On a new class of combinatorial identities [1]

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Abstract. In this paper we interpret a generalized basic series as the generating function of two different combinatorial objects,viz., a restricted n-colour partition function which we call a two-colour partition function and a weighted lattice path function. This leads to infinitely many combinatorial identities. Our main result has the potential of yielding many Rogers-Ramanujan-MacMahon type combinatorial identities. This is illustrated by an example.

1 Introduction, Definitions and the Main Results

A series involving factors like rising q-factorial $(a,q)_n$ defined by

$$(a;q)_n = \prod_{i=o}^{\infty} \frac{(1-aq^i)}{(1-aq^{n+i})}.$$

is called a basic series (or q-series, or Eulerian series). The following two "sum-product" basic series identities are known as the Rogers-Ramanujan identities

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \prod_{n=1}^{\infty} (1-q^{5n-1})^{-1}(1-q^{5n-4})^{-1}, \tag{1.1}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \prod_{n=1}^{\infty} (1-q^{5n-2})^{-1}(1-q^{5n-3})^{-1}. \tag{1.2}$$

They were first discovered by Rogers [17] and rediscovered by Ramanujan in 1913. MacMahon [15] gave the following partition theoretic interpretations of (1.1) and (1.2), respectively:

Theorem 1.1. The number of partitions of n into parts with the minimal difference 2 equals the number of partitions of n into parts $\equiv \pm 1 (mod 5)$.

Theorem 1.2. The number of partitions of n with minimal part 2 and minimal difference 2 equals the number of partitions of n into parts $\equiv \pm 2 (mod 5)$.

Partition theoretic interpretations of many more q-series identities like (1.1) and (1.2) have been given by several mathematicians. See, For instance, Göllnitz

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[11,12], Gordon [13], Connor [10], Hirschhorn [14], Agarwal and Andrews [5], Subbarao [19], Subbarao and Agarwal [20].

In all these results ordinary partitions were used. In [6] n-colour partitions were defined. Using these partitions several more basic series identities were interpreted combinatorially (see, for instance,[1,2,3,4,16]). In this paper we interpret the basic series

$$\sum_{n=0}^{\infty} \frac{q^{n(n+k-1)}(-q;q^2)_n}{(q^4;q^4)_n},$$

where k is a positive integer, as generating function of two different combinatorial objects, viz., a restricted n-colour partition function which we call a two-colour partition function and a weighted lattice path function. This leads to an infinite family of combinatorial identities. These identities have the potential of yielding many Rogers-Ramanujan-MacMahon type combinatorial identities like Theorems 1.1-1.2. First we recall the following definitions from [6]:

Definition 1.1. An n-colour partition (also called a partition with "n copies of n") is a partition in which a part of size n can come in n different colours denoted by subscripts: $n_1, n_2, \cdots, n_n$.
Thus, for example, the n-colour partitions of 3 are

$$3_1, \;\; 3_2, \;\; 3_3, \;\; 2_1 1_1, \;\; 2_2 1_1, \;\; 1_1 1_1 1_1.$$

Definition 1.2. The weighted difference of two parts m_i, n_j $(m \geq n)$ is defined by $m - n - i - j$ and is denoted by $((m_i - n_j))$.

We remark that in this paper we do not use the whole class of n-colour partitions but only a sub-class containing two-colour partitions which we define as follows:

Definition 1.3. A two-colour partition of a positive integer ν is a partition in which a part of size $m > 1$ can come in two different colours denoted by subscripts : m_1, m_2 and 1 appears only in one colour denoted by 1_1.

Next, we recall the following description of lattice paths from [7] which we shall be considering in this paper:

All paths will be of finite length lying in the first quadrant. They will begin on the y-axis and terminate on the x-axis. Only three moves are allowed at each step:
northeast: from (i,j) to $(i+1, j+1)$
southeast: from (i,j) to $(i+1, j-1)$, only allowed if $j > 0$,
horizontal: from $(i,0)$ to $(i+1,0)$, only allowed along x-axis.

The following terminology will be used in describing lattice paths:

PEAK: Either a vertex on the y-axis which is followed by a southeast step or a vertex preceded by a northeast step and followed by a southeast step.

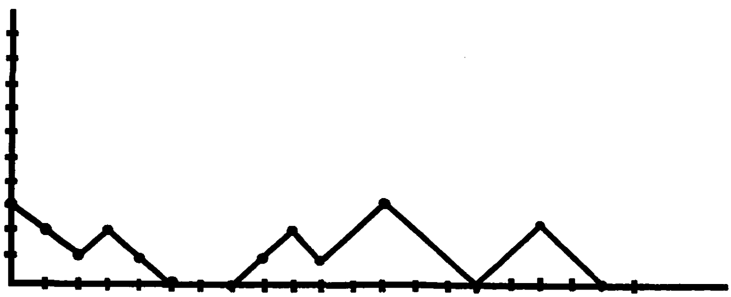
VALLEY: A vertex preceded by a southeast step and followed by a northeast step. Note that a southeast step followed by a horizontal step followed by a northeast step does not constitute a valley.

MOUNTAIN: A section of the path which starts on either the x- or y-axis, which ends on the x-axis, and which does not touch the x-axis anywhere in between the end points. Every mountain has at least one peak and may have more than one.

PLAIN: A section of the path consisting of only horizontal steps which starts either on the y-axis or at a vertex preceded by a southeast step and ends at a vertex followed by a northeast step.

The **HEIGHT** of a vertex is its y-coordinate. The **WEIGHT** of a vertex is its x-coordinate. The **WEIGHT OF A PATH** is the sum of the weights of its peaks.

Example: The following path has five peaks, three valleys, three mountains and one plain.



Graph 1

In this example, there are two peaks of height three and three of height two, two valleys of height one and one of height zero.
The weight of this path is $0 + 3 + 9 + 12 + 17 = 41$.

Now we state our main result.

Theorem 1.1. For a positive integer k, let $A_k(\nu)$ denote the number of two-colour partitions of ν with parts $\geq k$ such that (i) the parts m_i satisfy $m \equiv i + k - 1(mod2)$, with $m > 1$ if k is even, (ii) if m_i is the smallest or the only part in the partition, then $m \equiv i + k - 1(mod4)$ and (iii) the weighted difference between any two consecutive parts is nonnegative and is $\equiv 0(mod4)$. Let $B_k(\nu)$ denote the number of lattice paths of weight ν which start at $(0,0)$, such that (iv) they have no valley above height 0, (v) there is a plain of length $\equiv k - 1(mod4)$ in the beginning of the path, other plains, if any, are of lengths which are multiples of 4 and (vi) the height of each peak of odd (resp., even) weight is 1 (resp., 2) if k is odd and 2 (resp., 1) if k is even. Then

$$A_k(\nu) = B_k(\nu), \quad for\ all\ \nu\ , \tag{1.3}$$

and

$$\sum_{\nu=0}^{\infty} A_k(\nu)q^{\nu} = \sum_{\nu=0}^{\infty} B_k(\nu)q^{\nu} = \sum_{n=0}^{\infty} \frac{q^{n(n+k-1)}(-q;q^2)_n}{(q^4;q^4)_n} . \tag{1.4}$$

In the next section we prove Theorem 1.1. Our method consists in proving that both the functions $A_k(\nu)$ and $B_k(\nu)$ are generated by the extreme right hand side of (1.4). We shall also prove (1.3) bijectively. In our **Section 3** we illustrate by an example that our Theorem has the potential of yielding Rogers-Ramanujan-MacMahon type combinatorial identities.

2 Proof of Theorem 1.1

Step I. We shall prove that

$$\sum_{\nu=0}^{\infty} A_k(\nu)q^{\nu} = \sum_{n=0}^{\infty} \frac{q^{n(n+k-1)}(-q;q^2)_n}{(q^4;q^4)_n} . \tag{2.1}$$

Let $A_k(m,\nu)$ denote the number of partitions enumerated by $A_k(\nu)$ into m parts. We shall first prove the identity,

$$A_k(m,\nu) = A_k(m-1,\nu-k-2m+2)+A_k(m-1,\nu-k-4m+3)+A_k(m,\nu-4m). \tag{2.2}$$

We give the proof of (2.2) for odd k as the proof for even k is similar and hence is omitted.

To prove (2.2) for odd k, we split the partitions enumerated by $A_k(m,\nu)$ into three classes:

(i) those that have least part equal to k_1,

(ii) those that have least part equal to $(k+1)_2$, and

(iii) those that have least part greater than or equal to $(k+2)_1$.

We note that in class (iii) the parts are $\geq 5_1$ because if $k = 1$ then 3_1 can not be a part in view of the condition (ii) of the theorem.

We now transform the partitions in class (i) by deleting the least part k_1 and then subtracting 2 from all the remaining parts ignoring the subscripts. This produces a partition of $\nu - k - 2(m-1)$ into exactly $(m-1)$ parts each of which is $\geq k_1$ (since originally the second smallest part was $\geq (k+2)_1$). Obviously, this transformation does not disturb the weighted difference condition (iii) between the parts and so the transformed partition is of the type enumerated by $A_k(m-1, \nu-k-2m+2)$.

Next, we transform the partitions in class (ii) by deleting the least part $(k+1)_2$ and then subtracting 4 from all the remaining parts.This produces a partition of $\nu - (k+1) - 4(m-1) = \nu - k - 4m + 3$ into $m-1$ parts, each of which is $\geq k_1$ (since originally the second smallest part was $\geq (k+4)_1$). Note that originally $(k+2)_1$ and $(k+3)_2$ could not be the second smallest part because of the weighted difference condition (iii). Furthermore, since the weighted difference condition between the parts is not disturbed, we see that the transformed partition is of the type enumerated by $A_k(m-1, \nu-k-4m+3)$.

Finally, we transform the partitions in class (iii) by subtracting 4 from each part ignoring the subscripts. This produces a partition of $\nu - 4m$ into m parts, each $\geq k_1$. Since the weighted difference condition (iii) between the parts is again not disturbed, we see that the transformed partition is of the type enumerated by $A_k(m, \nu - 4m)$.

The above transformations establish a bijection between the partitions enumerated by $A_k(m,\nu)$ and those enumerated by $A_k(m-1, \nu-k-2m+2) + A_k(m-1, \nu-k-4m+3) + A_k(m, \nu-4m)$.

This proves the Identity (2.2).

For $|z| < |q|^{-1} and \; |q| < 1$, let

$$f_k(z;q) = \sum_{\nu=0}^{\infty} \sum_{m=0}^{\infty} A_k(m,\nu) z^m q^\nu. \tag{2.3}$$

Substituting for $A_k(m,\nu)$ from (2.2) in (2.3) and then simplifying, we get the following q- functional equation

$$f_k(z;q) = zq^k f_k(zq^2;q) + zq^{k+1} f_k(zq^4;q) + f_k(zq^4;q). \tag{2.4}$$

Since $f_k(0;q) = 1$, we may easily check by coefficient comparison in (2.4) that

$$f_k(z;q) = \sum_{n=0}^{\infty} \frac{q^{n(n+k-1)}(-q;q^2)_n z^n}{(q^4;q^4)_n}. \tag{2.5}$$

Now

$$\sum_{\nu=0}^{\infty} A_k(\nu) q^{\nu} = \sum_{\nu=0}^{\infty} \left(\sum_{m=0}^{\infty} A_k(m,\nu) \right) q^{\nu}$$
$$= f_k(1;q)$$
$$= \sum_{n=0}^{\infty} \frac{q^{n(n+k-1)}(-q;q^2)_n}{(q^4;q^4)_n} .$$

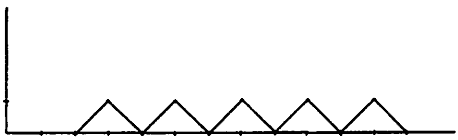
This completes the proof of (2.1).

Step II. We shall prove that

$$\sum_{\nu=0}^{\infty} B_k(\nu) q^{\nu} = \sum_{m=0}^{\infty} \frac{q^{m(m+k-1)}(-q;q^2)_m}{(q^4;q^4)_m} . \tag{2.6}$$

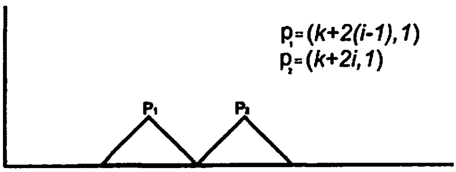
In $\frac{q^{m(m+k-1)}(-q;q^2)_m}{(q^4;q^4)_m}$ the factor $q^{m(m+k-1)}$ generates a lattice path from (0,0) to $(k+2m-1, 0)$ having m peaks each of height 1 and a plain of length $k-1$ in the beginning of the path.

For $k = 3$ and $m = 5$, the path begins as



Graph 2

In the above graph we consider two successive peaks say, ith and $(i+1)$th and denote them by p_1 and p_2, respectively.

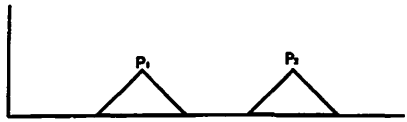


Graph 3

The factor $\frac{1}{(q^4;q^4)_m}$ generates m nonnegative multiples of 4, say $a_1 \geq a_2 \geq \cdots a_m \geq 0$, which are encoded by inserting a_m horizontal steps in front of the first mountain, and $a_i - a_{i+1}$ horizontal steps in front of the $(m-i+1)st$ mountain, $1 \leq i \leq m-1$. Thus the x-coordinate of the ith peak is increased by

$a_m + (a_{m-1} - a_m) + (a_{m-2} - a_{m-1}) + \cdots (a_{m-i+1} - a_{m-i+2}) = a_{m-i+1}$ and the x-coordinate of the $(i+1)$th peak is increased by a_{m-i}.

Graph 3 now becomes Graph 4.

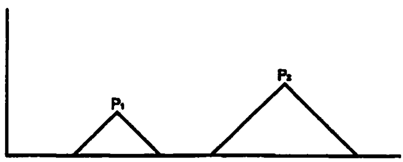


Graph 4

$p_1 \equiv (k + 2(i-1) + a_{m-i+1}, 1)$,
$p_2 \equiv (k + 2i + a_{m-i}, 1)$.

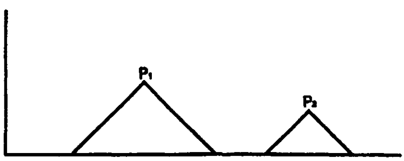
The factor $(-q; q^2)_m$ generates nonnegative multiples of $(2i-1)$, $1 \leq i \leq m$, say, $b_1 \times 1$, $b_2 \times 3, \cdots, b_m \times (2m-1)$, where each b_i $(1 \leq i \leq m)$ is 0 or 1. This is encoded by having the ith peak grow to height $b_{m-i+1}+1$. Each increase by one in the height of a given peak increases its weight by one and the weight of each subsequent peak by two.

Graph 4 now changes to Graph 5 or Graph 6 depending on whether $b_{m-i} > b_{m-i+1}$ or $< b_{m-i+1}$. In the case when $b_{m-i} = b_{m-i+1}$, the new graph looks like Graph 4.



Graph 5

or



Graph 6

Every lattice path enumerated by $B_k(\nu)$ is uniquely generated in this manner. This proves (2.6).

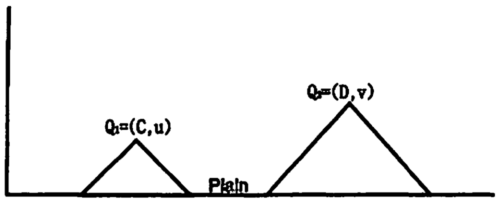
Step III. We now establish a $1-1$ correspondence between the lattice paths enumerated by $B_k(\nu)$ and the two-colour partitions enumerated by $A_k(\nu)$.
We do this by encoding each path as the sequence of the weights of the peaks with each weight subscripted by the height of the respective peak.
Thus, if we denote the two peaks in Graph 5 (or Graph 6) by A_x and B_y, $(B \geq A)$, respectively, then

$$A = k + 2(i-1) + a_{m-i+1} + 2(b_m + b_{m-1} + \cdots + b_{m-i+2}) + b_{m-i+1}$$
$$x = b_{m-i+1} + 1$$
$$B = k + 2i + a_{m-i} + 2(b_m + b_{m-1} + \cdots + b_{m-i+1}) + b_{m-i}$$
$$y = b_{m-i} + 1.$$

Clearly, when k is odd and A (*or* B) is odd (resp. even), x (*or* y) is 1 (resp. 2). Similarly, when k is even and A (*or* B) is odd (resp. even), x (*or* y) is 2 (resp. 1).
The weighted difference of these two parts is $((B_y - A_x)) = B - A - x - y = a_{m-i} - a_{m-i+1}$ which is nonnegative and is a multiple of 4.

To see the reverse implication, we consider two n-colour parts of a partition enumerated by $A_k(\nu)$, say, C_u and D_v with $D \geq C \geq k$; $1 \leq u, v \leq 2$.

Let $Q_1 \equiv (C, u)$ and $Q_2 \equiv (D, v)$ be the corresponding peaks in the associated lattice path.



Graph 7

The length of the plain between the two peaks is $D - C - u - v$ which is the weighted difference between the two parts C_u and D_v and is therefore a nonnegative multiple of 4. (by condition (iii) of the Theorem)
If C_u were the smallest part of the partition, the corresponding peak in the associated path would be the first peak preceded by a plain of length $(k-1)+a$, where a is a non-negative multiple of 4 since

$$C = (k-1)+a+1+b, \qquad (by(\text{ii}))$$

$$u = 1 + b\ ,$$

where b is 0 or 1.

The following two cases are also clear:

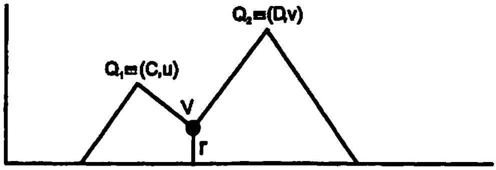
Case 1. when k is odd.

The parts are of the form $(2l-1)_1$ or $(2l)_2$ and the corresponding peaks are of odd weight with height 1 or even weight with height 2.

Case 2. when k is even.

In this case the parts are of the form $(2l)_1$ or $(2l-1)_2$ and the corresponding peaks are of even weight and height 1 or odd weight and height 2.

Finally, we show that there can not be a valley above height 0.
Suppose, there is a valley V of height h $(0 \leq h \leq 1)$ between the peaks Q_1 and Q_2.



Graph 8

In this case there is a descent of $u-h$ from Q_1 to V and an ascent of $v-h$ from V to Q_2. This implies that

$D = C + (u-h) + (v-h)$

$\Rightarrow D - C - u - v = -2h.$

But since the weighted difference is nonnegative, therefore, h=0.

This completes the bijection between the lattice paths enumerated by $B_k(\nu)$ and the two-colour partitions enumerated by $A_k(\nu)$.

3 A particular case of Theorem 1.1

By a little series manipulation, the following identity of Slater [16,p.154,Eq.25)]

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q^4;q^4)_n} = \frac{(-q;q^2)_\infty (q^3;q^3)_\infty (q^3;q^6)_\infty}{(q^2;q^2)_\infty}, \tag{3.1}$$

can be written in the following form:

$$\prod_{\substack{n=1 \\ n\equiv \pm 1, \pm 2 (mod 6)}}^{\infty} \frac{1}{(1-q^n)} = \left(\sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q^4;q^4)_n} \right) \left(\prod_{\substack{n=1 \\ n\equiv \pm 2, \pm 3, 6 (mod 12)}}^{\infty} \frac{1}{(1-q^n)} \right), \tag{3.2}$$

Now an appeal to Theorem 1.1 gives the following 3-way combinatorial interpretation of Identity (3.2)

Theorem 3.1. Let $C_1(\nu)$ denote the number of partitions of ν into parts $\equiv \pm 2, \pm 3, 6 (mod 12)$. And let $D_1(\nu)$ denote the number of partitions of ν into parts not divisible by 3. Then

$$D_1(\nu) = \sum_{k=0}^{\nu} A_1(k) C_1(\nu - k) = \sum_{k=0}^{\nu} B_1(k) C_1(\nu - k), \tag{3.3}$$

where $A_1(k)$ and $B_1(k)$ are as defined in Theorem 1.1.

Example.

$$D_1(7) = 9$$

Also

$$\sum_{k=0}^{7} A_1(k) C_1(7 - k) = 9$$

Table I shows the relevant partitions enumerated by $A_1(\nu)$, $C_1(\nu)$ and $D_1(\nu)$ for $0 \leq \nu \leq 7$.

ν	$A_1(\nu)$	Partitions enumerated by $A_1(\nu)$	$C_1(\nu)$	Partitions enumerated by $C_1(\nu)$	$D_1(\nu)$	Partitions enumerated by $D_1(\nu)$
0	1	empty partition	1	empty partition	1	empty partition
1	1	1_1	0	-	1	1
2	1	2_2	1	2	2	2, 1+1
3	0	-	1	3	2	2+1, 1+1+1
4	1	$3_1 + 1_1$	1	2+2	4	4, 2+2, 2+1+1, 1+1+1+1
5	2	$5_1, 4_2 + 1_1$	1	3+2	5	5, 4+1, 2+2+1, 2+1+1+1, 1+1+1+1+1
6	1	6_2	3	6, 3+3, 2+2+2	7	5+1, 4+2, 4+1+1, 2+2+2, 2+2+1+1, 2+1+1+1+1, 1+1+1+1+1+1
7	1	$5_1 + 2_2$	1	3+2+2	9	7, 5+2, 5+1+1, 4+2+1, 4+1+1+1, 2+2+2+1, 2+2+1+1+1, 2+1+1+1+1+1, 1+1+1+1+1+1+1

Table I

Remark. For the brevity of the paper we have not given the table of the lattice paths enumerated by $B_1(\nu)$. However, we know from Theorem 1.1 that $A_1(\nu) = B_1(\nu)$.

4 Conclusion.

The work done in this paper shows a nice interaction between the theory of basic series and combinatorics. Theorem 1.1 gives a combinatorial identity for each value of k. Thus we get infinitely many combinatorial identities from this theorem. In one particular case, viz., $k = 1$ we get 3-way combinatorial interpretation of a well known basic series identity of L.J. Slater. In the case $k = 3$, the basic series identity analogous to (3.1) was found by Andrews [9] and its combinatorial interpretation different from what we get from our Theorem 1.1 was given by Alladi and Berkovich [8]. It would be of interest if more applications of Theorem 1.1 can be found.

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References.

1. A.K. Agarwal, Rogers-Ramanujan identities for n-color partitions, J. Number Theory 28(1988), 299-305.

2. A.K. Agarwal, New combinatorial interpretations of two analytic identities, Proc. Amer. Math. Soc., Vol.107(2), (1989), 561-567.

3. A.K. Agarwal, q-functional equations and some partition identities, combinatorics and theoretical computer science (Washington, DC, 1989), Discrete Appl. Math. 34(1991), no. 1-3, 17-26.

4. A.K. Agarwal, New classes of infinite 3-way partition identities, ARS Combinatoria, 44(1996), 33-54.

5. A.K. Agarwal and G.E. Andrews, Hook differences and lattice paths, J. Statist. Plann. Inference, Vol.14, No.1 (1986), 5-14.

6. A.K. Agarwal and G.E. Andrews, Rogers-Ramanujan identities for partitions with "N copies of N", J. Combin. Theory Ser.A., 45, No. 1 (1987), 40-49.

7. A.K. Agarwal and D.M. Bressoud, Lattice paths and multiple basic hypergeometric series, Pacific J. Math., 136(2), (1989), 209-228.

8. K. Alladi and A. Berkovich, Göllnitz-Gordon partitions with weights and parity conditions, Zeta functions, topology and quantum physics, 1-17, Dev.Math. 14, Springer, New York,(2005)

9. G.E. Andrews, An introduction to Ramanujan's "Lost" notebook, Amer. Math. Monthly, 86(1979), 89-108.

10. W.G. Connor, Partition theorems related to some identities of Rogers and Watson, Trans. Amer. Math. Soc., 214 (1975), 95-111.

11. H. Göllnitz, Einfache Partitionen (unpublished), Diplomarbeit W.s.(1960), Gotttingen, 65pp.

12. H. Göllnitz, Partitionen mit Differenzenbedingungen, J. Reine Angew. Math., 225(1967), pp. 154-190.

13. B. Gordon, Some Continued Fractions of the Rogers -Ramanujan type, Duke J. Math., 32(1965), 741-748.

14. M.D. Hirschhorm, Some partition theorems of the Rogers-Ramanujan type, J. Combin. theory Ser A, 27(1) (1979), 33-37.

15. P.A. MacMahon, Combinatory Analysis, Vol.2, Cambridge Univ. Press, London and New York (1916).

16. M. Goyal and A.K. Agarwal, Further Rogers-Ramanujan identities for n-color partitions, Utilitas Mathematica, (to appear).

17. L.J. Rogers, Second memoir on the expansion of certain infinite products, Proc. Lond. Math. Soc., 25 (1894), 318-343.

18. L.J. Slater, Further identities of the Rogers-Ramanujan type, Proc. London Math. Soc. 54 (1952), 147-167.

19. M.V. Subbarao, Some Rogers-Ramanujan type partition theorems, Pacific J. Math. 120 (1985), 431-435.

20. M.V. Subbarao and A.K. Agarwal, Further theorems of the Rogers-Ramanujan type, Canad. Math. Bull. 31(2), (1988), 210-214.

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