^pP_{p-1} < {}^{p+2}P_{p+1} < {}^{p+3}P_{p+2} < \cdots < {}^{n+1}P_n$.
$A_3 \cap A_4 = \varphi$, as explained in the following :
Since ${}^{p+1}P_{p-1} < {}^{p+1}P_p < {}^{p+2}P_p < {}^{p+3}P_p < \cdots < {}^{n+1}P_p$, ${}^{p+2}P_{p+1} < {}^{p+3}P_{p+2} < \cdots < {}^{n+1}P_n$,i.e. $(p+2)! < (p+3)! < \dots < (n+1)!$ and since ${}^{p+i}P_p < (p+i)!$ $\forall i > 1$, it follows that if there exists a label in the intersection of A_3 and A_4 it will be of the following form ${}^{p+i}P_p = (p+i-t)!$ for some $i = 3,4, \dots, n-p+1$ and for some $t = 1,2, \dots, i-2$. Now we will prove that ${}^{p+i}P_p \neq (p+i-t)!$, i.e. $(p+i)(p+i-1)\dots(p+i-t+1) \neq i!$, for every $i = 3,4, \dots, n-p+1$ and for every $t = 1,2, \dots, i-2$ by using induction on t. Firstly we will get a relation between p and i.We have two cases:

$p > i$ and $p \le i$ .Let $p \le i$.Since $i \le n-p+1$, it follows that $p \le n-p+1$, $p \le \left\lfloor \frac{n+1}{2} \right\rfloor \rightarrow (1)$. From Bertrand's postulate there exists a prime number $\bar{p}$ such that $\left\lfloor \frac{n+1}{2} \right\rfloor < \bar{p} < n+1 \rightarrow (2)$.From (1)and (2) $\bar{p} > p$ which is a contradiction since p is the greatest prime number less than $n+1$.So $p > i$.

It is clear that at $t = 1$: $(p+i) \neq i!$, since if $(p+i) = i!$, then $p = i((i-1)! - 1)$ which is contradiction. At $t = k$: let $(p+i)(p+i-1)\dots(p+i-k+1) \neq i! \rightarrow (*)$ and we will prove that at $t = k+1$: $(p+i)(p+i-1)\dots(p+i-k+1)(p+i-k) \neq i!$ .Suppose to the contrary that $(p+i)(p+i-1)\dots(p+i-k+1)(p+i-k) = i! \rightarrow (**)$ From $(*)$ we have two cases:

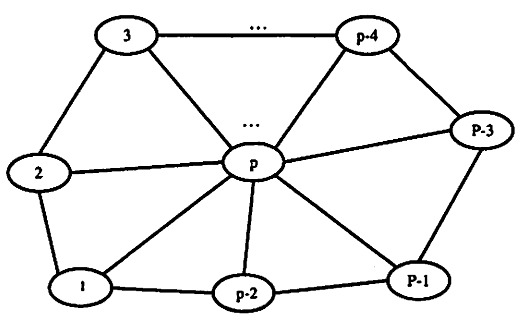
Case(I): $(p+i)(p+i-1)\dots(p+i-k+1) > i!$.Therefore from$(**)$ we get $(p+i-k) = \frac{i!}{(p+i)(p+i-1)\dots(p+i-k+1)} < 1$ which is a contradiction.

Case(II): $(p+i)(p+i-1)\dots(p+i-k+1) < i!$.By using division algorithm $i! = q(p+i)(p+i-1)\dots(p+i-k+1) + r, 0 \le r < (p+i)(p+i-1)\dots(p+i-k+1)$,and it follows by substitution in $(**)$that $p+i-k-q = \frac{r}{(p+i)(p+i-1)\dots(p+i-k+1)}$ which is clearly greater than 0 and less than 1,therefore $p+i-k-q$ is not integer which is a contradiction.

Hence ${}^{p+i}P_p \neq (p+i-t)!$ for every $i = 3,4,\dots,n-p+1$ and for every $t = 1,2,\dots,i-2$. Therefore $A_3 \cap A_4 = \varphi$.

Also $A_5 \cap (\cup_{i=1}^{4} A_i) = \varphi$, since all labels in $(\cup_{i=1}^{4} A_i)$ are even numbers except ${}^{p}P_1$ and $n+1$ is odd.

<u>Case (2):</u> $n+1$ is an odd prime number. Let $n+1 = p$. We label W_n as in the following figure.



We will divide the set of edge labels into three sets which are:

$B_1 = \{ {}^{2}P_1, {}^{3}P_2, \dots, {}^{p-3}P_{p-4}, {}^{p-1}P_{p-3}, {}^{p-1}P_{p-2} \}$

$\quad = \{2!, 3!, \dots, (p-3)!, {}^{p-1}P_{p-3}, (p-1)!\}$.

$B_2 = \{ {}^{p}P_1, {}^{p}P_2, \dots, {}^{p}P_{p-1} \}$.

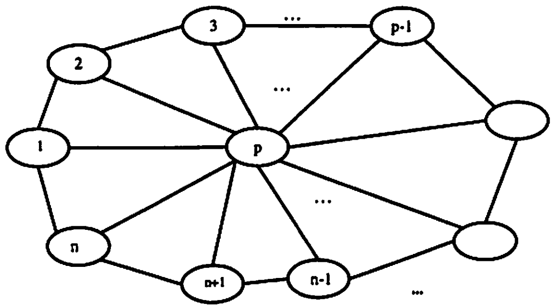
$B_3 = \{{}^{p-2}P_1\}$.

Clearly the labels in each set of the previous sets are distinct.

$B_1 \cap B_2 = \varphi$, since each label in B_2 is divisible by p while each label in $B_1$is not divisible by p.

Also $B_3 \cap (B_1 \cup B_2) = \varphi$, since all labels in $(B_1 \cup B_2)$ are even numbers except ${}^pP_1$ which is greater than $p-2$.

<u>Case (3):</u> $n+1$ is an even number, and p, the greatest prime number less than $n+1$ is such that $p \neq n$. We label W_n as in the following figure.



We will divide the set of labels into five sets which are:

$C_1 = \{ {}^2P_1, {}^3P_2, \dots, {}^{p-1}P_{p-2} \} = \{2!, 3!, \dots, (p-1)!\}$.

$C_2 = \{ {}^pP_1, {}^pP_2, \dots, {}^pP_{p-1} \}$.

$C_3 = \{ {}^{p+1}P_{p-1}, {}^{p+1}P_p, {}^{p+2}P_p, {}^{p+3}P_p, \dots, {}^{n+1}P_p \}$.

$C_4 = \{ {}^{p+2}P_{p+1}, {}^{p+3}P_{p+2}, \dots, {}^{n-1}P_{n-2}, {}^{n+1}P_{n-1}, {}^{n+1}P_n \}$

$= \{(p+2)!, (p+3)!, \dots, (n-2)!, {}^{n+1}P_{n-1}, (n+1)!\}$.

$C_5 = \{{}^nP_1\}$.

Clearly the labels in each set of the previous sets are distinct.

As in case(1) the labels in C_1, C_2, C_3 are distinct.

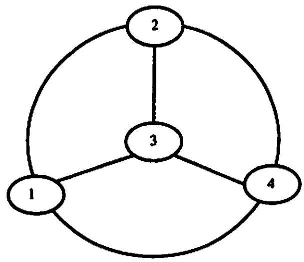
Also $C_5 \cap (\cup_{i=1}^4 C_i) = \varphi$, since all labels in $(\cup_{i=1}^4 C_i)$ are even numbers except ${}^pP_1$ and $n \neq p$.

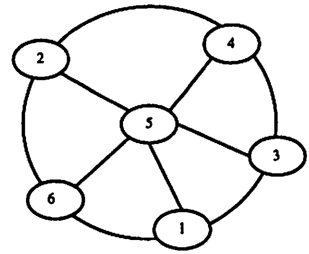
$C_1 \cap C_4 = \varphi$, since ${}^2P_1 < {}^3P_2 < \cdots < {}^{p-1}P_{p-2} < {}^{p+2}P_{p+1} < {}^{p+3}P_{p+2} < \cdots < {}^{n-3}P_{n-2} < {}^nP_{n-2} < {}^nP_{n-1}$.

$C_2 \cap C_4 = \varphi$, since ${}^pP_1 < {}^pP_2 < \cdots < {}^pP_{p-1} < {}^{p+2}P_{p+1} < {}^{p+3}P_{p+2} < \cdots < {}^{n-1}P_{n-2} < {}^{n+1}P_{n-1} < {}^{n+1}P_n$.

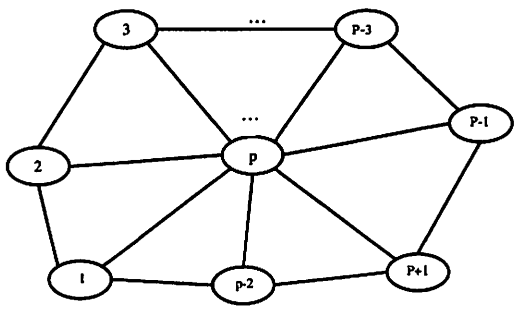
$C_3 \cap C_4 = \varphi$, since $C_3 = A_3$ and $C_4 - \{ {}^{n+1}P_{n-1} \} \subset A_4$ and since $A_3 \cap A_4 = \varphi$, $C_3 \cap (C_4 - \{ {}^{n+1}P_{n-1} \}) = \varphi$. Also this label ${}^{n+1}P_{n-1}$ is greater than all labels in C_3.

<u>Case(4):</u> $n+1$ is an even number, and p, the greatest prime number less than $n+1$ is such that $p = n$.First we will label the two cases where $p = 3,5$ as follows:





Second we label W_n , $p > 5$ as in the following figure.



We will divide the set of labels into three sets which are:

$D_1 = \{ {}^{2}P_1 , {}^{3}P_2 , \dots , {}^{p-3}P_{p-4} , {}^{p-1}P_{p-3} \} = \{2! , 3! , \dots , (p-3)! \ , {}^{p-1}P_{p-3} \}$.

$D_2 = \{ {}^{p}P_1 , {}^{p}P_2 , \dots , {}^{p}P_{p-1} , {}^{p+1}P_{p-2} , {}^{p+1}P_{p-1} , {}^{p+1}P_p \}$.

$D_3 = \{ {}^{p-2}P_1 \}$.

Clearly the labels in each set of the previous sets are distinct.

Also $D_3 \cap (D_1 \cup D_2) = \varphi$, since all labels in $D_1 \cup D_2$ are even numbers except ${}^{p}P_1$ and which is greater than $p-2$.

$D_1 \cap D_2 = \varphi$, since each label in D_2 is divisible by p and each label in D_1 is not divisible by p.

In all cases W_n is a permutation graph.

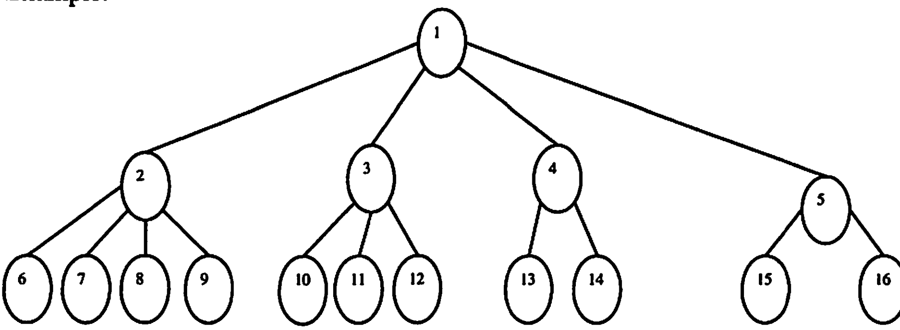
Corollary (4.2): F_n is a permutation graph.

Proof: Since F_n is a subgraph of W_n with the same number of vertices and W_n is a permutation graph, F_n is a permutation graph.

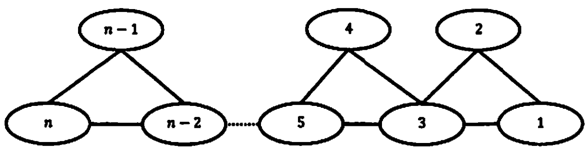
Theorem (4.3): T_n is a Permutation graph for every n.

Proof: we will introduce a labeling for T_n by using the Breadth-First algorithm. Choose a vertex and labeled it by 1 and label the adjacent vertices to this vertex by $2,3, \dots , n_1$ and then label the vertices that are adjacent to the vertex labeled 2 by $m = n_1 + 1 , m + 1 , \dots , m_1$ and then label the vertices that adjacent to the vertex labeled 3 by $m_1 + 1 , m_1 + 2, \dots , m_2$ and so on. By this way all the labels are distinct, since the permutation function is increasing.

Example:



Corollary (4.4): The triangular snakeT_n , $n \geq 3$, is a permutation graph.
Proof: We introduce a labeling for the triangular snake for every n as follows:



Therefore the triangular snake is a Permutation graph for every n.

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