RECONSTRUCTION NUMBER OF SEPARABLE SELF COMPLEMENTARY GRAPHS AND OTHERS

P.Bhanumathy
APMD/VSSC
Thiruvananthapuram-22

S.Ramachandran*
N.I.Centre for Higher Education
Nagercoil-629180. INDIA
dr.s.ramachandran@gmail.com

Abstract

We prove that each graph in two infinite families is fixed uniquely by just two of its maximal induced subgraphs, with each of which the degree of the missing vertex is also given. One of these families contains all separable self complementary graphs and a self complementary graph of diameter 3 and order n for each $n \geq 5$ such that $n \equiv 0$ or 1 (mod 4). The other contains a Hamiltonian self complementary graph of diameter 2 and order n for each admissible $n \geq 8$.

* **Research Supported by DST, Govt. Of India SR/S4/MS:609/09.**
MSC (2010): 05C 60, 05C76

1. INTRODUCTION

The collection of (n−1)-vertex induced subgraphs without labels of a graph G with $n \geq 3$ vertices is called the **deck** of G. Members of the deck of G are called **cards** of G. Harary and Plantholt [5] defined the **reconstruction number** *rn*(G) to be the size of the smallest subcollection of the deck of G which is not contained in the deck of any other graph H, H $\not\cong$ G. Myrvold [7] referred to to this parameter as **ally-reconstruction number** and proved that the reconstruction number of a tree with five or more vertices is three. Class reconstruction number has also been defined [5] this way: Let C be a class of graphs. Then the class reconstruction number of *a graph* G in C, denoted by Crn(G) , is the minimum number of cards of G which, *together with the information that G is in C*, determine the graph G uniquely. Harary and Lauri [4] have proved that the class reconstruction number of a maximal planar graph is either 2 or 3. A survey of work on *rn*(G) is given in Lauri [6].

Motivated by reconstruction questions for digraphs, a **degree associated card** or **dacard** of a graph G was defined [8] as an ordered pair (*C*,*d*) where C is a card of G and *d* is the degree of the vertex of G whose deletion has given the card C. The collection of dacards of G is called the **dadeck** of G. **Degree associated reconstruction number** *drn*(G) of a graph G was defined [8] as the size of the smallest subcollection of the dadeck of G which is not contained in the dadeck of any other graph H, H $\not\cong$ G. In fact, *drn*(G) = *Crn*(G), where C is the class of all graphs with a given number *m* of vertices.

In [8], *drn*(G) for complete graphs, cycles, complete bipartite graphs and disjoint union of identical graphs have been studied. Barrus and West [1] have studied the *drn* of vertex transitive graphs and caterpillars of special form. Here we prove that *drn*(G) = 2 for graphs in two infinite families and show that one of these families contains all separable self complementary graphs and a self complementary graph of diameter 3 and order n for each admissible n ≥ 5. The other contains a Hamiltonian self complementary graph of diameter 2 and order n for each admissible n ≥ 5. We use the terminology in Harary [3]. Self complementary graph is written in short as 'sc-graph'.

2. THE GRAPHS Bu(H) and Lo(H)

For a graph H, the graphs Bu(H) and Lo(H) are defined in [2] as follows. The graph formed from H by adding four new vertices v_1, v_2, v_3 and v_4, the edges of the path $v_1v_2v_3v_4$, and the edges joining v_2 and v_3 with all the vertices of H is called the **bull on H** and is denoted by **Bu(H)** (Figure 1a). The graph formed from H by adding four new vertices v_1, v_2, v_3 and v_4, the edges of the path $v_2v_1v_4v_3$ and the edges joining v_2 and v_3 with all the vertices of H is called the **lock on H** and is denoted by **Lo(H)** (Figure 1b).

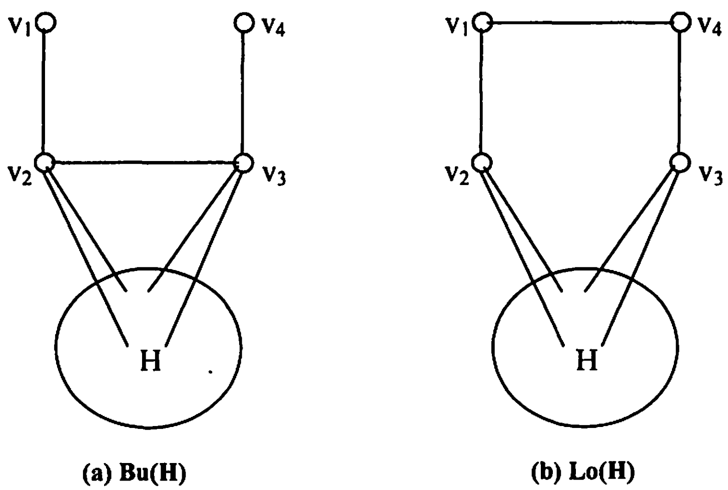

Figure 1: The construction of graphs Bu(H) and Lo(H).

Obviously Bu(H) and Lo(H) are sc-graphs when H is self complementary. Also Bu(H) contains all separable sc-graphs by the following theorem.

Theorem 1 (Theorem 1.11 [2]): For a self complementary graph G of order n, the following statements are equivalent:

(i) G has cut-vertices.

(ii) G has end-vertices.

(iii) G = Bu(H), where H is a self complementary graph of order n–4. In this case, G has exactly two cut-vertices, exactly two end-vertices, and diameter 3. □

In addition, the family of graphs Bu(H) (respectively, Lo(H)) contains an n-vertex sc-graph of diameter 3 and with endvertices (respectively, Hamiltonian sc-graphs of diameter 2) for all feasible values of $n \geq 5$ as follows [2]. Starting from $G_1 = P_4$ and repeatedly performing the operation $G_{i+1} = Bu(G_i)$, we get an infinite family of sc-graphs of diameter 3 on n vertices with $n \equiv 0 \pmod 4$. If we start with $G_1 = K_1$, we get a similar family for $n \equiv 1 \pmod 4$. Similarly, starting with $G_1 = Lo(P_4)$ or $Lo(K_1)$ and repeatedly performing the operation $G_{i+1} = Lo(G_i)$, we can form two new families of Hamiltonian sc-graphs of diameter 2 corresponding to $n \equiv 0$ or $1 \pmod 4$ and $n \geq 5$.

3. DETERMINATION OF *drn*(Bu(H)) and *drn*(Lo(H))

Theorem 2.4 of [1] gives the following as a corollary.

Result 2: drn(G) = 1 only if one of the following holds.

(i) G is complete or edgeless

(ii) G has an isolated vertex or a vertex of degree n–1

(iii) G hs a vertex v such that G–v is vertex transitive and d(v) = 1 or n–2. □

Theorem 3: *drn* (Bu(H)) = 2, for every graph H.

Proof: Bu(H) has $n \geq 5$ vertices. Choose a dacard $D_1 = (A, n-2)$ from the dadeck of Bu(H). Now A has a vertex of degree n–3 and has no component isomorphic to K_2. ...(1)

Choose another dacard $D_2 = (B,1)$. B has *only one* vertex of degree n–2 and *only one* endvertex and they are adjacent. Moreover, if w_1 and w_2 are vertices of degree n–3 in B, then w_1 and w_2 must be adjacent to all vertices of B other than its unique endvertex and hence the transposition $(w_1\ w_2)$ of B is an automorphism of B. ... (2)

Suppose G is a graph having dacards D_1 and D_2 in its dadeck. Hence G can be obtained from B by annexing a vertex v to it and joining with a single vertex of B. ... (3)

If v is joined to a vertex of B of degree n–3, then $G \cong Bu(H)$ (because all graphs obtained in this process are isomorphic by (2) and the foot of the *endvertex whose deletion from Bu(H) has given B* remain as a vertex of degree n–3 in B).

If v is joined to the unique endvertex in B to obtain G, then G will have only one vertex of degree n–2 and its only dacard with attached degree n–2 will have a component K_2 so that D_1 is not a dacard of G, giving contradiction.

If v is joined to the unique vertex of B of degree n–2 to obtain G, then G will have no vertex of degree n–2 and so D_1 can not be a dacard of G, again giving contradiction.

If v is joined to a vertex of B of degree d, $2 \le d \le n-4$ to get G, then G will have exactly one vertex of degree n–2 (which must be adjacent to all vertices of G other than the endvertex v of G) and hence by (2), the only dacard of G with attached degree n–2 has no vertex of degree n–3 ($n \ge 2$). So by (1), D_1 is not a dacard of G, again giving a contradiction.

Now (3) implies G ≅ Bu(H). Hence drn(Bu(H)) ≤ 2 and so by Result 2, drn(Bu(H)) = 2. □

Now Theorem 1 and Theorem 3 give the following.

Theorem 4: Self complelmentary graphs with cutvertices are reconstructible and have *drn* equal to two. □

When (A,a) and (B,b) are two dacards of G, A can be placed over B such that precisely an (n−2)-vertex induced subgraph of them coincide and the noncoinciding vertices u of A and v of B satisfy either (i) $deg_A(u) = b-1$ and $deg_B(v) = a-1$, or (ii) $deg_A(u) = b$ and $deg_B(v) = a$. Cases (i) and (ii) correspond, respectively, to when u and v are or are not adjacent in G.

Definition: A superimposition of two dacards (A,a) and (B,b) of a graph as above is called a **pasting** of the dacards (A,a) and (B,b). Vertex *u* is called the **aloof vertex** of A and *v* is called the aloof vertex of B in the pasting. A pasting is called **type 1** or **type 2** according as it satisfies (i) or (ii). Two pastings are called **identical** (and counted as a single pasting) if there is an isomorphism from one pasting to the other taking the aloof vertices of one to the aloof vertices of the other.

If G has a pair of dacards (A,a) and (B,b) such that there is only one pasting of them, then obvioulsy these two dacards determine G uniquely and *drn*(G) ≤ 2. However, two dacards can have more than one pasting. Lo(H) has a pair of dacards having a pasting of type 1 and a pasting of type 2. For the graph C_5, *drn* is 3 and each of its dacards is $(P_4,2)$. The 'Y' shaped pasting when augmented give a graph different from C_5.

Theorem 5: *drn*(Lo(H)) = 2 when H has at least three vertices and neither H nor H^C has an isolated vertex.

Proof: Now Lo(H) has $n \ge 7$ vertices and $\Delta(Lo(H)) = n-3$. Choose a dacard $D_1 = (A, 2)$. D_1 corresponds to the deletion of vertex v_1 or v_4 from Lo(H) and A has exactly one endvertex (say x) and its foot (say x_1) has degree n–3 (Figure 2(A)).

The only vertex x_2 of A which is not adjacent to x_1 has exactly n–4 neighbors and they constitute the set of neighbors of x_1 other than x also and the "subgraph E of A induced by these n–4 vertices has no isolated vertex and no vertex of degree $\geq$ n–5 (in E)". Thus a vertex of E can have maximum degree n–4 in A. (E corresponds to and is isomorphic to H of Lo(H). But it is not labeled as H so that only the properties of H in Lo(H) that descend to D_1 and noted alone are used while trying to obtain the graphs having the chosen dacards). . . . (1)

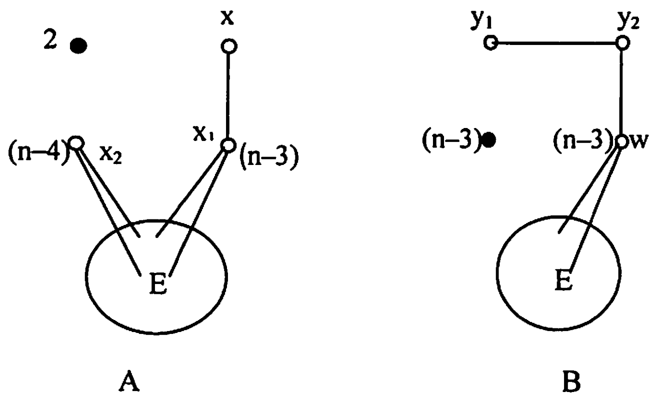

Figure 2: The dacards D_1 = (A, 2) and D_2 = (B, n–3).

The graph obtained from A by adjoining a vertex and joining it precisely with x_2 and x, is isomorphic to Lo(H). . . . (2)

Choose another dacard D_2 = (B, n–3) such that B has a vertex of degree n–3. D_2 corresponds to the deletion of one among the vertices v_2 and v_3 from Lo(H). B has exactly one end vertex (say y_1) and its foot (say y_2) has degree 2 (Figure 2(B)). The only neighbor (say w) of y_2 other than y_1 has degree n–3. The neighbors of w other than y_2 in B, which are n–4 in number induce a subgraph isomorphic to E (mentioned in (1) above). As E has no isolated vertex, B has no endvertex other than y_1. E has no vertex of degree $\geq$ n–5 and so each vertex of B other than w has degree less than n–4 in B. . . . (3)

There is at least one pasting of D_1 and D_2 and let u and v denote the aloof vertices of A and B respectively.

Case 1. There is a pasting of D_1 and D_2 of type 1.

$Deg_A(u)$ = n–4 and $deg_B(v)$ = 1. As B has only one end vertex y_1, v = y_1.

Case 1.1. u has a neighbor of degree n–3 in A.

By (1), x_1 is the only vertex of degree n–3 in A and so *u must be a vertex of E* (Figure 2(A)). Hence the foot of the only endvertex *x* of A–u has degree n–4.

But the foot of the only endvertex y_2 of B–v (= B–y_1) has degree n–3 (Figure 2(B)). This contradicts A–u ≅ B–v, and so this case does not arise.

Case 1.2: u does not have a neighbor of degree n–3 in A.

From Figure 2(A), **u** = $\mathbf{x_2}$. We try to fit "B with v marked" over "A with u marked" such that B–v fits exactly over A–u to get the pasting. The lone endvertices x and y_2 of A–u and B–v respectively must coincide. Their respective neighbors x_1 and w must also coincide. Hence the neighbors of x_1 other than x must coincide with the neighbors of w other than y_2 such that A–u coincides with B–v (= B–y_1). This gives a unique fitting because of the structure of A and B. (Note that u (= x_2) is adjacent with all the vertices of E). As it is a pasting of type 1, the corresponding graph for which both D_1 and D_2 are dacards is obtained by joining u and v in the pasting. Such a graph is same as the graph obtained from A by adjoining a vertex and joining it precisely with x_2 and x and hence is Lo(H) by (2).

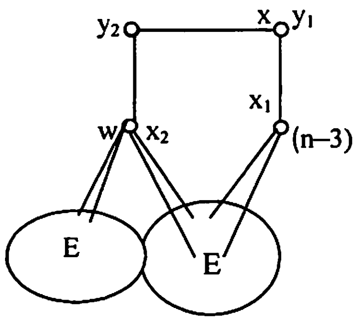

Figure 3. B is being pasted over A as in Case 2.2., Step (4).

Case 2: There is a pasting of D_1 and D_2 of type 2.

By(1), x_1 is the only vertex of degree n–3 in A and so **u** = $\mathbf{x_1}$.

Case 2.1: v has no neighbor of degree 1.

Now B–v is connected, whereas A–u is disconnected, giving contradiction.

Case 2.2: v has a neighbor of degree 1 also.

Now **v is identified as $\mathbf{y_2}$ in B**. A pasting (of B over A) can be obtained in only one way as follows: The vertices x of A and y_1 of B must coincide as they are the lone isolated vertices of A–u and B–v. The vertex w of B has degree n–4 in B–v and hence the vertex of A that can coincide with w must have degree n–4 in A–u. By (1), no vertex of E has degree ≥ n–4 in A–u and so x_2 is the only vertex of degree n–4 in A–u and hence w of B must coincide with x_2 of A in the pasting considered. (Figure 3). . . . (4)

Vertices of B other than the already accounted for vertices y_1, y_2 and w (there are n–4 of them) are adjacent with w, and the vertices of A other than the already accounted for vertices x_1, x and x_2 (there are n–4 of them) are adjacent with x_2 and they must coincide. The result of the pasting is a graph that is same as the one obtained from A by adjoining a vertex (y_2 in Figure 3) and joining it precisely with x and x_2 and hence is Lo(H) by (2).

So in both Case 1 and Case 2, Lo(H) is the only graph having dacards D_1 and D_2 and hence by Result 2(iii), *drn*(Lo(H)) = 2. □

Corollary : There are Hamiltonian sc-graphs of diameter two on n vertices for each $n \geq 8$ of the form $n \equiv 0$ or 1(mod 4) and having *drn* equal to 2. □

Acknowledgement: We are thankful to the referee for pointing out that '*drn*(G) = C*rn*(G), where C is the class of graphs with a given number *m* of edges' and for his other comments which have improved the presentation.

REFERENCES

1. M.D.Barrus and D.B.West. Degree-associated reconstruction number of graphs, *Disc. Math.* 310(2010) 2600-2612.

2. A.Farrugia, Self-complementary graphs and generalizations: a comprehensive reference manual, *M.Sc Thesis* written under S.Fiorini, University of Malta, August 1999.

3. F.Harary, *Graph Theory*. Addison –Wesley, Mass., 1969.

4. F.Harary and J.Lauri, The class reconstruction number of maximal planar graphs, *Graphs and Combinatorics* 3 (1987) 45-53

5. F.Harary and M.Plantholt, The graph reconstruction number, *J. Graph Theory* 9(1985) 451-454.

6. J.Lauri, The reconstruction problem, Section 2.3 in *Handbook of Graph Theory*, (Eds. J.L.Gross and J.Yellen) CRC press 2004. 79-98.

7. W.J.Myrvold, The ally-reconstruction number of a tree with five or more vertices is three, *J. Graph Theory* 14(1990) 149-166.

8. S.Ramachandran, Reconstruction number for Ulam's conjecture. *Ars Combin.* 78(2006) 289-296.

****