A characterization of the graphs with high degree sum that are not covered by three cycles

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Abstract

Let G be a graph of order n. In [A. Saito, Degree sums and graphs that are not covered by two cycles, J. Graph Theory **32** (1999), 51–61.], Saito characterized the graphs with $\sigma_3(G) \geq n-1$ that are not covered by two cycles. In this paper, we characterize the graphs with $\sigma_4(G) \geq n-1$ that are not covered by three cycles. Moreover, to prove our main theorem, we show several new results which are useful in the study of this area.

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1 Introduction

In this paper, all graphs are finite undirected graphs without loops or multiple edges. For standard graph theoretic terminology not explained in this paper, we refer the reader to [2]. Let G be a graph. For $X \subseteq V(G)$,

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we let $G[X]$ denote the subgraph induced by X in G, and let $G - X := G[V(G) - X]$. We denote by $N_G(x)$ the neighborhood of a vertex x in G, and let $d_G(x) := |N_G(x)|$. Let H be a subgraph of G. For $x \in V(G)$, let $N_H(x) := N_G(x) \cap V(H)$ and $d_H(x) := |N_H(x)|$. For $X \subseteq V(G)$, let $N_H(X) := \bigcup_{x \in X} N_H(x)$. If there is no fear of confusion, we often identify a subgraph of G with its vertex set. For a positive integer k and $X \subseteq V(G)$, if there exists an independent set of order k in $G[X]$, then we let $\sigma_k(X; H) := \min\{\sum_{x \in S} d_H(x) : S \subseteq X$ is an independent set of G with $|S| = k\}$; otherwise, we let $\sigma_k(X; H) := +\infty$. If $X = V(H)$, then we abbreviate $\sigma_k(V(H); H)$ by $\sigma_k(H)$. Let $p(G)$ and $c(G)$ be the order of a longest path and a longest cycle of G, respectively. We define $\mathrm{diff}(G) := p(G) - c(G)$, and is called a *relative length* of G. We write a cycle (or a path) C with a given orientation by $\overrightarrow{C}$, and we denote by $\overleftarrow{C}$ a cycle (or a path) C with a reverse orientation. If there is no fear of confusion, we abbreviate $\overrightarrow{C}$ by C. Let $\overrightarrow{C}$ be an oriented cycle or a path. For $u, v \in V(C)$, we denote by $u\overrightarrow{C}v$ a path from u to v along $\overrightarrow{C}$. The reverse sequence of $u\overrightarrow{C}v$ is denoted by $v\overleftarrow{C}u$. For $u \in V(C)$ and a positive integer h, we denote the h-th successor and the h-th predecessor of u on $\overrightarrow{C}$ by u^{+h} and u^{-h}, respectively, and let $u^{+0} = u^{-0} := u$. For $X \subseteq V(C)$ and a positive integer h, we define $X^{+h} := \{x^{+h} : x \in X\}$ and $X^{-h} := \{x^{-h} : x \in X\}$, respectively. We abbreviate u^{+1}, u^{-1}, X^{+1} and X^{-1} by u^+, u^-, X^+ and X^-, respectively. In this paper, we regard K_1 and K_2 as cycles.

In [9], Ore gave a degree sum condition for the existence of a Hamiltonian cycle.

Theorem A (Ore [9]) *Let G be a graph of order n. If $\sigma_2(G) \geq n$, then G is Hamiltonian.*

If there exist cycles $C_1, \ldots, C_t$ in G such that $\bigcup_{i=1}^{t} V(C_i) = V(G)$, then we say that G *is covered by t cycles*. Note that if G is Hamiltonian, then G is covered by one cycle. In [5], Enomoto, Kaneko and Tuza proved the following theorem.

Theorem B (Enomoto et al. [5]) *Let G be a graph of order n. If $\sigma_3(G) \geq n$, then G is covered by two cycles.*

In [6], Kouider and Lonc generalized these results as follows. (Later, Ainouche and Kouider [1] improved this result.)

Theorem C (Kouider et al. [6]) *Let G be a graph of order n. If $\sigma_k(G) \geq n$, then G is covered by $k-1$ cycles.*

On the other hand, for $k = 2, 3$, a characterization of the graphs with $\sigma_k(G) \geq n-1$ that are not covered by $k-1$ cycles has been determined.

Theorem D (Nara [8]) *Let G be a 2-connected graph of order n. If $\sigma_2(G) \geq n-1$, then one of the following holds.*

(i) *G is Hamiltonian.*

(ii) *$K_{k,k+1} \subseteq G \subseteq K_k + (k+1)K_1$ with $k = (n-1)/2 \geq 2$.*

Theorem E (Saito [10]) *Let G be a graph of order n. If $\sigma_3(G) \geq n-1$, then one of the following holds.*

(i) *G is covered by two cycles.*

(ii) *$K_{k,2k+1} \subseteq G \subseteq K_k + (2k+1)K_1$ with $k = (n-1)/3 \geq 2$.*

(iii) *G belongs to one of two exceptional classes whose element has connectivity at most one.*

In this paper, we characterize the graphs with $\sigma_4(G) \geq n-1$ that are not covered by three cycles as follows.

Theorem 1 *Let G be a graph of order n. If $\sigma_4(G) \geq n-1$, then one of the following holds.*

(i) *G is covered by three cycles.*

(ii) *$K_{k,3k+1} \subseteq G \subseteq K_k + (3k+1)K_1$ with $k = (n-1)/4 \geq 2$.*

(iii) *G belongs to one of three exceptional classes $\mathcal{G}_i$ $(1 \leq i \leq 3)$.*

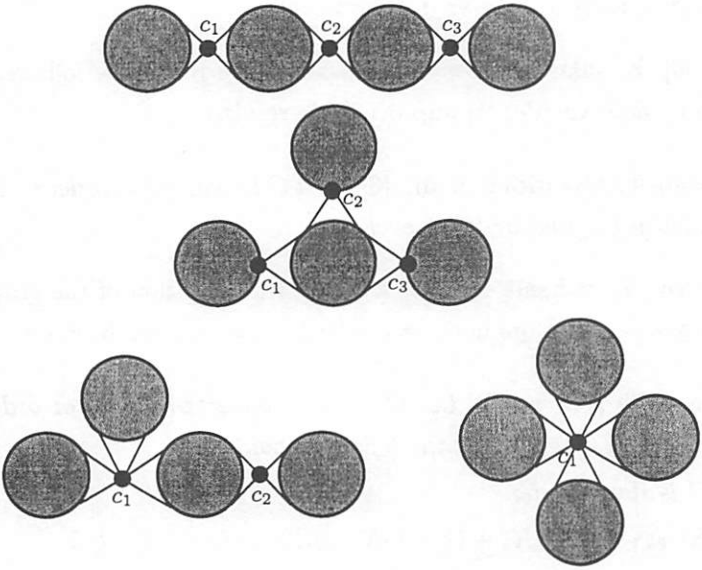

Figure 1: The graphs in $\mathcal{G}_1$

Exceptional classes in Theorem 1. Let $\mathcal{G}$ be a set of graphs with connectivity one. For $1 \leq i \leq 3$, let $\mathcal{G}_i$ be the subset of $\mathcal{G}$ whose element G_i satisfies the following properties:

(I) $\mathcal{G}_1$: There exist s cut vertices $c_1, \ldots, c_s$ in G_1 for some $1 \leq s \leq 3$ such that $G_1 - \{c_1, \ldots, c_s\}$ has exactly four components, and each component is complete (see Figure 1).

(II) $\mathcal{G}_2$: G_2 has exactly two blocks H_1 and H_2 which satisfy one of the following (i) and (ii) (see Figure 2).
(i) $K_{k,2k+1} \subseteq H_1 \subseteq K_k + (2k+1)K_1$ for some $k \geq 2$ and H_2 is complete, and further the cut vertex of G_2 has degree at least $2k+1$ in H_1.
(ii) $K_{k,k+1} \subseteq H_i \subseteq K_k + (k+1)K_1$ for some $k \geq 2$ $(i = 1, 2)$, and further the cut vertex of G_2 has degree at least $k+1$ in H_i $(i = 1, 2)$.

(III) $\mathcal{G}_3$: G_3 has exactly three or four blocks, and if G_3 has four blocks, then there exist two distinct blocks of G_3 such that each of them is not an end block of G_3 and has order just 2. Moreover, there exist two blocks H_1

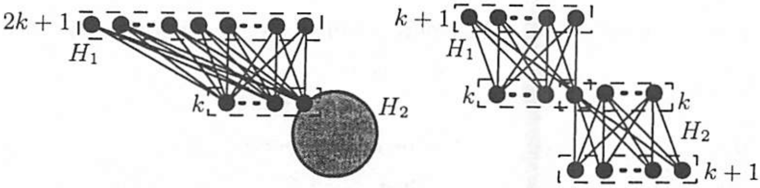

Figure 2: The graphs in $\mathcal{G}_2$

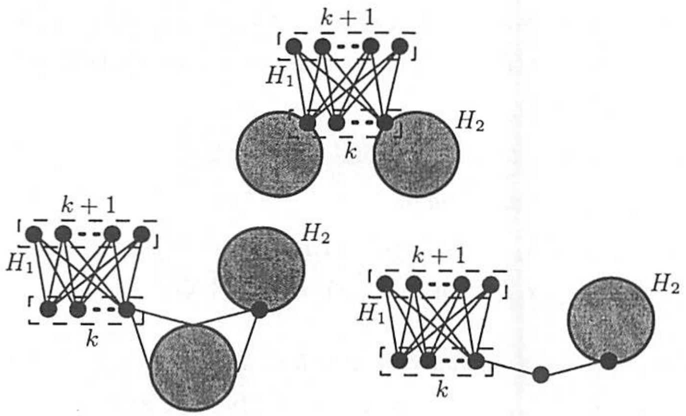

Figure 3: The graphs in $\mathcal{G}_3$

and H_2 in G_3 such that $K_{k,k+1} \subseteq H_1 \subseteq K_k + (k+1)K_1$ for some $k \geq 2$, H_2 is complete and $G - (H_1 \cup H_2)$ is complete, and further the cut vertex of G_3 which belongs to H_1 has degree at least $k+1$ in H_1 (see Figure 3).

In the proof of Theorem 1, we divide the proof into three cases according to the connectivity of the graph G. In case that G is 2-connected, we use the arguments based on the relative length. In particular, the following lemma is a most important in this case.

Lemma 1 *Let G be a 2-connected graph of order n. If $\sigma_4(G) \geq n-1$, then G is covered by three cycles or* $\mathrm{diff}(G) \leq 2$.

In case that the connectivity of G is one, we consider a degree sum of some vertices which are not cut vertices. Therefore, we need the following lemmas concerning the degree sum conditions except for specified vertices for graphs to be covered by some cycles.

Lemma A (**Saito** [10]) *Let G be a 2-connected graph of order n, and let $c \in V(G)$. If $\sigma_2(V(G) - \{c\}; G) \geq n$, then G is Hamiltonian.*

The case $c_1 = c_2$ of the following lemma was proved in [10].

Lemma 2 *Let G be a 2-connected graph of order n, and let $c_1, c_2 \in V(G)$. If $\sigma_2(V(G) - \{c_1, c_2\}; G) \geq n - 1$, then one of the following holds.*

(i) *G is Hamiltonian.*

(ii) *$G - \{c_i\}$ is Hamiltonian for some $i = 1$ or 2.*

(iii) *$G - \{c_1, c_2\}$ is Hamiltonian.*

(iv) *$K_{k,k+1} \subseteq G \subseteq K_k + (k+1)K_1$ with $k = (n-1)/2 \geq 2$ and $d_G(c_i) \geq k+1$ for $i = 1, 2$. (Then $\sigma_2(V(G) - \{c_1, c_2\}; G) = n - 1$.)*

Lemma 3 *Let G be a 2-connected graph of order n, and let $c \in V(G)$. If $\sigma_3(V(G) - \{c\}; G) \geq n$, then G is covered by two cycles.*

Lemma 4 *Let G be a 2-connected graph of order n, and let $c \in V(G)$. If $\sigma_3(V(G) - \{c\}; G) \geq n - 1$, then one of the following holds.*

(i) *G is covered by two cycles.*

(ii) *$G - \{c\}$ is covered by two cycles.*

(iii) *$K_{k,2k+1} \subseteq G \subseteq K_k + (2k+1)K_1$ with $k = (n-1)/3 \geq 2$ and $d_G(c) \geq 2k+1$.*

Since the proofs of Lemmas 1–4 are long, we prove these lemmas after the section of the proof of Theorem 1 for the convenience of the reader (see Sections 4–7).

2 Lemmas to prove Theorem 1 (and Lemmas 1–4)

In this section, we prepare some lemmas to prove Theorem 1 (and Lemmas 1–4). We first prepare the results concerning the existence of cycles covering specified vertices. In [5], Enomoto, Kaneko and Tuza proved the following. (In fact, they deal with the case where $d_C(x) \geq |X| + 1$ for all $x \in X$. However, the same argument also works for the case where G is 2-connected and $d_C(x) \geq |X|$ for all $x \in X$.)

Lemma B (Enomoto et al. [5]) *Let G be a 2-connected graph and C be a cycle of G, and let $X \subseteq V(G - C)$. If $d_C(x) \geq |X|$ for all $x \in X$, then there exists a cycle containing X.*

As a generalization of Lemma B for the case $G - C \cong sK_2 \cup tK_1$, we prove the following lemma.

Lemma 5 *Let G be a 2-connected graph and C be a cycle of G. Suppose that $G - C \cong sK_2 \cup tK_1$ for some integers s and t with $s, t \geq 0$. Let $X \subseteq V(G - C)$. If $\sigma_k(X; C) \geq |X|$, then there exist k cycles $C_1, \ldots, C_k$ such that $X \subseteq \bigcup_{i=1}^{k} V(C_i)$.*

Proof. We prove it by an induction on k. Let x_0 be a vertex in X such that $d_C(x_0) = \min\{d_C(x) : x \in X\}$, and let $a := d_C(x_0)$. By Lemma B and the minimality of $d_C(x_0)$ and since G is 2-connected, there exist a cycle C' and $X' \subseteq X$ such that $|X'| \geq a$ and $\{x_0\} \cup (N_G(x_0) \cap X) \subseteq X' \subseteq V(C')$. Then $\sigma_{k-1}(X - X'; C) \geq \sigma_k(X; C) - d_C(x_0) \geq |X| - a \geq |X - X'|$. Hence by the hypothesis of induction, we obtain the conclusion. □

Furthermore, we need the following two lemmas. We omit the proof of Lemma 6 since the proof is easy. A cycle C of a graph G is called a *maximal cycle* of G if there exists no cycle C' of G such that $V(C) \subsetneq V(C')$.

Lemma 6 *Let G be a graph and C be a maximal cycle of G, and let H be a component of $G - C$. Then the following hold.*

(i) $N_C(H) \cap N_C(H)^+ = \emptyset$, *specifically,* $d_C(x) \leq |N_C(H)| \leq |C|/2$ *for all* $x \in V(H)$.

(ii) *For* $u_1, u_2 \in N_C(H)$ *with* $u_1 \neq u_2$, $u_1^+ u_2^+, u_1^- u_2^- \notin E(G)$.

(iii) *For* $x, y \in V(G-C)$, $|N_C(x)^+ \cap N_C(y)| \leq 1$.

Lemma 7 *Let* G *be a graph and* C *be a maximal cycle of* G*, and let* $X \subseteq V(G-C)$. *Suppose that there exists a vertex* x *in* X *such that* $d_C(x) \geq 2$ *and* $d_C(x') = |C|/2$ *for all* $x' \in X - \{x\}$. *If* $|C| \geq 2|X|$, *then there exists a cycle containing* X.

Proof. If $|X| = 1$, then the assertion clearly holds. Thus we may assume that $|X| \geq 2$. Let $\{x_1, \dots, x_s\} := X$, and suppose that $d_C(x_s) \geq 2$ and $d_C(x_i) = |C|/2$ for each $1 \leq i \leq s-1$. Let $\{u_1, \dots, u_l\} := N_C(x_1)$. We may assume that $u_1, \dots, u_l$ occur in this order along $\overrightarrow{C}$. By Lemma 6 (i), $|u_i \overrightarrow{C} u_{i+1}| = 3$ for each $1 \leq i \leq l$, where we let $u_{l+1} := u_1$. By Lemma 6 (i) and (iii), we also have $N_C(x_1) = N_C(x_i)$ for each $2 \leq i \leq s-1$. By Lemma 6 (iii), and by changing the label if necessary, we can take $u, u' \in N_C(x_s)$ such that $u = u_1$ and $u' \in \{u_i, u_i^+\}$ for some i with $i \geq 2$. Note that $l \geq s \geq 2$ because $l = |C|/2 \geq |X| = s \geq 2$. If $s = 2$, then we can easily obtain the desired conclusion. Hence we may assume that $s \geq 3$. Let $v_1, \dots, v_{s-2}$ be $s-2$ distinct vertices in $N_C(x_1) - \{u_1, u_i\}$. Since $N_C(x_1) = N_C(x_i)$ for each $2 \leq i \leq s-1$, if $u' = u_i$, then $v_1 x_1 v_2 x_2 \dots v_{s-2} x_{s-2} u_1 x_s u_i x_{s-1} v_1$ is a cycle containing X; otherwise, $v_1 x_1 v_2 x_2 \dots v_{s-2} x_{s-2} u_1 x_s u_i^+ u_i x_{s-1} v_1$ is a cycle containing X. □

3 Proof of Theorem 1

Let G be a graph of order n, and suppose that $\sigma_4(G) \geq n-1$ and G is not covered by three cycles.

Case 1. G is 2-connected.

Then by Lemma 1, diff$(G) \leq 2$. Let $\overrightarrow{C}$ be a longest cycle of G, and let $X := V(G-C)$. Since diff$(G) \leq 2$, $G - C \cong sK_2 \cup tK_1$ for some integers

s and t with $s, t \geq 0$. Since G is 2-connected and G is not covered by three cycles, $s + t \geq 5$. Choose C so that (i) s is as small as possible, (ii) $\sum_{x \in X} d_G(x)$ is as small as possible, subject to (i). Let $X^* := \{x \in X : N_G(x) \cap X \neq \emptyset\}$. For each $x \in X$, let $\{x^*\} := N_G(x) \cap X$ if $x \in X^*$; otherwise, let $x^* := x$. Let $x_1, x_2 \in X$ with $x_1 \neq x_2$, $x_1x_2 \notin E(G)$, $d_C(x_1) \leq d_C(x_2)$ and $d_C(x_1) + d_C(x_2) = \sigma_2(X; C)$. By Lemma 5, $d_C(x_1) + d_C(x_2) \leq |X| - 1$. Let $Y := \{x_1, x_2, x_1^*, x_2^*\}$.

There exists no cycle containing $X - Y$ because G is not covered by three cycles and there exists a cycle containing Y. Hence by Lemma B, the following fact holds.

Fact 3.1 *There exists a vertex z in $X - Y$ such that $d_C(z) \leq |X - Y| - 1$.*

By the maximality of $|C|$ and since diff$(G) \leq 2$, $\sigma_4(G) \geq n - 1$ and $d_C(x_1) + d_C(x_2) \leq |X| - 1$, the following claim holds.

Claim 3.2 *Let $x_3, x_4 \in X$ with $x_3 \neq x_4$. Then the following hold.*

(i) *If $x_3 \in X^*$, then $N_C(x_3)^+ \cap N_C(x_4) = N_C(x_3)^- \cap N_C(x_4) = \emptyset$.*

(ii) *If $x_3, x_4 \in X - Y$ and $x_3x_4 \notin E(G)$, then $d_C(x_3) + d_C(x_4) \geq |C| - \sum_{i=1}^{4} |N_G(x_i) \cap X|$.*

Case 1.1. $s \geq 1$.

Note that $X^* \neq \emptyset$ because $s \geq 1$. We first consider about the upper bound of the degree of a vertex in X^* and the relations between the degree of a vertex in X^* and the degree of an other vertex in X (see Claims 3.3 and 3.4).

Claim 3.3 *Let $x_3 \in X^*$. Then the following hold.*

(i) $d_C(x_3) \leq |C|/3$.

(ii) $d_C(x_4) \leq |C| - 2d_C(x_3)$ *for all* $x_4 \in X - \{x_3\}$.

(iii) $d_C(x_3) \leq |C| - 2d_C(x_4) + 2 - |\{3 \leq i \leq 4 : x_i \in X^*\}|$ *for all* $x_4 \in X - \{x_3, x_3^*\}$.

Proof. Suppose that there exists $v \in N_C(x_3)^+ \cap N_C(x_3)^-$. Then $C' := v^- x_3 v^+ \overrightarrow{C} v^-$ is a longest cycle of G with $V(C') = (V(C) - \{v\}) \cup \{x_3\}$. Since $xv \notin E(G)$ for all $x \in X - \{x_3\}$ by Claim 3.2 (i), this contradicts the choice (i) of C. Thus $N_C(x_3)^+ \cap N_C(x_3)^- = \emptyset$. By Lemma 6 (i), we also have $N_C(x_3)^+ \cap N_C(x_3) = \emptyset$. These imply that (i) holds.

Let $x_4 \in X - \{x_3\}$. Then by Claim 3.2 (i), $N_C(x_4) \subseteq V(C) - (N_C(x_3)^+ \cup N_C(x_3)^-)$. Since $N_C(x_3)^+ \cap N_C(x_3)^- = \emptyset$, $d_C(x_4) \leq |C - (N_C(x_3)^+ \cup N_C(x_3)^-)| = |C| - 2|N_C(x_3)| = |C| - 2d_C(x_3)$, and hence (ii) holds.

Let $x_4 \in X - \{x_3, x_3^*\}$. If $x_4 \in X^*$, then by Claim 3.3 (ii), clearly (iii) holds. Thus we may assume that $x_4 \notin X^*$. To prove (iii), we show $|N_C(x_4)^{+2} \cap N_C(x_3)| \leq 1$. Suppose that there exist $v_1, v_2 \in N_C(x_4)^{+2} \cap N_C(x_3)$ with $v_1 \neq v_2$. Then $C' := v_1 x_3 v_2 \overrightarrow{C} v_1^{-2} x_4 v_2^{-2} \overleftarrow{C} v_1$ is a longest cycle of G with $V(C') = (V(C) - \{v_1^-, v_2^-\}) \cup \{x_3, x_4\}$. Since $v_1^- v_2^- \notin E(G)$ and $v_i^- x \notin E(G)$ for each $i = 1, 2$ and each $x \in X - \{x_3, x_4\}$ by Lemma 6 (ii) and Claim 3.2 (i), this contradicts the choice (i) of C. Thus $|N_C(x_4)^{+2} \cap N_C(x_3)| \leq 1$. Hence by Claim 3.2 (i), we obtain (iii). □

In fact, we can obtain a slightly stronger statement than Claim 3.3 (i).

Claim 3.4 *$d_C(x) \leq (|C| - 1)/3$ for all $x \in X^*$.*

Proof. Let $x \in X^*$. Then by Claim 3.3 (i), $d_C(x) \leq |C|/3$. Suppose that $d_C(x) = |C|/3$. Then by Claim 3.3 (ii), $d_C(x') \leq |C| - 2d_C(x) = |C|/3$ for all $x' \in X - \{x\}$. Let $x_3, x_4 \in X - Y$ with $x_3 \neq x_4$ and $x_3 x_4 \notin E(G)$. Then $|C| + |X| - 1 = n - 1 \leq \sum_{i=1}^4 d_G(x_i) \leq |X| - 1 + 2|C|/3 + \sum_{i=1}^4 |N_G(x_i) \cap X|$, that is, $|C| \leq 3\sum_{i=1}^4 |N_G(x_i) \cap X|$. On the other hand, $|C| + |X| - 1 = n - 1 \leq \sum_{i=1}^4 d_G(x_i) \leq 4|C|/3 + \sum_{i=1}^4 |N_G(x_i) \cap X|$, that is, $|C| \geq 3|X| - 3 - 3\sum_{i=1}^4 |N_G(x_i) \cap X|$. Hence $3|X| - 3 - 3\sum_{i=1}^4 |N_G(x_i) \cap X| \leq 3\sum_{i=1}^4 |N_G(x_i) \cap X|$, that is, $|X| \leq 1 + 2\sum_{i=1}^4 |N_G(x_i) \cap X|$, in particular, $|X| \leq 9$. Since $s + t \geq 5$, $|X| \geq 5 + \sum_{i=1}^4 |N_G(x_i) \cap X|$. Hence $5 + \sum_{i=1}^4 |N_G(x_i) \cap X| \leq 1 + 2\sum_{i=1}^4 |N_G(x_i) \cap X|$, that is, $\sum_{i=1}^4 |N_G(x_i) \cap X| \geq 4$. This implies that $x_i \in X^*$ for each $1 \leq i \leq 4$. Since x_3 and x_4 are arbitrary vertices in $X - Y$ with $x_3 \neq x_4$ and $x_3 x_4 \notin E(G)$, we obtain

$X^* = X$. Then since $s = s + t \geq 5$, $|X| \geq 2s \geq 10$, a contradiction. □

Let $x_3, x_4 \in X - Y$ with $x_3 \neq x_4$ and $x_3x_4 \notin E(G)$. Let $l := |\{1 \leq i \leq 4 : x_i \in X^*\}|$ and $l' := |\{3 \leq i \leq 4 : x_i \in X^*\}|$. Choose x_3 and x_4 so that l' is as large as possible.

Claim 3.5 (i) *If $l' \geq 1$, then $|C| \leq 3l + 2 - l'$.*
(ii) *If $l' = 0$, then $|C| \leq 2|X| - 6$.*

Proof. Suppose that $l' \geq 1$. We may assume that $x_3 \in X^*$. Then by Claim 3.3 (ii) and (iii), $d_C(x_3) \leq |C| - 2d_C(x_4) + 2 - l'$ and $d_C(x_4) \leq |C| - 2d_C(x_3)$. Hence $|C| + |X| - 1 = n - 1 \leq \sum_{i=1}^{4} d_G(x_i) \leq |X| - 1 + 2|C| - 2(d_C(x_3) + d_C(x_4)) + l + 2 - l'$, that is, $d_C(x_3) + d_C(x_4) \leq (|C| + l + 2 - l')/2$. Therefore, by Claim 3.2 (ii), $|C| - l \leq d_C(x_3) + d_C(x_4) \leq (|C| + l + 2 - l')/2$, that is, $|C| \leq 3l + 2 - l'$.

Suppose that $l' = 0$. Then by the choice of x_3 and x_4, $(X - Y) \cap X^* = \emptyset$. We may assume that $d_C(x_4) \leq d_C(x)$ for all $x \in X - Y$. Then by Lemma 6 (i), Fact 3.1 and Claim 3.2 (ii), $|X| - l - 3 = |X - Y| - 1 \geq d_C(x_4) \geq |C| - d_C(x_3) - l \geq |C| - |C|/2 - l = |C|/2 - l$, that is, $|C| \leq 2|X| - 6$. □

Claim 3.6 $l' = 0$.

Proof. Suppose that $l' \geq 1$. We may assume that $x_3 \in X^*$. Then by Lemma 6 (i) and Claim 3.4, $|C| + |X| - 1 = n - 1 \leq \sum_{i=1}^{4} d_G(x_i) \leq l(|C| - 1)/3 + (4 - l)|C|/2 + l$, that is, $|X| \leq (1 - l/6)|C| + 2l/3 + 1$. By Claim 3.5 (i) and since $s + t \geq 5$, $5 + l \leq |X| \leq (1 - l/6)|C| + 2l/3 + 1 \leq (1 - l/6)(3l + 2 - l') + 2l/3 + 1$. If $l' = 2$, then $3l^2 - 16l + 24 \leq 0$. Since $2 \leq l \leq 4$, this is a contradiction. Thus $l' = 1$, that is, $x_4 \notin X^*$. Then $l/3 + 4 \leq (1 - l/6)(3l + 1)$. Since $1 \leq l \leq 3$, either $l = 2$ or $l = 3$ holds, and the equalities hold in the above inequalities. This implies that $|C| = 3l + 1$, $d_C(x_3) = (|C| - 1)/3$ and $d_C(x_4) = |C|/2$. Then by Claim 3.3 (ii), $(3l + 1)/2 = |C|/2 = d_C(x_4) \leq |C| - 2d_C(x_3) = |C| - 2l = l + 1$, a contradiction. □

By the choice of x_3 and x_4 and Claim 3.6, $(X - Y) \cap X^* = \emptyset$. Choose

$v_1 \in X - Y$ so that $d_G(v_1) \geq d_G(v)$ for all $v \in X - Y$. Let $v_2, v_3 \in X - (Y \cup \{v_1\})$ with $v_2 \neq v_3$. Since $s \geq 1$, there exists $1 \leq h \leq 2$ such that $x_h \in X^*$. Note that $|X - Y| \leq |X| - 3$ since $x_h \in X^*$. Then by Claim 3.3 (iii), $n - 1 \leq d_G(x_h) + \sum_{i=1}^{3} d_G(v_i) \leq d_G(x_h) + 3d_G(v_1) = (1 + d_C(x_h)) + 3d_C(v_1) \leq (1 + (|C| - 2d_C(v_1) + 1)) + 3d_C(v_1) = |C| + d_C(v_1) + 2$, that is, $d_C(v_1) \geq |X| - 3$. Therefore by Lemma 6 (i) and Claim 3.5 (ii), $(|C| + 6)/2 - 3 \leq |X| - 3 \leq d_C(v_1) \leq |C|/2$, that is, $d_C(v_1) = |C|/2$, and the equalities hold in the above inequalities. Hence $d_C(v_i) = |X| - 3$ for each $1 \leq i \leq 3$. Since v_2 and v_3 are arbitrary vertices in $X - (Y \cup \{v_1\})$, $d_C(v) = |X| - 3 \geq |X - Y|$ for all $v \in X - Y$, which contradicts Fact 3.1.

Case 1.2. $s = 0$.

Since $t = s + t \geq 5$, we can take $x_3, x_4 \in X - \{x_1, x_2\}$ with $x_3 \neq x_4$. Then by Lemma 6 (i) and Claim 3.2 (ii), $d_G(x_3) = d_G(x_4) = |C|/2$, and hence we obtain $d_G(x_1) + d_G(x_2) = |X| - 1$. Since x_3 and x_4 are arbitrary vertices in $X - \{x_1, x_2\}$ with $x_3 \neq x_4$, $d_G(x) = |C|/2$ for all $x \in X - \{x_1, x_2\}$. Note that $|C|$ is even. Then by Lemma 7 and since there exist no cycles C_1 and C_2 such that $X \subseteq V(C_1) \cup V(C_2)$, we obtain $|C| \leq |X| - 1$. Hence $d_G(x) = |C|/2 \leq (|X| - 1)/2$ for all $x \in X - \{x_1, x_2\}$. Since $d_G(x_1) + d_G(x_2) = |X| - 1$, and by the choice of x_1 and x_2, we obtain $d_G(x) = (|X| - 1)/2$ for all $x \in X$. Let $\{u_1, \ldots, u_k\} := N_G(x_1)$. We may assume that $u_1, \ldots, u_k$ occur in this order along $\overrightarrow{C}$. Then by Lemma 6 (i) and (iii), $|u_i \overrightarrow{C} u_{i+1}| = 3$ for each $1 \leq i \leq k$, where let $u_{k+1} := u_1$, and $N_G(x_1) = N_G(x)$ for all $x \in X$. Then $|C| = 2k$ and $|X| = 2k + 1$. Let $D_i := u_i x_1 u_{i+1} \overrightarrow{C} u_i$ for each $1 \leq i \leq k$. Since $u_i^+ \notin N_G(x)$ for each $1 \leq i \leq k$ and each $x \in X$, D_i is a longest cycle of G such that $V(G - D_i)$ is an independent set of G and $V(D_i) = (V(C) - \{u_i^+\}) \cup \{x_1\}$. Hence by the choice (ii) of C, $d_G(u_i^+) \geq d_G(x_1) = k$ for each $1 \leq i \leq k$. By Lemma 6 (i) and (ii), $d_G(u_i^+) = k$ and $N_G(u_i^+) = N_G(x_1)$ for each $1 \leq i \leq k$. Hence $K_{k,3k+1} \subseteq G \subseteq K_k + (3k + 1)K_1$.

Case 2. G is connected, but not 2-connected.

Let C be a set of cut vertices of G, and let $\mathcal{B} := \{B : B$ is a block of G with $V(B) - C \neq \emptyset\}$. Let B_1 and B_2 be dintinct end blocks of G, and let $c_i \in V(B_i) \cap C$ for $i = 1, 2$. Note that $B_1, B_2 \in \mathcal{B}$. For convenience, let $I_i := V(B_i) - \{c_i\}$ for $i = 1, 2$ and $I := I_1 \cup I_2$.

Since G is not covered by three cycles and $\sigma_4(G) \geq n-1$, if $V(G) - (I \cup \{c_1, c_2\}) \neq \emptyset$, then B_1, B_2 and $G - (I \cup \{c_1, c_2\})$ become denser as follows.

Claim 3.7 *Let $B_3 := G - I$, and suppose that $V(B_3) - \{c_1, c_2\} \neq \emptyset$. Let i be an integer with $1 \leq i \leq 3$. Then $d_G(x) + d_G(y) \geq |B_i| - 1$ for all $x, y \in V(B_i) - \{c_1, c_2\}$ with $x \neq y$ and $xy \notin E(G)$, i.e., $\sigma_2(B_i - \{c_1, c_2\}; B_i) \geq |B_i| - 1$. Furthermore, if $d_G(x) + d_G(y) = |B_i| - 1$ for some $x, y \in V(B_i) - \{c_1, c_2\}$ with $x \neq y$ and $xy \notin E(G)$, then the following hold.*

(i) *$N_G(z) = V(B_j) - \{z\}$ for each $z \in V(B_j) - \{c_1, c_2\}$ and each $1 \leq j \leq 3$ with $j \neq i$, specifically, if $j \neq 3$, then B_j is complete.*

(ii) *If $i = 3$ and there exist $B_x, B_y \in \mathcal{B}$ such that $x \in V(B_x) - C$, $y \in V(B_y) - C$ and $B_x \neq B_y$, then there exists $c \in C$ such that $V(B_3) = V(B_x) \cup V(B_y)$, $V(B_x) \cap V(B_y) = \{c\}$ and $N_G(x) = V(B_x) - \{x\}$, $N_G(y) = V(B_y) - \{y\}$.*

Proof. Suppose that $d_G(x) + d_G(y) \leq |B_3| - 1$ for some $x, y \in V(B_3) - \{c_1, c_2\}$ with $x \neq y$ and $xy \notin E(G)$. Let $x_i \in I_i$ for $i = 1, 2$. Then $\{x_1, x_2, x, y\}$ is an independent set of G. Hence $n - 1 \leq d_G(x_1) + d_G(x_2) + d_G(x) + d_G(y) \leq \sum_{i=1}^{3}(|B_i| - 1) \leq n - 1$. Thus the equalities hold in the above inequalities. Since x_i is an arbitrary vertex in I_i for $i = 1, 2$, and $N_G(x) \cup N_G(y) \subseteq V(B_3) - \{x, y\}$, we obtain the desired conclusions for $i = 3$. For $i = 1, 2$, we can also obtain the desired conclusions by the similar argument. □

Let $s := |\mathcal{B}|$, and let $\{B_3, \ldots, B_s\} := \mathcal{B} - \{B_1, B_2\}$ if $s \geq 3$. Let $X := V(G) - \bigcup_{B \in \mathcal{B}} V(B)$.

Claim 3.8 *We may assume that $2 \leq s \leq 3$. Furthermore, if $s = 3$, then we may assume that $X = \emptyset$.*

Proof. Suppose $s \geq 4$, and let $x_i \in V(B_i) - C$ for $i = 3, 4$. Then $d_G(x_3) + d_G(x_4) \leq |B_3| + |B_4| - 2 \leq |G - I| - 1$. Hence by Claim 3.7, $d_G(x_3) + d_G(x_4) =$

$|G-I|-1$. Then by Claim 3.7 (i), B_i is complete for $i=1,2$. Since x_i is an arbitrary vertex in $B_i - C$ for $i=3,4$, this together with Claim 3.7 (ii) implies that $B_i - C$ is complete for $i=3,4$, $V(G-I) = V(B_3) \cup V(B_4)$ and $1 \leq |C| \leq 3$, and hence $G \in \mathcal{G}_1$. In case $s=3$ and $X \neq \emptyset$, by taking $x \in X$ instead of x_4, we can obtain $G \in \mathcal{G}_1$ by the similar argument (note that, in this case, $V(G-(B_1 \cup B_2 \cup B_3)) = \{x_4\}$). □

Case 2.1. $s=3$.

By Claim 3.8, $X=\emptyset$. Thus $V(G) = \bigcup_{i=1}^{3} V(B_i)$. Note that $|V(B_3) \cap C| \leq 2$ and $|B_3| \geq 3$.

Suppose first that $V(B_3) \cap C = \{c_1, c_2\}$, i.e., $B_3 = G - I$. Then by Lemmas A and 2 and since G is not covered by three cycles, there exists $1 \leq i \leq 3$ such that $d_G(x)+d_G(y) \leq |B_i|-1$ for some $x, y \in V(B_i)-\{c_1,c_2\}$ with $x \neq y$ and $xy \notin E(G)$. Then by Claim 3.7, $d_G(x)+d_G(y) = |B_i|-1$, and hence by Claim 3.7 (i), $B_j - \{c_1, c_2\}$ is complete for each $1 \leq j \leq 3$ with $j \neq i$, in particular, B_j is complete if B_j is an end block of G. Hence $G - (B_i - \{c_1, c_2\})$ is covered by two cycles. By Claim 3.7, $\sigma_2(B_i - \{c_1,c_2\}; B_i) \geq |B_i|-1$. Therefore, by Lemma 2, we obtain $K_{k,k+1} \subseteq B_i \subseteq K_k + (k+1)K_1$ for some $k \geq 2$ and $d_{B_i}(c_j) \geq k+1$ for $j=1,2$, and hence $G \in \mathcal{G}_3$. Thus we may assume that $V(B_3) \cap C \neq \{c_1,c_2\}$, i.e., $B_3 \neq G-I$.

Then there exists j with $1 \leq j \leq 2$ such that $c_j z \notin E(G)$ for all $z \in V(B_3) - C = V(G-I) - (C \cup \{c_1,c_2\})$. Hence by Claim 3.7 (i), $\sigma_2(B_i - \{c_i\}; B_i) = \sigma_2(B_i - \{c_1,c_2\}; B_i) \geq |B_i|$ for each $i=1,2$. This together with Lemma A implies that B_i is covered by one cycle for each $i=1,2$. Since $V(G) = \bigcup_{i=1}^{3} V(B_i)$ and G is not covered by three cycle, B_3 is not covered by one cycle. Let $x_i \in I_i$ for each $i=1,2$. Note that $n \geq |B_1|+|B_2|+|B_3|-1$ because $B_3 \neq G-I$. Since $\sigma_4(G) \geq n-1$, we have that for each $z_1, z_2 \in V(B_3)-C$ with $z_1 \neq z_2$ and $z_1z_2 \notin E(G)$, $d_{B_3}(z_1)+d_{B_3}(z_2) = d_G(z_1)+d_G(z_2) \geq \sigma_4(G) - (d_G(x_1)+d_G(x_2)) \geq (|B_1|+|B_2|+|B_3|-2) - (|B_1|+|B_2|-2) = |B_3|$. This implies that

$$\sigma_2(B_3 - C; B_3) \geq |B_3|. \tag{3.1}$$

Let c_3 be a vertex in $(V(B_3) \cap C) - \{c_1,c_2\}$. Then $d_G(c_3)-2 \leq d_{B_3}(c_3) \leq$

$d_G(c_3) - 1$, in particular, if $|V(B_3) \cap C| = 2$, then $d_{B_3}(c_3) = d_G(c_3) - 1$. Let ε be an integer with $d_{B_3}(c_3) = d_G(c_3) - 1 - \varepsilon$. Note that $n \geq |B_1| + |B_2| + |B_3| - 1 + \varepsilon$. Then for each $z \in V(B_3) - C$ with $c_3 z \notin E(G)$, $d_{B_3}(c_3) + d_{B_3}(z) = (d_G(c_3) - 1 - \varepsilon) + d_G(z) \geq \sigma_4(G) - (d_G(x_1) + d_G(x_2)) - 1 - \varepsilon \geq (|B_1| + |B_2| + |B_3| - 2 + \varepsilon) - (|B_1| + |B_2| - 2) - 1 - \varepsilon \geq |B_3| - 1$. This and (3.1) imply that

$$\sigma_2(B_3 - \{c_3^*\}; B_3) \geq |B_3| - 1, \tag{3.2}$$

where let c_3^* be a vertex in $V(B_3) \cap C$ with $c_3^* \neq c_3$ if $|V(B_3) \cap C| = 2$; otherwise, let c_3^* be a vertex in $G - B_3$. Note that if $|V(B_3) \cap C| = 1$, i.e., $c_3^* \in V(G - B_3)$, then (3.2) implies that $\sigma_2(B_3) \geq |B_3| - 1$. Suppose that $|V(B_3) \cap C| = 1$ or $\sigma_2(B_3 - \{c_3^*\}; B_3) \geq |B_3|$. Then by Theorem D and Lemma A, G_3 is covered by one cycle, or $|V(B_3) \cap C| = 1$ and $K_{k,k+1} \subseteq B_3 \subseteq K_k + (k+1)K_1$ for some $k \geq 2$. Since G_3 is not covered by one cycle, the latter case holds, but this contradicts (3.1). Thus $|V(B_3) \cap C| = 2$ and the equality holds in (3.2). Since $|V(B_3) \cap C| = 2$, $d_{B_3}(c_3) = d_G(c_3) - 1$. Hence the equality in (3.2) implies that $n = |B_1| + |B_2| + |B_3| - 1$, and hence $c_3^* \in \{c_1, c_2\}$. Since G is not covered by one cycle, $B_3 - \{c_3^*\}$ is not covered by one cycle. Hence by Lemma 2 and (3.2), $K_{k,k+1} \subseteq B_3 \subseteq K_k + (k+1)K_1$ and $d_{B_3}(c_3^*) = k + 1$ for some $k \geq 2$. But this contradicts (3.1) again.

Case 2.2. $s = 2$.

Then there exists a path P in G such that $G = B_1 \cup B_2 \cup P$ and $V(B_i) \cap V(P) = \{c_i\}$ for $i = 1, 2$. Let $P := x_1 x_2 \ldots x_p$, where $x_1 = c_1$ and $x_p = c_2$.

Suppose first that $p \geq 5$. Then $d_G(x_2) + d_G(x_4) = 4 \leq p - 1 = |G - I| - 1$. Thus by Claim 3.7, $p = 5$ and B_i is complete for $i = 1, 2$, and hence $G \in \mathcal{G}_1$.

Suppose next that $3 \leq p \leq 4$. Since G is not covered by three cycles, we may assume that B_1 has no Hamiltonian cycle. Then by Lemma A, $d_G(x) + d_G(y) \leq |B_1| - 1$ for some $x, y \in I_1$ with $x \neq y$ and $xy \notin E(G)$. Hence by Claim 3.7, $d_G(x) + d_G(y) = |B_1| - 1$. Then by Claim 3.7 (i), B_2 is complete and $p = 3$. Therefore, by Lemma 2 and since $d_G(x) + d_G(y) = |B_1| - 1$, we obtain $K_{k,k+1} \subseteq B_1 \subseteq K_k + (k+1)K_1$ for some $k \geq 2$ and $d_{B_1}(c_1) \geq k + 1$, and hence $G \in \mathcal{G}_3$.

Now suppose that $p \leq 2$. Since G is not covered by three cycles, we may assume that B_1 has no Hamiltonian cycle of B_1. Hence $d_G(x) + d_G(y) \leq |B_1| - 1$ for some $x, y \in I_1$ with $x \neq y$ and $xy \notin E(G)$. Since $\sigma_4(G) \geq n-1$ and by Lemmas A and 2, one of the following holds:

(I) $p = 2$ and B_2 is Hamiltonian, or

(II) $p = 1$, and B_2 or $B_2 - \{c_1\}$ is Hamiltonian, or

(III) $p = 1$, $K_{k,k+1} \subseteq B_2 \subseteq K_k + (k+1)K_1$ and $d_{B_2}(c_1) \geq k+1$ for some $k \geq 2$. (Then $\sigma_2(I_2; B_2) = |B_2| - 1$.)

If (I) or (II) holds, then B_1 is not covered by two cycles, and hence by Lemma 3, $\sigma_3(I_1; B_1) \leq |B_1| - 1$. Then $n - 1 \leq \sigma_4(G) \leq \sigma_3(I_1; B_1) + \sigma_1(I_2; B_2) \leq |B_1| + |B_2| - 2$. This implies that $p = 1$ and the equalities hold. The equality $\sigma_1(I_2; B_2) = |B_2| - 1$ implies that B_2 is complete. Therefore, by Lemma 4 and since G is not covered by three cycles, we obtain $K_{k,2k+1} \subseteq B_1 \subseteq K_k + (2k+1)K_1$ for some $k \geq 2$ and $d_{B_1}(c_1) \geq k+1$. Hence $G \in \mathcal{G}_2$.

If (III) holds, then $\sigma_2(I_1; B_1) = |B_1| - 1$. By Lemma 2 and since G is not covered by three cycles, we obtain $K_{l,l+1} \subseteq B_1 \subseteq K_l + (l+1)K_1$ and $d_{B_1}(c_1) \geq l+1$ for some $l \geq 2$. Since $\sigma_4(G) \geq n-1$, $k = l$, and hence $G \in \mathcal{G}_2$.

Case 3. G is disconnected.

Let $H_1, \ldots, H_s$ be components of G. If $s \geq 4$, then we can easily see that $\sigma_4(G) \leq n-4$, a contradiction. Thus $s \leq 3$. Then we may assume that H_1 has no Hamiltonian cycle. Hence by Theorem A, $\sigma_2(H_1) \leq |H_1| - 1$. Since $\sigma_4(G) \geq n-1$, this implies that $\sigma_2(G - H_1) \geq |G - H_1|$. Hence $s = 2$, and by Theorem A, H_2 has a Hamiltonian cycle. Then H_1 is not covered by two cycles, and hence by Theorem C, $\sigma_3(H_1) \leq |H_1| - 1$. Let $x \in V(H_2)$. Then $n - 1 \leq \sigma_4(G) \leq \sigma_3(H_1) + d_{H_2}(x) \leq |H_1| + |H_2| - 2 = n - 2$, a contradiction.

This completes the proof of Theorem 1. □

4 Lemmas to prove Lemmas 1–4

In this section, we prepare some lemmas to prove Lemmas 1–4. As is mentioned in Section 1, we use the arguments based on the relative length in the proof of Theorem 1. Similarly, to prove Lemmas 3 and 4, we will use this arguments. So the following two lemmas analogous to Lemma 1 are important. We prove Lemmas 1, 8 and 9 in Section 6.

Lemma 8 *Let G be a 2-connected graph of order n, and let $c \in V(G)$. If $\sigma_3(V(G) - \{c\}; G) \geq n$, then G is covered by two cycles or* $\mathrm{diff}(G) \leq 1$.

Lemma 9 *Let G be a 2-connected graph of order n, and let $c \in V(G)$. If $\sigma_3(V(G) - \{c\}; G) \geq n - 1$, then G or $G - \{c\}$ is covered by two cycles, or* $\mathrm{diff}(G) \leq 1$.

We use the following lemma in the proofs of Lemmas 1, 8 and 9.

Lemma 10 *Let G be a graph. Let $\overrightarrow{P_1}$, $\overrightarrow{P_2}$ and $\overrightarrow{P_3}$ be three internally disjoint (x, y)-paths of G with $x \neq y$, and let $Q := P_1 \cup P_2 \cup P_3$. Let $\overrightarrow{R}$ be an (a, b)-path of $G - Q$. If $d_Q(a) \geq 1$, $d_Q(b) \geq 1$ and $|N_Q(a) \cup N_Q(b)| \geq 2$, then $G[V(Q) \cup V(R)]$ is covered by two cycles.*

Proof. By the assumption, there exist two distinct vertices c, d in Q such that $ac, bd \in E(G)$. By the symmetry of P_1, P_2 and P_3, we may assume that either $c, d \in V(P_1)$ (arranged in this order along $\overrightarrow{P_1}$), or $c \in V(P_1)$ and $d \in V(P_2)$ hold. If $c, d \in V(P_1)$, then $x\overrightarrow{P_1}ca\overrightarrow{R}bd\overrightarrow{P_1}y\overleftarrow{P_3}x$ and $x\overrightarrow{P_1}y\overleftarrow{P_2}x$ are cycles which cover $G[V(Q) \cup V(R)]$. Hence we may assume that $c \in V(P_1)$, $d \in V(P_2)$ and $\{c, d\} \cap \{x, y\} = \emptyset$. Then $x\overrightarrow{P_1}ca\overrightarrow{R}bd\overrightarrow{P_2}y\overleftarrow{P_3}x$ and $x\overrightarrow{P_1}y\overleftarrow{P_2}x$ are cycles which cover $G[V(C) \cup V(R)]$. □

Furthermore, to prove Lemmas 1 and 2, we use the following two lemmas.

Lemma C (Dirac [3]) *Let G be a 2-connected graph of order n. If $\sigma_2(G) \geq d$, then there exists a cycle of order at least* $\min\{d, n\}$.

Lemma D (Fraisse and Jung [4]) *Let G be a graph with connectivity one, and let x, y be two distinct vertices in G such that x and y are not cut vertices and x and y belong to distinct end blocks of G. Then there exist $z_1, z_2 \in V(G)$ with $z_1 \neq z_2$ and an (x, y)-path P such that z_1 and z_2 are not cut vertices of G, z_1 and z_2 belong to distinct end blocks of G, and $|P| \geq d_G(z_1) + d_G(z_2)$.*

5 Useful tools for the proofs of Lemmas 1, 8 and 9

In this section, we give some claims to prove Lemmas 1, 8 and 9. These techniques are useful for the investigation of properties of graphs with high relative length.

Let $k \geq 2$ be an integer and G be a 2-connected graph. Let Q be a longest path of G, and let $H := G - Q$ and $H_1, \ldots, H_l$ be components of H. Let $\overrightarrow{C}$ be a cycle and P_0 be a path with end vertex x such that $V(C) \cup V(P_0) = V(Q)$, $V(C) \cap V(P_0) = \emptyset$ and $N_C(x) \neq \emptyset$. (Note that there exist such a cycle C and a path P_0, because the end vertex of Q has a neighbor in $V(Q)$.) Choose Q, C and P_0 so that $|C|$ is as large as possible. A vertex $y \in V(P_0)$ is called *endable for* x if there exists an (x, y)-path P such that $V(P) = V(P_0)$. Let $L := \{y \in V(P_0) : y \text{ is endable for } x\}$ and $L' := L \cup \{x\}$. Let $\mathcal{T} := \{(y, P) : y \in L \text{ and } P \text{ is an } (x, y)\text{-path such that } V(P) = V(P_0)\}$. For each $(y, P) \in \mathcal{T}$, we give an orientation P from x to y along the edges of P.

By the maximality of $|Q|$ and $|C|$, the following two claims hold. Since the proof of Claim 5.1 is easy, we omit the proof.

Claim 5.1 (i) $N_H(L) = \emptyset$.

(ii) *For $u \in N_C(L')$, $N_{G-C}(u^+) = N_{G-C}(u^-) = \emptyset$, furthermore, $N_{H_i}(u^{+2}) = N_{H_i}(u^{-2}) = \emptyset$ for each $1 \leq i \leq l$ with $|H_i| \geq 2$.*

(iii) *For $1 \leq i \leq l$ and $(y, P) \in \mathcal{T}$, $(N_P(H_i)^+ \cup N_P(y)^+) \cap N_P(H_i) = \emptyset$.*

(iv) *For $1 \leq i \leq l$, $(N_C(H_i)^+ \cup N_C(H_i)^-) \cap N_C(H_i) = \emptyset$.*

(v) *For* $1 \le i \le l$ *and* $u, v \in N_C(H_i)$ *with* $u \neq v$, $u^+v^+, u^-v^- \notin E(G)$.

Claim 5.2 *Let* $u_1 \in N_C(L')$ *and* $u_2 \in N_C(G-C)$ *with* $u_1 \neq u_2$. *Let* $C_1 := u_1^+\overrightarrow{C}u_2$ *and* $C_2 := u_2^+\overrightarrow{C}u_1$. *Then the following hold.*

(i) $N_{C_i}(u_i^+)^- \cap N_{C_i}(u_{3-i}^+) = \emptyset$ *for* $i = 1, 2$.

(ii) *If* $|H_i| \ge 2$ *for some* i *with* $1 \le i \le l$, *then* $N_C(u_1^+)^- \cap N_C(H_i)^+ = \emptyset$.

(iii) *If* $u_1^{+2} \in N_C(H_i)$ *for some* i *with* $1 \le i \le l$, *then* $N_{P_0}(H_i) = \emptyset$.

(iv) *If* $u_1 \in N_C(x)$ *and* $u_2 \in N_C(L)$, *then* $N_{C_i}(u_{3-i}^+)^+ \cap N_{C_i}(H_j) = \emptyset$ *for* $i = 1, 2$ *and* $1 \le j \le l$, *furthermore, if* $|H_j| \ge 2$, *then* $N_{C_i}(u_{3-i}^+)^{+2} \cap N_{C_i}(H_j) = \emptyset$ *for* $i = 1, 2$.

Proof. By the maximality of $|Q|$, we can easily obtain (ii) and (iv). Let $z \in V(G-C)$ with $zu_2 \in E(G)$. Suppose that there exists $v \in N_{C_1}(u_1^+)^- \cap N_{C_1}(u_2^+)$. If $u_1 \in N_C(y)$ for some $y \in L$ with $(y, P) \in \mathcal{T}$ and $z \notin V(P)$, then $Q' := zu_2\overleftarrow{C}v^+u_1^+\overrightarrow{C}vu_2^+\overrightarrow{C}u_1y\overleftarrow{P}x$ is a path of G such that $|Q'| > |Q|$, a contradiction. If $u_1 \in N_C(y)$ for some $y \in L$ with $(y, P) \in \mathcal{T}$ and $z \in V(P)$, then $C' := zu_2\overleftarrow{C}v^+u_1^+\overrightarrow{C}vu_2^+\overrightarrow{C}u_1y\overleftarrow{P}z$ is a cycle of G and $P_0' := z^-\overleftarrow{P}x$ is a path of G such that $V(C') \cup V(P_0') = V(Q)$, $V(C') \cap V(P_0') = \emptyset$, $N_{C'}(z^-) \neq \emptyset$ and $|C'| > |C|$, which contradicts the choice of Q, C and P_0. Similarly, in case $u_1 \in N_C(x)$, we can also get a contradiction. Thus $N_{C_1}(u_1^+)^- \cap N_{C_1}(u_2^+) = \emptyset$. Similarly, we obtain $N_{C_2}(u_1^+) \cap N_{C_2}(u_2^+)^- = \emptyset$, and hence (i) holds. By the similar argument, if $u_1^{+2} \in N_C(H_i)$ for some i with $1 \le i \le l$ and $N_{P_0}(H_i) \neq \emptyset$, then we can see that this contradicts the choice Q, C and P_0, and hence we can obtain (iii). □

Claim 5.3 *Let* j *be an integer with* $1 \le j \le k$, $u_1 \in N_C(x)$ *and* $u_2 \in N_C(L)$ *with* $u_1 \neq u_2$, *and let* $z_1, z_2 \in V(H)$ *with* $z_1 \neq z_2$. *Let* $C_1 := u_1^+\overrightarrow{C}u_2$ *and* $C_2 := u_2^+\overrightarrow{C}u_1$. *If* $\mathrm{diff}(G) \ge k$, *then the following hold.*

(i) $N_{C_i}(u_i^+)^- \cap N_{C_i}(u_{3-i}^+)^{+(j-1)} = \emptyset$ *for* $i = 1, 2$.

(ii) $N_{C_1}(u_1^+)^- \cap N_{C_1}(y)^{+(j-1)} = \emptyset$ *for all* $y \in L$.

(iii) *If* $u_i^{+2} \in N_C(z_i)$ *for* $i = 1, 2$, *then* $N_{C_i}(z_i)^- \cap N_{C_i}(z_{3-i})^{+(j-1)} = \emptyset$ *for* $i = 1, 2$.

(iv) *If* $u_1^{+2} \in N_C(z_1)$, *then* $N_{C_1}(z_1)^- \cap N_{C_1}(u_2^+)^{+(j-1)} = N_{C_2}(z_1)^+ \cap N_{C_2}(u_2^+)^{-(j-1)} = \emptyset$.

Proof. Let $(y, P) \in \mathcal{T}$. If $yu_2 \in E(G)$ and there exists $v \in N_{C_1}(u_1^+)^- \cap N_{C_1}(u_2^+)^{+(j-1)}$, then $C' := u_1^+ \overrightarrow{C} v^{-(j-1)} u_2^+ \overrightarrow{C} u_1 x \overrightarrow{P} y u_2 \overleftarrow{C} v^+ u_1^+$ is a cycle of G such that $|C'| = |Q| - (j-1)$, and hence $\mathrm{diff}(G) = |Q| - c(G) \leq |Q| - |C'| = j - 1 < k$, a contradiction. Similarly, if $yu_2 \in E(G)$ and $N_{C_2}(u_1^+)^{+(j-1)} \cap N_{C_2}(u_2^+)^- \neq \emptyset$, then we can get a contradiction. Thus (i) holds. By the similar argument, we can obtain (ii), (iii) and (iv). □

By Claim 5.1 (iii), we can obtain the following claim.

Claim 5.4 *Let* $(y, P) \in \mathcal{T}$ *and* $z \in V(H)$. *Then* $d_P(y) + d_P(z) \leq |P|$. *Furthermore, if* $x \notin N_P(z)$, *then* $d_P(y) + d_P(z) \leq |P| - 1$.

Proof. By Claim 5.1 (iii), $N_P(y)^+ \cap N_P(z) = \emptyset$. Since $N_P(y)^+ \cup N_P(z) \subseteq V(P)$ and $x \notin N_P(y)^+$, we can get the desired conclusion. □

6 Proofs of Lemmas 1, 8 and 9

Let G be a 2-connected graph of order n, and suppose that Q, C, H, $H_1, \ldots, H_l$, P_0, x, L, L' and $\mathcal{T}$ are the same as those in Section 5. Let $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}_3$ be the sets of graphs which satisfy the assumptions of Lemmas 1, 8 and 9, respectively.

Proof of Lemma 1. Let $G \in \mathcal{F}_1$, and suppose that G is not covered by three cycles and $\mathrm{diff}(G) \geq 3$. Then $|P_0| \geq 3$. Since G is not covered by three cycles, the following fact holds.

Fact 6.1 *If* $N_G(y) \cap (V(C) \cup \{x\}) \neq \emptyset$ *for some* $y \in L$, *then* $|H| \geq 3$.

Case 1. There exists $(y, P) \in \mathcal{T}$ such that there are two independent edges joining x and C, and y and C.

Since G is not covered by three cycles, there exists no cycle containing $V(H)$. Note that by Fact 6.1 and the assumption of Case 1, $|H| \geq 3$.

Claim 6.2 *For all $z_1, z_2 \in V(H)$, there exist no $v_1 \in N_C(x)^+ \cap N_C(z_1)^-$ and $v_2 \in N_C(y)^+ \cap N_C(z_2)^-$ such that $v_1 \neq v_2$.*

Proof. Suppose not, and there exist $v_1 \in N_C(x)^+ \cap N_C(z_1)^-$ and $v_2 \in N_C(y)^+ \cap N_C(z_2)^-$ such that $v_1 \neq v_2$ for some $z_1, z_2 \in V(H)$. Since diff$(G) \geq 3$, we can easily see $z_1 \neq z_2$. Let $C_1 := v_1 \overrightarrow{C} v_2^-$ and $C_2 := v_2 \overrightarrow{C} v_1^-$. By Claim 5.1 (v), $N_{C_i}(v_i) \cap N_{C_i}(z_i)^- = \emptyset$ for $i = 1, 2$. By Claim 5.3 (i), $N_{C_i}(v_i) \cap N_{C_i}(v_{3-i})^+ = \emptyset$ for $i = 1, 2$. By Claim 5.2 (i), $N_{C_i}(v_{3-i})^+ \cap N_{C_i}(z_{3-i})^{+2} = \emptyset$ for $i = 1, 2$. By Claim 5.3 (iii), $N_{C_i}(z_i)^- \cap N_{C_i}(z_{3-i})^{+2} = \emptyset$ for $i = 1, 2$. By Claim 5.3 (iv) and the symmetry of x and y, $N_{C_i}(z_i)^- \cap N_{C_i}(v_{3-i})^+ = N_{C_i}(v_i) \cap N_{C_i}(z_{3-i})^{+2} = \emptyset$ for $i = 1, 2$. Hence $N_{C_i}(z_i)^-, N_{C_i}(v_i), N_{C_i}(v_{3-i})^+$ and $N_{C_i}(z_{3-i})^{+2}$ are pairwise disjoint for $i = 1, 2$. Since $N_{C_i}(z_i)^- \cup N_{C_i}(v_i) \cup N_{C_i}(v_{3-i})^+ \cup N_{C_i}(z_{3-i})^{+2} \subseteq V(C_i) \cup \{v_{3-i}, v_{3-i}^+\}$ for $i = 1, 2$, we obtain $d_{C_i}(z_i) + d_{C_i}(z_{3-i}) + d_{C_i}(v_i) + d_{C_i}(v_{3-i}) \leq |C_i| + 2$ for $i = 1, 2$. By Claim 5.1 (ii), $N_{G-C}(v_i) = \emptyset$ for $i = 1, 2$. By Claims 5.1 (ii) and 5.2 (iii), $N_{G-C}(z_i) = \emptyset$ for $i = 1, 2$. Hence $d_G(z_i) + d_G(z_{3-i}) + d_G(v_i) + d_G(v_{3-i}) = d_C(z_1) + d_C(z_2) + d_C(v_1) + d_C(v_2) \leq |C| + 4 = n - (|P| + |H|) + 4 \leq n - 2$, a contradiction. □

By changing the label of x and y if necessary, we take $u \in N_C(x)$ so that (i) $u^{+2} \in N_C(H)$ if possible, (ii) $N_C(y) - \{u\} \neq \emptyset$, subject to (i). Then by Claim 6.2, $N_C(H) \cap N_C(y)^{+2} \subseteq \{u^{+2}\}$. Choose $w \in N_C(y) - \{u\}$ so that $|w^+ \overrightarrow{C} u|$ is as small as possible. Let $z \in V(H)$. By Claims 5.1 (i), (ii) and 5.2 (i), $\{u^+, w^+, y, z\}$ is an independent set of G. Let $C_1 := u^+ \overrightarrow{C} w$ and $C_2 := w^+ \overrightarrow{C} u$. We show that $d_{C_i}(u^+) + d_{C_i}(w^+) + d_{C_i}(y) + d_{C_i}(z) \leq |C_i| + 2$ for $i = 1, 2$.

We first consider the case of $i = 1$. By Claim 5.2 (i), $N_{C_1}(u^+)^- \cap N_{C_1}(z) = \emptyset$. By Claim 5.3 (i), $N_{C_1}(u^+)^- \cap N_{C_1}(w^+)^+ = \emptyset$. By Claim 5.3 (ii), $N_{C_1}(u^+)^- \cap N_{C_1}(y)^{+2} = \emptyset$. By Claim 5.2 (iv), $N_{C_1}(z) \cap N_{C_1}(w^+)^+ = \emptyset$. By the choice of x and y, $N_{C_1}(z) \cap N_{C_1}(y)^{+2} = \emptyset$. By Claim 5.2 (i), $N_{C_1}(w^+)^+ \cap N_{C_1}(y)^{+2} = \emptyset$. Hence $N_{C_1}(u^+)^-, N_{C_1}(z), N_{C_1}(w^+)^+$ and $N_{C_1}(y)^{+2}$ are pairwise disjoint. Since $N_{C_1}(u^+)^- \cup N_{C_1}(z) \cup N_{C_1}(w^+)^+ \cup$

$N_{C_1}(y)^{+2} \subseteq V(C_1) \cup \{w^+, w^{+2}\}$,

$$d_{C_1}(u^+) + d_{C_1}(w^+) + d_{C_1}(y) + d_{C_1}(z) \leq |C_1| + 2. \qquad (6.1)$$

We next consider the case of $i = 2$. By the choice of x, y and w, $N_{C_2}(y) \subseteq \{u\}$, and hence $d_{C_2}(y) \leq 1$. By Claims 5.2 (i), (iv) and 5.3 (i), $N_{C_2}(w^+)^-$, $N_{C_2}(z)$ and $N_{C_2}(u^+)^+$ are pairwise disjoint. Since $N_{C_2}(w^+)^- \cup N_{C_2}(z) \cup N_{C_2}(u^+)^+ \subseteq V(C_2) \cup \{u^+\}$,

$$d_{C_2}(u^+) + d_{C_2}(w^+) + d_{C_2}(y) + d_{C_2}(z) \leq |C_2| + 2. \qquad (6.2)$$

By Claims 5.1 (i), (ii), 5.4 and the symmetry of x and y,

$$d_P(u^+) + d_P(w^+) + d_P(y) + d_P(z) = d_P(y) + d_P(z) \leq |P| - 1. \qquad (6.3)$$

Hence by Claim 5.1 (i), (ii) and (6.1)–(6.3),

$$d_G(u^+) + d_G(w^+) + d_G(y) + d_G(z) \leq n - (|H| - d_H(z)) + 3. \qquad (6.4)$$

Since $\sigma_4(G) \geq n - 1$ and z is an arbitrary vertex in H, this together with (6.4) implies that

$$|H| - d_H(z) \leq 4 \text{ for all } z \in V(H). \qquad (6.5)$$

By (6.5), we have $l \leq 4$. We divide the proof of Case 1 into three cases.

Case 1.1. $l = 4$.

Then by (6.5), $H = 4K_1$ and the equality holds in (6.4), and hence the equalities hold in (6.1)–(6.3) for all $z \in V(H)$. Write $V(H) = \{z_1, z_2, z_3, z_4\}$. Then $N_{C_1}(u^+)^- \cup N_{C_1}(z_i) \cup N_{C_1}(w^+)^+ \cup N_{C_1}(y)^{+2} = V(C_1) \cup \{w^+, w^{+2}\}$, $N_{C_2}(w^+)^- \cup N_{C_2}(z_i) \cup N_{C_2}(u^+)^+ = V(C_2) \cup \{u^+\}$ and $N_P(y)^+ \cup N_P(z_i) = V(P) - \{x\}$ hold for each $1 \leq i \leq 4$. Since these are disjoint unions, we have $N_G(z_1) = N_G(z_2) = N_G(z_3) = N_G(z_4)$.

Suppose that $d_G(z_1) \geq 4$. Let z_1', z_2', z_3', z_4' be four distinct vertices in $N_G(z_1)$. Then $D := z_1' z_1 z_2' z_2 z_3' z_3 z_4' z_4 z_1'$ is a cycle containing $V(H)$, a contradiction. Thus $d_G(z_i) \leq 3$ for each $1 \leq i \leq 4$, and hence $n - 1 \leq \sum_{i=1}^{4} d_G(z_i) \leq 12$, that is, $n \leq 13$. On the other hand, since $|P| \geq 3$

and $|C| \geq 8$ by the maximality of $|C|$ and the assumption of Case 1, $n = |P| + |C| + |H| \geq 15$, a contradiction.

Case 1.2. $2 \leq l \leq 3$.

We may assume that $|H_1| \leq \cdots \leq |H_l|$. Then by (6.5), if $l = 3$, then $|H_1| = |H_2| = 1$ and $|H_3| \leq 2$; if $l = 2$, then $|H_i| \leq 3$ for each $i = 1, 2$, in particular, if $l = 2$ and $|H_i| = 3$ for each $i = 1, 2$, then H_i is complete for each $i = 1, 2$. Therefore since G is 2-connected, there exist $x_1, y_1, x_2, y_2 \in V(H)$ and an (a, b)-Hamiltonian path of $H - \{x_1, y_1, x_2, y_2\}$ such that $x_i = y_i$ or $x_i y_i \in E(G)$ for each $i = 1, 2$, and $d_Q(a) \geq 1$, $d_Q(b) \geq 1$ and $|N_Q(a) \cup N_Q(b)| \geq 2$ (note that (a, b)-Hamiltonian path may consist of one vertex). Then by Lemma 10, $G[V(Q) \cup (V(H) - \{x_1, y_1, x_2, y_2\})]$ is covered by two cycles. Since G is 2-connected, there exists a cycle containing $\{x_1, y_1, x_2, y_2\}$, and hence G is covered by three cycles, a contradiction.

Case 1.3. $l = 1$.

Note that for two nonadjacent vertices z_1, z_2 of H, $\{u^+, w^+, z_1, z_2\}$ is an independent set of G by Claims 5.1 (ii) and 5.2 (i). In this case, we will use such an independent set when we calculate the degree sum of independent four vertices. Since $N_{G-C}(u^+) = \emptyset$ and $N_{G-C}(w^+) = \emptyset$, we have only to calculate the degree sum of z_1 and z_2 when we calculate the degree sum of u^+, w^+, z_1 and z_2 on $G - C$. We first consider about the degree sum of z_1 and z_2 on P and H (see Claims 6.3 and 6.4).

Claim 6.3 *$d_P(z_1) + d_P(z_2) \leq \max\{|P| - 5, 0\}$ for all $z_1, z_2 \in V(H)$ with $z_1 \neq z_2$ and $z_1 z_2 \notin E(G)$.*

Proof. Since $z_1 z_2 \notin E(G)$, there exists a path of order at least three from z_1 to z_2 in H. Hence by the maximality of $|Q|$ and Claim 5.1 (iii), $(N_P(z_1) \cup N_P(z_2)) \cap \{x, x^+, x^{+2}, y, y^-, y^{-2}\} = \emptyset$ and $N_P(z_1) \cap N_P(z_2)^- = \emptyset$. Therefore the desired conclusion holds. □

Claim 6.4 *There exist $z_1, z_2 \in V(H)$ with $z_1 \neq z_2$ such that $z_1 z_2 \notin E(G)$ and $d_H(z_1) + d_H(z_2) \leq \min\{|P|, |H| - 1\}$.*

Proof. Suppose that $\sigma_2(H) \geq \min\{|P|+1, |H|\}$. By the maximality of $|Q|$ and since there exists no cycle containing $V(H)$, it follows from Theorem A and Lemma C that $|P|+1 \leq |H|-1$ and the connectivity of H is one. Let v_1 and v_2 be two distinct vertices in H such that v_1 and v_2 are not cut vertices of H, and v_1 and v_2 belong to distinct end blocks of H. Since G is 2-connected, we can take v_1 so that $N_Q(v_1) \neq \emptyset$. Then by Lemma D, there exist $z_1, z_2 \in V(H)$ with $z_1 \neq z_2$ and $z_1z_2 \notin E(G)$ and a (v_1, v_2)-path of order at least $d_H(z_1)+d_H(z_2) \geq |P|+1$. Since $N_Q(v_1) \neq \emptyset$, this contradicts the maximality of $|Q|$. $\square$

Let $z_1, z_2 \in V(H)$ be as in Claim 6.4. By Claim 5.3 (i), $N_{C_1}(u^+)^- \cap N_{C_1}(w^+)^{+2} = \emptyset$. By Claim 5.2 (i), (ii) and (iv), $N_{C_1}(u^+)^- \cap N_{C_1}(z_1)^+ = N_{C_1}(u^+)^- \cap N_{C_1}(z_2) = N_{C_1}(w^+)^{+2} \cap N_{C_1}(z_1)^+ = N_{C_1}(w^+)^{+2} \cap N_{C_1}(z_2) = \emptyset$. By Claim 5.1 (iv), $N_{C_1}(z_1)^+ \cap N_{C_1}(z_2) = \emptyset$. Therefore we have that $N_{C_1}(u^+)^-$, $N_{C_1}(z_2)$, $N_{C_1}(z_1)^+$ and $N_{C_1}(w^+)^{+2}$ are pairwise disjoint. Since $N_{C_1}(u^+)^- \cup N_{C_1}(z_2) \cup N_{C_1}(z_1)^+ \cup N_{C_1}(w^+)^{+2} \subseteq V(C_1) \cup \{w^+, w^{+2}\}$, $d_{C_1}(u^+) + d_{C_1}(w^+) + d_{C_1}(z_1) + d_{C_1}(z_2) \leq |C_1| + 2$. Similarly, $d_{C_2}(u^+) + d_{C_2}(w^+) + d_{C_2}(z_1) + d_{C_2}(z_2) \leq |C_2| + 2$. Hence by Claim 5.1 (ii), $n-1 \leq d_G(u^+)+d_G(w^+)+d_G(z_1)+d_G(z_2) = (d_C(u^+)+d_C(w^+)+d_C(z_1)+d_C(z_2))+ d_{G-C}(z_1) + d_{G-C}(z_2) \leq |C| + 4 + d_{G-C}(z_1) + d_{G-C}(z_2)$. Hence by Claim 6.4, $d_P(z_1) + d_P(z_2) \geq \max\{|P|-4, |H|-5\}$. By Claim 6.3, $|P| \leq 4$ and $|H| \leq 5$.

Since G is 2-connected, there exists a path R in H connecting a and b with $a, b \in V(H)$ such that $d_Q(a) \geq 1$, $d_Q(b) \geq 1$ and $|N_Q(a) \cup N_Q(b)| \geq 2$. Then by Lemma 10, $G[V(Q) \cup V(R)]$ is covered by two cycles. Choose such a path R so that $|R|$ is as large as possible. Since G is 2-connected and G is not covered by three cycles, it follows that $|R| = 2$, $|H-R| = 3$ and $3K_1 \subseteq H-R \subseteq K_1 \cup K_2$. Since H is connected, we may assume that $N_{H-R}(a) \neq \emptyset$. Let $z, z' \in V(H-R)$ with $z \neq z'$, $zz' \notin E(G)$ and $z \in N_{H-R}(a)$. Then by the maximality of $|R|$, $d_H(z)+d_H(z') \leq 3$ and $d_Q(z) \leq 1$. By Claim 6.3 and since $|P| \leq 4$, $d_P(z') = 0$. By Claim 5.3 (i), $N_{C_1}(u^+)^- \cap N_{C_1}(w^+)^+ = \emptyset$. By Claim 5.2 (i) and (iv), $N_{C_1}(u^+)^- \cap N_{C_1}(z') = N_{C_1}(z') \cap N_{C_1}(w^+)^+ = \emptyset$. Since $N_{C_1}(u^+)^- \cup N_{C_1}(z') \cup N_{C_1}(w^+)^+ \subseteq V(C_1) \cup \{w^+\}$, $d_{C_1}(u^+) +$

$d_{C_1}(w^+) + d_{C_1}(z') \le |C_1| + 1$. Similarly, $d_{C_2}(u^+) + d_{C_2}(w^+) + d_{C_2}(z') \le |C_2| + 1$. Therefore, since $|P| \ge 3$ and $|H| = 5$ and by Claim 5.1 (ii), $d_G(u^+)+d_G(w^+)+d_G(z)+d_G(z') = (d_C(u^+)+d_C(w^+)+d_C(z'))+d_Q(z)+(d_H(z) + d_H(z')) \le (|C| + 2) + 1 + 3 = |C| + 6 \le n - 2$, a contradiction.

Case 2. For any $(y, P) \in \mathcal{T}$, there exist no two independent edges joining x and C, and y and C.

Let $u \in N_C(x)$. By the assumption of Case 2 and Claim 5.1 (i), $N_G(L) \subseteq V(P) \cup \{u\}$. Since G is 2-connected, there exist $z \in V(G - C)$ and $w \in V(C) - \{u\}$ such that $zw \in E(G)$. Choose z and w so that $|w^+\overrightarrow{C}u|$ is as small as possible. By the choice of z and w and Claim 5.1 (ii), $d_{G-C}(u^+)+d_{G-C}(w^+) = 0$. By Claim 5.2 (i), $d_C(u^+)+d_C(w^+) \le |C|$. Hence the following fact holds since $\sigma_4(G) \ge n-1$, which plays an important role in the proof of this case. Note that for each $y \in L$, $\{u^+, w^+, y\}$ is an independent set by Claims 5.1 (ii) and 5.2 (i).

Fact 6.5 *There exist no $y \in L$ and $z' \in V(G) - \{u^+, w^+, y\}$ such that $d_G(y) + d_G(z') \le |P_0| + |H| - 2$ and $\{u^+, w^+, y, z'\}$ is an independent set of G.*

In case of $V(H) = \emptyset$, to obtain a contradiction, we have only to show that there exist two cycles whose union contains $V(P_0)$.

Claim 6.6 $V(H) \neq \emptyset$.

Proof. Suppose that $V(H) = \emptyset$. Then $N_C(y_0) = \emptyset$. Since $G - \{u\}$ is connected, there exists $v \in V(x^+\overrightarrow{P_0}y_0)$ such that $N_{C-\{u\}}(v^-) \neq \emptyset$. Choose such a vertex v so that $|v\overrightarrow{P_0}y_0|$ is as small as possible. By the choice of v, $N_{C-\{u\}}(v\overrightarrow{P_0}y_0) = \emptyset$.

Subclaim 6.6.1 *$N_G(v') \cap \{u, x\} = \emptyset$ for all $v' \in V(v\overrightarrow{P_0}y_0)$.*

Proof. Suppose that there exists $v' \in V(v\overrightarrow{P_0}y_0)$ such that $N_G(v')\cap\{u, x\} \neq \emptyset$. Take such a vertex v' so that $|v'\overrightarrow{P_0}y_0|$ is as small as possible. Note that $v' \in V(v\overrightarrow{P_0}y_0^{-2})$ since G is not covered by three cycles. Let $P_1 := x\overrightarrow{P_0}v'$ and $P_2 := v'^+\overrightarrow{P_0}y$. Since $V(H) = \emptyset$ and G is not covered by three cycles, $N_{P_1}(y_0) = \emptyset$ and $N_{P_2}(v'^+) \cap N_{P_2}(y_0)^+ = \emptyset$, in particular, $v'^+y_0 \notin E(G)$. By the choice of v', $N_{P_1}(v'^+) \subseteq V(P_1) - \{x\}$ and $u \notin N_C(v'^+)$. Moreover, $N_{P_2}(v'^+) \cup N_{P_2}(y_0)^+ \subseteq V(P_2) - \{v'^+\}$. Hence $d_{P_i}(v'^+) + d_{P_i}(y_0) \leq |P_i| - 1$ for $i = 1, 2$ and $N_C(v'^+) = \emptyset$. Hence $d_G(y_0) + d_G(v'^+) = d_{P_0}(y_0) + d_{P_0}(v'^+) \leq |P_0| - 2$, which contradicts Fact 6.5. □

Let $P_1 := x\overrightarrow{P_0}v^-$ and $P_2 := v\overrightarrow{P_0}y_0$. Since $V(H) = \emptyset$ and G is not covered by three cycles, $N_{P_1}(y_0) = \emptyset$ and $N_{P_2}(y_0)^+ \cap N_{P_2}(v) = \emptyset$, specifically, $vy_0 \notin E(G)$. By Subclaim 6.6.1, $N_{P_1}(v) \subseteq V(P_1) - \{x\}$ and $u \notin N_C(v)$. Moreover, $N_{P_2}(y_0)^+ \cup N_{P_2}(v) \subseteq V(P_2) - \{v\}$. Hence $d_{P_i}(v) + d_{P_i}(y_0) \leq |P_i| - 1$ for $i = 1, 2$ and $N_C(v) = \emptyset$. Hence $d_G(y_0) + d_G(v) = d_{P_0}(y_0) + d_{P_0}(v) \leq |P_0| - 2$, which contradicts Fact 6.5. This completes the proof of Claim 6.6. □

Our proof needs crossing argument on P by using nonadjacent two vertices of L', and also needs the fact $x \notin N_G(y)$ for any $y \in L$. Therefore, the following claim is useful.

Claim 6.7 *There exists no cycle D such that $V(D) = V(P_0)$.*

Proof. Suppose not. We may assume that there exists $(y, P) \in \mathcal{T}$ such that $x \in N_G(y)$. By Fact 6.1, $|H| \geq 3$.

Subclaim 6.7.1 *We may assume that $N_{(C-\{u\})\cup H}(x^{+2}\overrightarrow{P}y) \neq \emptyset$.*

Proof. Suppose that $N_{G-P}(x^{+2}\overrightarrow{P}y) \subseteq \{u\}$. Since $G - \{x\}$ and $G - \{u\}$ are connected and $x^+ \in L$ and by the assumption of Case 2, there exists $a \in V(x^{+2}\overrightarrow{P}y)$ such that $N_{G-P}(a) = \{u\}$, and $N_{C-\{u\}\cup H}(x) \neq \emptyset$. By the maximality of $|Q|$ or the assumption of Case 2, $a \neq y$. Hence we can replace $(y, P) \in \mathcal{T}$ by $(y', P') \in \mathcal{T}$, where $P' := a\overrightarrow{P}yx\overrightarrow{P}a^-$ and $y' := a^-$. □

Let $v \in V(x^{+2}\overrightarrow{P}y)$ with $N_{(C-\{u\})\cup H}(v) \neq \emptyset$. Choose such a vertex v so that $|x\overrightarrow{P}v|$ is as small as possible. Then $N_{(C-\{u\})\cup H}(v^-) = \emptyset$. Let $P_1 := x\overrightarrow{P}v^-$ and $P_2 := v\overrightarrow{P}y$. By the assumption of Case 2 or the maximality of $|Q|$, $u \notin N_C(v^-)$, $N_{P_1}(v^-)^+ \cap N_{P_1}(y) = \emptyset$ and $N_{P_2}(v^-) \cap N_{P_2}(y)^+ = \emptyset$, specifically, $v^-y \notin E(G)$. Moreover, $N_{P_1}(v^-)^+ \cup N_{P_1}(y) \subseteq V(P_1)$ and $N_{P_2}(v^-) \cup N_{P_2}(y)^+ \subseteq V(P_2)$. These imply that $d_{P_i}(v^-) + d_{P_i}(y) \leq |P_i|$ for $i = 1, 2$ and $N_C(v^-) = \emptyset$. Hence $d_G(y) + d_G(v^-) = (d_P(y) + d_P(v^-)) + d_C(y) \leq |P|+1 \leq |P|+|H|-2$, which contradicts Fact 6.5. This completes the proof of Claim 6.7. □

Claim 6.8 *For any $(y, P) \in \mathcal{T}$ and any $v \in L - \{y\}$, $v^+ \in L$.*

Proof. Suppose that there exist $(y, P) \in \mathcal{T}$ and $v \in L - \{y\}$ such that $v^+ \notin L$. Since $v^+ \notin L$, $yv \notin E(G)$. Since $y, v \in L$, it follows from Claim 5.1 (i) that $d_H(y) + d_H(v) = 0$.

By the assumption of Case 2, $N_C(y) \cup N_C(v) \subseteq \{u\}$. Hence by Fact 6.1 and Claim 6.6, $d_C(y) + d_C(v) \leq |H| - 1$.

Let $P_1 := x\overrightarrow{P}v$ and $P_2 := v^+\overrightarrow{P}y$. Since $v^+ \notin L$, $N_{P_1}(y) \cap N_{P_1}(v)^+ = \emptyset$ and $N_{P_2}(y)^+ \cap N_{P_2}(v) = \emptyset$. By Claim 6.7, $N_{P_1}(y) \cup N_{P_1}(v)^+ \subseteq V(P_1) - \{x\}$. Moreover, $N_{P_2}(y)^+ \cup N_{P_2}(v) \subseteq V(P_2)$. These imply that $d_{P_1}(y) + d_{P_1}(v) \leq |P_1| - 1$ and $d_{P_2}(y) + d_{P_2}(v) \leq |P_2|$.

Hence by the above four inequalities, $d_G(y) + d_G(v) \leq |P| + |H| - 2$, which contradicts Fact 6.5. □

Claim 6.9 *For any $(y, P) \in \mathcal{T}$, $N_C(y) = \emptyset$.*

Proof. Suppose that there exists $(y, P) \in \mathcal{T}$ such that $N_C(y) \neq \emptyset$. Then by the assumption of Case 2 and Fact 6.1, $N_C(x) = N_C(y) = \{u\}$ and $|H| \geq 3$. By Claim 6.7, $N_P(x)^- \cap N_P(y) = \emptyset$, specifically, $xy \notin E(G)$. Since $N_P(x)^- \cup N_P(y) \subseteq V(P) - \{y\}$, $d_P(x) + d_P(y) \leq |P| - 1$. Since x is endable for y, it follows from Claim 5.1 (i) that $d_H(x) + d_H(y) = 0$. Hence $d_G(x) + d_G(y) = (d_P(x) + d_P(y)) + (d_C(x) + d_C(y)) \leq (|P| - 1) + 2 \leq |P| + |H| - 2$, which contradicts Fact 6.5. □

By Claims 6.8, 6.9 and the assumption of Case 2, we can easily obtain the following claim.

Claim 6.10 *For any* $(y, P) \in \mathcal{T}$ *and any* $v \in L$, $V(v\overrightarrow{P}y) \subseteq L$ *and* $N_{G-P}(v\overrightarrow{P}y) = \emptyset$.

We choose $(y, P) \in \mathcal{T}$ and $a \in N_P(y)$ so that $|x\overrightarrow{P}a|$ is as small as possible. By Claims 6.7, 6.9 and 6.10, $N_G(a^+\overrightarrow{P}y) - V(a^+\overrightarrow{P}y) \subseteq V(x^+\overrightarrow{P}a)$. Since $G - \{a\}$ is connected, there exists an edge bc such that $b \in V(a^+\overrightarrow{P}y)$ and $c \in V(x^+\overrightarrow{P}a^-)$. Suppose that $yb^- \in E(G)$. Then $\overrightarrow{P'} := x\overrightarrow{P}b^-y\overleftarrow{P}b$ is a path such that $(b, P') \in \mathcal{T}$, $c \in N_{P'}(b)$ and $|x\overrightarrow{P}a| > |x\overrightarrow{P'}c|$, which contradicts the choice of y and a. Thus $yb^- \notin E(G)$ and $b^- \in V(a^+\overrightarrow{P}y)$. By Claim 6.10, $b^- \in L$. By Claim 5.1 (ii), $\{u^+, w^+, y, b^-\}$ is an independent set. By Claim 6.10, $d_{G-P}(y) + d_{G-P}(b^-) = 0$.

Let $P_1 := x\overrightarrow{P}a^-$, $P_2 := a\overrightarrow{P}b^-$ and $P_3 := b\overrightarrow{P}y$. Then the choice of y and a implies $N_{P_1}(y) = \emptyset$, and hence $d_{P_1}(y) + d_{P_1}(b^-) = d_{P_1}(b^-) \leq |P_1| - 1$ because $xb^- \notin E(G)$ by Claim 6.7.

If there exists $d \in N_{P_2}(y) \cap N_{P_2}(b^-)^+$, then $\overrightarrow{P'} := x\overrightarrow{P}d^-b^-\overleftarrow{P}dy\overleftarrow{P}b$ is a path such that $(b, P') \in \mathcal{T}$, $c \in N_{P'}(b)$ and $|x\overrightarrow{P}a| > |x\overrightarrow{P'}c|$, which contradicts the choice of y and a. Thus $N_{P_2}(y) \cap N_{P_2}(b^-)^+ = \emptyset$. Since $N_{P_2}(y) \cup N_{P_2}(b^-)^+ \subseteq V(P_2)$, $d_{P_2}(y) + d_{P_2}(b^-) \leq |P_2|$.

If there exists $d \in N_{P_3}(y)^+ \cap N_{P_3}(b^-)$, then we can find $P' := x\overrightarrow{P}b^-d\overrightarrow{P}y d^-\overleftarrow{P}b$. This contradicts the choice of y and a, again. Thus $N_{P_3}(y)^+ \cap N_{P_3}(b^-) = \emptyset$. Since $N_{P_3}(y)^+ \cup N_{P_3}(b^-) \subseteq V(P_3)$, $d_{P_3}(y) + d_{P_3}(b^-) \leq |P_3|$.

Hence by the above four inequalities and Claim 6.6, we obtain $d_G(y) + d_G(b^-) = d_P(y) + d_P(b^-) \leq |P| - 1 \leq |P| + |H| - 2$, which contradicts Fact 6.5.

This completes the proof of Lemma 1. □

Proofs of Lemmas 8 and 9. Let $G \in \mathcal{F}_2 \cup \mathcal{F}_3$, and suppose that G is not covered by two cycles and $\mathrm{diff}(G) \geq 2$. Suppose furthermore that if $G \notin \mathcal{F}_2$, then $G - \{c\}$ is not covered by two cycles.

Case 1. There exists $(y, P) \in \mathcal{T}$ such that there are two independent edges joining x and C, and y and C.

Since G is 2-connected and G is not covered by two cycles, it follows from Lemma 10 that H is not complete, in particular, $V(H) - \{c\} \neq \emptyset$.

Let $z \in V(H) - \{c\}$. By the assumption of Case 1, there exist two distinct vertices $u \in N_C(x)$ and $w \in N_C(y)$. Since diff$(G) \geq 2$, $N_C(x)^+ \cap N_C(y)^- = \emptyset$. Therefore, by changing the orientation of $\overrightarrow{C}$ if necessary, we may assume that $c \notin \{u^+, w^+\}$. Let $C_1 := u^+\overrightarrow{C}w$ and $C_2 := w^+\overrightarrow{C}u$. Then by Claim 5.1 (ii) and 5.2 (i), $\{u^+, w^+, z\}$ is an independent set of G. By Claim 5.1 (i) and the symmetry of x and y, $N_P(z) \cap \{x, y\} = \emptyset$, and hence $d_P(z) \leq |P| - 2$.

By Claims 5.2 (i), (iv) and 5.3 (i), $N_{C_1}(u^+)^-$, $N_{C_1}(z)$ and $N_{C_1}(w^+)^+$ are pairwise disjoint. Since $N_{C_1}(u^+)^- \cup N_{C_1}(z) \cup N_{C_1}(w^+)^+ \subseteq V(C_1) \cup \{w^+\}$, $d_{C_1}(u^+) + d_{C_1}(w^+) + d_{C_1}(z) \leq |C_1| + 1$. Similarly, we obtain $d_{C_2}(u^+) + d_{C_2}(w^+) + d_{C_2}(z) \leq |C_2| + 1$

Hence by the above three inequalities and Claims 5.1 (ii), $d_G(u^+) + d_G(w^+) + d_G(z) = (d_C(u^+) + d_C(w^+) + d_C(z)) + d_P(z) + d_H(z) \leq (|C| + 2) + (|P| - 2) + (|H| - 1) = n - 1$. This implies that $G \notin \mathcal{F}_2$ and $d_H(z) = |H| - 1$. Since z is an arbitrary vertex in $V(H) - \{c\}$, H is complete, a contradiction.

Case 2. For any $(y, P) \in \mathcal{T}$, there exist no two independent edges joining x and C, and y and C.

Let $u \in N_C(x)$. We may assume that $u^+ \neq c$. Note that by the assumption of Case 2 and Claim 5.1 (i), $N_G(y) \subseteq V(P_0) \cup \{u\}$ for all $y \in L$. For each $(y, P) \in \mathcal{T}$, let $\overrightarrow{D(y, P)} := ux\overrightarrow{P}y$. Choose $(y, P) \in \mathcal{T}$ and $v \in N_G(y)$ so that $|v\overrightarrow{D(y, P)}y|$ is as large as possible. If $v = u$, then we can use the symmetry of x and y, and hence we may assume that $y \neq c$. If $v \neq u$ and $y = c$, then let $P' := x\overrightarrow{P}vy\overleftarrow{P}v^+$ and $y' := v^+$. Then $(y', P') \in \mathcal{T}$, $|v\overrightarrow{D(y, P)}y| = |v\overrightarrow{D(y', P')}y'|$ and $y' \neq c$. Hence we may assume that $y \neq c$. For convenience, we abbreviate $D(y, P)$ to D if there is no fear of confusion.

Claim 6.11 *There exist $a \in V(C) - \{u, u^-\}$ and $b \in V(v^+\overrightarrow{D}y)$ such that $a \notin N_C(u^+)^-$ and $b \notin N_D(y)^+$.*

Proof. By Claim 5.1 (ii) and since $G - \{u\}$ is connected, there exist $w \in V(G - C)$ and $a \in V(C) - \{u, u^-\}$ such that $wa \in E(G)$. Then by Claim 5.2 (i), $a \notin N_C(u^+)^-$. Since $G - \{v\}$ is connected, there exist $w \in V(G - V(v\overrightarrow{D}y))$ and $b \in V(v^+\overrightarrow{D}y)$ such that $wb \in E(G)$. Then by Claim 5.1 (i), the assumption of Case 2 or the choice of (y, P) and v, we obtain

$b \notin N_D(y)^+$. $\square$

Claim 6.12 $V(H) - \{c\} = \emptyset$.

Proof. Suppose that $V(H) - \{c\} \neq \emptyset$, and let $z \in V(H) - \{c\}$. By Claim 5.2 (i), $N_{C-\{u\}}(u^+)^- \cap N_{C-\{u\}}(z) = \emptyset$. By Claim 5.1 (ii), $N_{C-\{u\}}(u^+)^- \cup N_{C-\{u\}}(z) \subseteq V(C)-\{u, u^-\}$. Therefore $d_{C-\{u\}}(u^+)+d_{C-\{u\}}(z) \leq |C|-2$. By Claim 5.1 (i), (iii) and the symmetry of x and y, $N_D(y)^+ \cap N_D(z) = \emptyset$. Since $N_D(y)^+ \cup N_D(z) \subseteq V(D)$, $d_D(y) + d_D(z) \leq |D| = |P| + 1$. Hence by Claim 5.1 (i) and (ii), $d_G(u^+)+d_G(y)+d_G(z) = (d_{C-\{u\}}(u^+)+d_{C-\{u\}}(z)) + d_D(u^+) + (d_D(y) + d_D(z)) + d_H(z) \leq (|C| - 2) + 1 + (|P| + 1) + (|H| - 1) = n - 1$. This implies that $G \notin \mathcal{F}_2$ and the equalities hold. The equality $d_{C-\{u\}}(u^+) + d_{C-\{u\}}(z) = |C| - 2$ implies that $N_{C-\{u\}}(u^+)^- \cup N_{C-\{u\}}(z) = V(C) - \{u, u^-\}$. The equality $d_D(y) + d_D(z) = |D|$ implies that $N_D(y)^+ \cup N_D(z) = V(D)$. Let a and b be as in Claim 6.11. Then since $a \notin N_{C-\{u\}}(u^+)^-$ and $b \notin N_D(y)^+$, $a \in N_{C-\{u\}}(z)$ and $b \in N_D(z)$. Since z is an arbitrary vertex in $V(H)-\{c\}$, we have $V(H)-\{c\} \subseteq N_G(a) \cap N_G(b)$. Moreover, the equality $d_H(z) = |H| - 1$ implies that H is complete. If $|H - \{c\}| \geq 2$, then we can choose $z_1, z_2 \in V(H) - \{c\}$ so that $z_1 \neq z_2$. If $|H - \{c\}| = 1$, then we let $z_1 = z_2$ be the unique vertex in $V(H) - \{c\}$. Let $\overrightarrow{R}$ be a path from z_1 to z_2 in H such that $V(H) - \{c\} \subseteq V(R)$. Then $C' := u\overrightarrow{C}az_1\overrightarrow{R}z_2b\overrightarrow{D}yv\overleftarrow{D}u$ and $C'' := a\overrightarrow{C}u\overrightarrow{D}bz_1\overrightarrow{R}z_2a$ are cycles such that $V(G) - \{c\} \subseteq V(C') \cup V(C'')$, a contradiction. $\square$

Claim 6.13 $c \in V(C)$.

Proof. Suppose that $c \notin V(C)$. Since G is 2-connected, there exist $z \in V(G - C)$ and $w \in V(C) - \{u\}$ such that $zw \in E(G)$. Choose such z and w so that $|w\overrightarrow{C}u|$ is as small as possible. Since $c \notin V(C)$, $w^+ \neq c$. Let $C_1 := u^+\overrightarrow{C}w$ and $C_2 := w^+\overrightarrow{C}u$. By Claim 5.2 (i), $N_{C_1}(u^+)^- \cap N_{C_1}(w) = N_{C_2}(u^+) \cap N_{C_2}(w^+)^- = \emptyset$. Since $N_{C_1}(u^+)^- \cup N_{C_1}(w^+) \subseteq V(C_1)$ and $N_{C_2}(u^+) \cup N_{C_2}(w^+)^- \subseteq V(C_2)$, $d_{C_i}(u^+) + d_{C_i}(w^+) \leq |C_i|$ for $i = 1, 2$. By Claim 6.11, $N_G(y) \subseteq V(D) - \{b^-, y\}$. Hence $d_G(u^+) + d_G(w^+) + d_G(y) \leq$

$|C| + |D| - 2 = |C| + |P| - 1 = n - |H| - 1$. This implies that $G \notin \mathcal{F}_2$, $V(H) = \emptyset$ and the equalities hold. The equality $d_G(y) = |D| - 2$ implies that $N_G(y) = V(D) - \{b^-, y\}$, and hence $u \in N_G(y)$. Hence G is covered by two cycles, a contradiction. □

By Claims 6.12 and 6.13, $V(H) = \emptyset$. Hence $u, x \notin N_G(y)$. Let $P_1 := x\overrightarrow{D}v^-$ and $P_2 := v\overrightarrow{D}y$, and let $z \in V(P_1)$. Suppose that there exists $g \in N_{P_2}(z) \cap N_{P_2}(y)^+$. Let $P' := x\overrightarrow{P}zg\overrightarrow{P}yg^-\overleftarrow{P}z^+$ and $y' := z^+$. Then $(y', P') \in \mathcal{T}$ and $|z\overrightarrow{D(y', P')}y'| > |v\overrightarrow{D(y, P)}y|$, which contradicts the choice of (y, P) and v. Thus $N_{P_2}(z) \cap N_{P_2}(y)^+ = \emptyset$. Since $N_{P_2}(z) \cup N_{P_2}(y)^+ \subseteq V(P_2)$, $d_{P_2}(z) + d_{P_2}(y) \le |P_2|$. By the choice of (y, P) and v, $N_{P_1}(y) = \emptyset$, and hence $d_{P_1}(z) + d_{P_1}(y) \le |P_1| - 1$. By Claim 5.2 (i), $N_C(u^+)^- \cap N_C(z) = \emptyset$. Hence by Claim 5.1 (ii), $d_C(u^+) + d_C(z) \le |C|$. Therefore $d_G(u^+) + d_G(y) + d_G(z) \le |C| + |P_1| + |P_2| - 1 = n - 1$. This implies that $G \notin \mathcal{F}_2$ and the equalities hold. Since z is an arbitrary vertex in $V(P_1)$, the equalities $d_C(u^+) + d_C(z) = |C|$, $d_{P_1}(z) = |P_1| - 1$ and $d_{P_2}(z) + d_{P_2}(y) = |P_2|$ imply that $N_C(u^+)^- \cup N_C(z) = V(C)$, $N_{P_1}(z) = V(P_1) - \{z\}$ and $N_{P_2}(z) \cup N_{P_2}(y)^+ = V(P_2)$ for all $z \in V(P_1)$. Note that $vz, uz, az, bz \in E(G)$ for all $z \in V(P_1)$, where let a and b be as in Claim 6.11.

If $|P_1| \ge 2$, then $uxb\overrightarrow{P}yv\overleftarrow{P}x^+a\overleftarrow{C}u$ and $uxb\overleftarrow{P}x^+a\overrightarrow{C}u$ are cycles which cover G, a contradiction. Thus $V(P_1) = \{x\}$. Since $G - \{x\}$ is connected, there exists an edge hg with $h \in V(C)$ and $g \in V(P_2)$. If $\{h, g\} \cap \{u, v\} = \emptyset$, then $u\overrightarrow{C}hg\overrightarrow{P}yv\overleftarrow{P}xu$ and $ux\overrightarrow{P}gh\overrightarrow{C}u$ are cycles which cover G, a contradiction. If $h = u$ and $g \ne v$, then $ug\overrightarrow{P}yv\overleftarrow{P}xa\overleftarrow{C}u$ and $ug\overleftarrow{P}xa\overrightarrow{C}u$ are cycles which cover G, a contradiction. If $h \ne u$ and $g = v$, then $uxb\overrightarrow{P}yvh\overleftarrow{C}u$ and $uxb\overleftarrow{P}vh\overrightarrow{C}u$ are cycles which cover G, a contradiction. Thus $h = u$ and $g = v$. Then $uv\overrightarrow{P}bxa\overleftarrow{C}u$ and $uvy\overleftarrow{P}bxa\overrightarrow{C}u$ are cycles which cover G, a contradiction.

This completes the proofs of Lemmas 8 and 9. □

7 Proofs of Lemmas 2, 3 and 4

Let $\mathcal{F}_4, \mathcal{F}_5$ and $\mathcal{F}_6$ be the sets of graphs which satisfy the assumptions of Lemmas 2, 3 and 4, respectively.

Proof of Lemma 2. Let $G \in \mathcal{F}_4$, and suppose that G does not satisfy (i)–(iii) of Lemma 2. Since the case of $c_1 = c_2$ is proved in [10], we may assume that $c_1 \neq c_2$. Let $I := \{c_1, c_2\}$, and let $\overrightarrow{C}$ be a cycle containing I. Let $H := G - C$. Note that $V(H) \neq \emptyset$ since G has no Hamiltonian cycle. Let $\{u_1, \ldots, u_k\} := N_C(H)$. We may assume that $u_1, \ldots, u_k$ occur in this order along $\overrightarrow{C}$. Note that $k \geq 2$ since G is 2-connected. Choose C so that (i) $|C|$ is as large as possible, (ii) $\sum_{x \in V(H)} d_G(x)$ is as small as possible, subject to (i). Note that by the choice (i), C is a maximal cycle of G.

Let $A_1 := \{i : u_i^+ \notin I\}$ and $A_2 := \{i : u_i^+ \in I \text{ and } u_i^{+2} \notin I\}$. We choose the orientation of C so that $A_1 \neq \emptyset$ if possible.

Case 1. $A_1 = \emptyset$.

Then by the choice of the orientation of C, we obtain $k = 2$, $|C| = 4$, $u_1^+ = u_2^- = c_1$ and $u_2^+ = u_1^- = c_2$. Let $G' := G - \{c_2\}$ if $u_1u_2 \in E(G)$; otherwise, let $G' := (G - \{c_2\}) \cup \{u_1u_2\}$. Then $d_{G'}(u_i) \geq d_G(u_i) - 1$ for $i = 1, 2$. Since $N_C(H) = \{u_1, u_2\}$, $d_{G'}(z) = d_G(z)$ for all $z \in V(H)$. Hence $\sigma_2(V(G') - \{c_1\}; G') \geq n - 2 = |G'| - 1$ since $u_1u_2 \in E(G')$. By applying Lemma 2 to G' as $c_1 = c_2$ and since $d_{G'}(c_1) = 2$, there exists a cycle C_1 of G' containing $V(G') - \{c_1\}$. If $u_1u_2 \in E(C_1)$, then $C_2 := (C_1 - \{u_1u_2\}) \cup \{u_1c_2, c_2u_2\}$ is a cycle of G containing $V(G) - \{c_1\}$, a contradiction. If $u_1u_2 \notin E(C_1)$, then C_1 is a cycle of G containing $V(G) - \{c_1, c_2\}$, a contradiction.

Case 2. $A_1 \neq \emptyset$.

Claim 7.1 (i) *$N_G(u_i^+) = V(C) - N_C(H)^+$ for all $i \in A_1$.*

(ii) *H is complete.*

(iii) *$N_C(z) = N_C(H)$ for all $z \in V(H)$.*

(iv) *$N_G(u_i^{+2}) = \big(V(C) - (N_C(H)^+ \cup \{u_i^{+2}\})\big) \cup \{u_i^+\}$ for all $i \in A_2$.*

Proof. Let $z \in V(H)$ and $i \in A_1$. By Lemma 6 (i), $N_H(u_i^+) = \emptyset$. By Lemma 6 (ii), $N_C(u_i^+) \subseteq V(C) - N_C(H)^+$. Hence $n-1 \leq d_G(u_i^+) + d_G(z) = d_C(u_i^+) + d_C(z) + d_H(z) \leq |V(C) - N_C(H)^+| + |N_C(H)| + |H| - 1 = n-1$. Thus the equalities hold. The equality $d_G(u_i^+) = |V(C) - N_C(H)^+|$ implies that (i) holds. The equality $d_H(z) = |H| - 1$ implies that $N_H(z) = V(H) - \{z\}$. Since z is an arbitrary vertex in H, (ii) holds. The equality $d_C(z) = |N_C(H)|$ implies that (iii) holds.

To prove (iv), let $i \in A_2$. By Claim 7.1 (ii) and (iii) and since (ii) of Lemma 2 does not hold, we obtain $N_H(u_i^{+2}) = \emptyset$ and $N_C(u_i^{+2}) \subseteq (V(C) - (N_C(H)^+ \cup \{u_i^{+2}\})) \cup \{u_i^+\}$. Hence we can obtain (iv) as in the proof of Claim 7.1 (i). □

Claim 7.2 (i) $V(u_i^+ \overrightarrow{C} u_{i+1}^-) \not\subseteq I$ *for each* $1 \leq i \leq k$, *where* $u_{k+1} := u_1$.
(ii) $d_G(u_i^+) + d_G(u_j^+) \leq |C|$ *for each* $i, j \in A_1$ *with* $i \neq j$.
(iii) $d_G(u_i^+) + d_G(u_j^{+2}) \leq |C|$ *for each* $i \in A_1$ *and* $j \in A_2$.

Proof. By Claim 7.1 (ii) and (iii) and since (ii) and (iii) of Lemma 2 do not hold, we can easily obtain (i). Let $i \in A_1$ and $j \in A_2$, and let $C_1 := u_i^+ \overrightarrow{C} u_j$ and $C_2 := u_j^+ \overrightarrow{C} u_i$. If $N_{C_1}(u_i)^- \cap N_{C_1}(u_j^{+2}) \neq \emptyset$ or $N_{C_2}(u_i) \cap N_{C_2}(u_j^{+2})^- \neq \emptyset$, then by Claim 7.1 (ii) and (iii), it is easy to see that $G - \{u_j^+\}$ is Hamiltonian, a contradiction. Thus $N_{C_1}(u_i^+)^- \cap N_{C_1}(u_j^{+2}) = N_{C_2}(u_i^+) \cap N_{C_2}(u_j^{+2})^- = \emptyset$. Note that by Claim 7.1 (i), $u_j^+ \notin N_{C_2}(u_i^+)$. Since $N_{C_1}(u_i^+)^- \cup N_{C_1}(u_j^{+2}) \subseteq V(C_1)$ and $N_{C_2}(u_i^+) \cup N_{C_2}(u_j^{+2})^- \subseteq (V(C_2) - \{u_j^+\}) \cup \{u_j\}$ and by Claim 7.1 (i) and (iv), we obtain $d_G(u_i^+) + d_G(u_j^{+2}) = d_C(u_i^+) + d_C(u_j^{+2}) \leq |C_1| + ((|C_2| - 1) + 1) = |C|$. Thus (iii) holds. By the similar argument, we can obtain (ii). □

Claim 7.3 $d_C(z) = |C|/2$ *for all* $z \in V(H)$.

Proof. By Claim 7.2 (i) and the assumption of Case 2, and by changing the orientation of $\overrightarrow{C}$ if necessary, we may assume that $A_1 \neq \emptyset$ and $|A_1 \cup A_2| \geq 2$. Let $i \in A_1$ and $j \in (A_1 \cup A_2) - \{i\}$. Let $u^* := u_j^+$ if $j \in A_1$; otherwise, let $u^* := u_j^{+2}$. Then by Claims 7.1 (i), (iv) and 7.2 (ii), (iii),

$2(|C| - |N_C(H)|) \leq d_G(u_i^+) + d_G(u^*) \leq |C|$. Hence by Lemma 6 (i) and Claim 7.1 (iii), $d_C(z) = |N_C(H)| = |C|/2$ for all $z \in V(H)$. □

By Lemma 6 (i) and Claim 7.3, $|u_i \overrightarrow{C} u_{i+1}| = 3$ for each $1 \leq i \leq k$. Hence by Claims 7.1 (ii), (iii), 7.2 (i) and the maximality of $|C|$, $I \subseteq N_C(H)$ and $|H| = 1$. Let z be an unique vertex in H, and let $D_i := u_i z u_{i+1} \overrightarrow{C} u_i$ for each $1 \leq i \leq k$. Then D_i is a cycle containing I and $|D_i| = |C|$, and hence by the choice (ii), $d_G(u_i^+) \geq d_G(z) = |C|/2 = k$ for each $1 \leq i \leq k$. By Lemma 6 (i) and (ii), $d_G(u_i^+) = k$ and $N_G(u_i^+) = N_G(z)$ for each $1 \leq i \leq k$. Hence $K_{k,k+1} \subseteq G \subseteq K_k + \overline{K_{k+1}}$ with $k = (n-1)/2 \geq 2$ and $d_G(c_i) \geq k+1$ for $i = 1, 2$.

This completes the proof of Lemma 2. □

Proofs of Lemmas 3 and 4. Let $G \in \mathcal{F}_5 \cup \mathcal{F}_6$, and suppose that G is not covered by two cycles. Suppose furthermore that if $G \notin \mathcal{F}_5$, then $G - \{c\}$ is not covered by two cycles. Then by Lemmas 8 and 9, diff$(G) \leq 1$. Let C be a longest cycle of G, and let $X := V(G - C)$ and $X' := X - \{c\}$. Note that $|X| \geq 3$ since G is 2-connected and G is not covered by two cycles. Moreover, if $G \notin \mathcal{F}_5$, then $|X'| \geq 3$ because $G - \{c\}$ is not covered by two cycles. Choose C so that (i) $c \in V(C)$ if possible, (ii) $\sum_{x \in X'} d_G(x)$ is as small as possible, subject to (i).

Since diff$(G) \leq 1$, the following fact holds.

Fact 7.4 (i) *X is an independent set of G.*

(ii) *$N_C(x)^+ \cap N_C(y) = N_C(x)^- \cap N_C(y) = \emptyset$ for all $x, y \in X$.*

(iii) *Let $u_1, u_2 \in N_C(x)$ with $u_1 \neq u_2$ for some $x \in X$. Let $C_1 := u_1^+ \overrightarrow{C} u_2$ and $C_2 := u_2^+ \overrightarrow{C} u_1$. Then $(N_{C_i}(u_i^+)^- \cup N_{C_i}(u_{3-i}^+)^+) \cap N_{C_i}(x') = \emptyset$ for each $x' \in X - \{x\}$ and $i = 1, 2$.*

Claim 7.5 *If $G \in \mathcal{F}_5$, then $c \notin X$.*

Proof. Suppose that $G \in \mathcal{F}_5$ and $c \in X$. By Fact 7.4 (i) and since G is 2-connected, we can take two vertices $u_1, u_2 \in N_C(c)$. Let $x \in X'$. Then by Lemma 6 (ii) and Fact 7.4 (ii) $\{u_1^+, u_2^+, x\}$ is an independent set of G. Let $C_1 := u_1^+ \overrightarrow{C} u_2$ and $C_2 := u_2^+ \overrightarrow{C} u_1$. By Fact 7.4 (iii), $(N_{C_i}(u_i^+)^- \cup$

$N_{C_i}(u_{3-i}^+)^+) \cap N_{C_i}(x) = \emptyset$ for $i = 1,2$. If there exists $v \in N_{C_1}(u_1^+)^- \cap N_{C_1}(u_2^+)^+$, then $C' := u_1 c u_2 \overleftarrow{C} v^+ u_1^+ \overrightarrow{C} v^- u_2^+ \overrightarrow{C} u_1$ is a cycle such that $|C'| = |C|$ and $c \in V(C')$, which contradicts the choice (i). Thus $N_{C_1}(u_1^+)^- \cap N_{C_1}(u_2^+)^+ = \emptyset$. Similarly, $N_{C_2}(u_1^+)^+ \cap N_{C_2}(u_2^+)^- = \emptyset$. Since $N_{C_i}(u_i^+)^- \cup N_{C_i}(u_{3-i}^+)^+ \cup N_{C_i}(x) \subseteq V(C_i) \cup \{u_{3-i}^+\}$ for $i = 1,2$, we obtain $d_{C_i}(u_i^+) + d_{C_i}(u_{3-i}^+) + d_{C_i}(x) \leq |C_i| + 1$ for $i = 1,2$. Hence by Fact 7.4 (i) and (ii), $d_G(u_1^+) + d_G(u_2^+) + d_G(x) = d_C(u_1^+) + d_C(u_2^+) + d_C(x) \leq |C| + 2 < |C| + |X| = n$, a contradiction. □

Note that by Claim 7.5, $|X'| \geq 3$. Let $x_1 \in X'$ with $d_G(x_1) = \min\{d_G(x) : x \in X'\}$. Let $\{u_1, \dots, u_k\} := N_G(x_1)$. We may assume that $u_1, \dots, u_k$ occur in this order along $\overrightarrow{C}$. Let $x_2, x_3 \in X' - \{x_1\}$ with $x_2 \neq x_3$. Then by Lemma 6 (i) and Fact 7.4 (i), we obtain $d_G(x_2) + d_G(x_3) = d_C(x_2) + d_C(x_3) \leq |C|/2 + |C|/2 = |C|$.

On the other hand, by Lemma B, Fact 7.4 (i) and Claim 7.5, and by the definition of x_1, we obtain $2 \leq d_G(x_1) = d_C(x_1) \leq |X'| - 1$. Then by Fact 7.4 (i), $d_G(x_2) + d_G(x_3) \geq n - 1 - d_G(x_1) \geq n - 1 - (|X'| - 1) = |C| + (|X| - |X'|) \geq |C|$. Thus the equalities hold. Hence $d_G(x_1) = |X'| - 1$, $d_G(x_2) = d_G(x_3) = |C|/2$, $d_G(x_2) + d_G(x_3) = n - 1 - d_G(x_1)$ and $X = X'$, in particular, $|C|$ is even and $G \notin \mathcal{F}_5$. Since x_2 and x_3 are arbitrary distinct vertices in $X' - \{x_1\}$, we have $d_G(x) = |C|/2$ for all $x \in X' - \{x_1\} = X - \{x_1\}$.

Since there exists no cycle containing X, it follows from Lemma 7 that $|C| \leq 2|X| - 1$. Since $|C|$ is even, $|C| \leq 2|X| - 2$. Hence by the definition of x_1, $|X| - 1 = d_G(x_1) \leq d_G(x) = |C|/2 \leq |X| - 1$ for all $x \in X - \{x_1\}$. This implies that $d_G(x) = |X| - 1 = |C|/2$ for all $x \in X$. By Fact 7.4 (ii), $N_G(x) = N_G(x_1)$ for all $x \in X - \{x_1\}$ and $|u_i \overrightarrow{C} u_{i+1}| = 3$ for all $1 \leq i \leq k$, where we let $u_{k+1} := u_1$. Then $|X| = k + 1$ and $|C| = 2k$. Set $X := \{x_1, x_2, x_3, \dots, x_{k+1}\}$.

Suppose that $c \in N_C(x_1)^+$. We may assume that $u_1^+ = c$. Then $C_1 := u_1 x_{k+1} u_2 \overrightarrow{C} u_1$ and $C_2 := u_1 x_1 u_2 x_2 \dots u_{k-1} x_{k-1} u_k x_k u_1$ are cycles which cover $G - \{c\}$, a contradiction. Thus $c \in N_C(x_1)$.

Let $D_i := u_i x_1 u_{i+1} \overrightarrow{C} u_i$ for each $1 \leq i \leq k$. Then D_i is a longest cycle such that $c \in V(C_i)$ and $V(C_i) = (V(C) - \{u_i^+\}) \cup \{x_1\}$. Hence by the choice (ii) of C, $d_G(u_i^+) \geq k = |C|/2$. By Lemma 6 (i) and (ii), $d_G(u_i^+) = k = |C|/2$ and $N_G(u_i^+) = N_G(x_1)$ for each $1 \leq i \leq k$. Hence $K_{k,2k+1} \subseteq G \subseteq K_k + (2k+1)K_1$ and $d_G(c) \geq 2k+1$.

This completes the proofs of Lemmas 3 and 4. □

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