On the $\{4, -k\}$-hypomorphy for digraphs

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Abstract

Let k be a non-negative integer, two digraphs $G = (V, A)$, $G' = (V, A')$ are $\{k\}$-hypomorphic if for all k-element subset K of V, the subdigraphs $G[K]$ and $G'[K]$ induced on K are isomorphic. The equivalence relation $D_{G,G'}$ on V is defined by : $xD_{G,G'}y$ if $x = y$ or there exists a sequence $x_0 = x, \ldots, x_n = y$ of elements of V satisfying $(x_i, x_{i+1}) \in A$ if and only if $(x_i, x_{i+1}) \notin A'$, for all i, $0 \leq i \leq n-1$. The main result of this paper is the following : Let v, k be non-negative integers with $v \geq k+6$, if G and G' are two digraphs, $\{4\}$-hypomorphic and $\{v-k\}$-hypomorphic on the same vertex set V of v vertices, and C is an equivalence class of the equivalence relation $D_{G,G'}$, then $G'[C-A]$ and $G[C-A]$ are isomorphic for all subset A of V of at most k vertices. In particular, $G'[C]$ and $G[C]$ are $\{v-h\}$-hypomorphic for all $h \in \{1, 2, \ldots, k\}$, $G'[C]$ and $G[C]$ (resp. G' and G) are isomorphic. In particular, for $k = 1$ and $k = 4$ we obtain the result of G. Lopez and C. Rauzy [7]. As an application of the main result, we have : If G and G' are $\{v-4\}$-hypomorphic on the same vertex set V of $v \geq 10$ vertices, then $G[X]$ and $G'[X]$ are isomorphic for all subset X of V ; the particular case of tournaments was obtained by Y. Boudabbous [2].

Key words : Digraph, Isomorphy, Hereditary isomorphy, Hypomorphy, Reconstruction, Tournament.

1 Introduction

A *directed graph* or simply *digraph* G consists of a finite non-empty set $V(G)$ of vertices together with a prescribed collection $A(G)$ of ordered pairs of distinct vertices, called the set of the arcs of G. Such a digraph is denoted by $(V(G), A(G))$ or simply by (V, A). The cardinality of

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G is that of V. We denote this cardinality by $|V(G)|$ as well as $|G|$. For example, given a non-empty set V, $(V, \emptyset)$ is the *empty digraph* on V whereas $(V, \{(x,y) : x \neq y \in V\})$ is the *complete digraph* on V. Given a digraph $G = (V, A)$, with each non-empty subset X of V associate the *subdigraph* $(X, A \cap (X \times X))$ *of* G induced by X denoted by $G[X]$. The subdigraph $G[V - X]$, where $X \subseteq V$, (resp. $G[V - \{x\}]$, where $x \in V$) is also denoted by $G - X$ (resp. $G - x$). Let $G = (V, A)$ be a digraph. For $x \neq y \in V$, $x \longrightarrow_G y$ or $y \longleftarrow_G x$ means $(x,y) \in A$ and $(y,x) \notin A$, $x \longleftrightarrow_G y$ means $(x,y) \in A$ and $(y,x) \in A$, $x \ldots_G y$ means $(x,y) \notin A$ and $(y,x) \notin A$. For $x \in V$ and $Y \subseteq V$, $x \longrightarrow_G Y$ signifies that for every $y \in Y$, $x \longrightarrow_G y$. For $X, Y \subseteq V$, $X \longrightarrow_G Y$ signifies that for every $x \in X$, $x \longrightarrow_G Y$. For $x \in V$ and for $X, Y \subseteq V$, $x \longleftarrow_G Y$, $x \longleftrightarrow_G Y$, $x \ldots_G Y$, $X \longleftrightarrow_G Y$ and $X \ldots_G Y$ are defined in the same way. Given a digraph $G = (V, A)$, distinct vertices x and y of G form a *directed pair* if either $x \longrightarrow_G y$ or $x \longleftarrow_G y$. Otherwise, $\{x, y\}$ is a *neutral pair*; it is *full* if $x \longleftrightarrow_G y$, and *void* when $x \ldots_G y$. A digraph $T = (V, A)$ is a *tournament* if all its pairs are directed. A *transitive tournament* or *total order* or *n-chain* is a tournament T of cardinality n such that for $x, y, z \in V(T)$, if $x \longrightarrow_T y$ and $y \longrightarrow_T z$ then $x \longrightarrow_T z$. Given a total order $O = (V, A)$, $x < y$ means $x \longrightarrow_O y$ for $x, y \in V$.
In another respect, given digraphs $G = (V, A)$ and $G' = (V', A')$, a bijection from V onto V' is an *isomorphism* from G onto G' provided that for any $x, y \in V$, $(x, y) \in A$ if and only if $(f(x), f(y)) \in A'$. Two digraphs are then *isomorphic* if there exists an isomorphism from one onto the other, in that case write $G \simeq G'$. In the contrary case, write $G \not\simeq G'$. With each digraph $G = (V, A)$ associate its *dual* $G^* = (V, A^*)$, defined as follows : for $x \neq y \in V$, $(x, y) \in A^*$ if $(y, x) \notin A$. Say that a digraph is *self-dual* if it is isomorphic to its dual. A digraph H *embeds into* a digraph G if H is isomorphic to a subdigraph of G. Given digraphs G and G' with the same vertex set V, G and G' are *hereditary isomorphic* if for each subset X of V, $G[X]$ and $G'[X]$ are isomorphic; given an integer k, the digraphs G and G' are *$\{k\}$-hypomorphic* if for every set K of k vertices, the subdigraphs $G[K]$ and $G'[K]$ are isomorphic. Then, a digraph G is *$\{k\}$-reconstructible* if every graph G' which is $\{k\}$-hypomorphic to G is in fact isomorphic to G. If $|V| > k$, we say that G and G' are *$\{-k\}$-hypomorphic* if they are $\{|V| - k\}$-hypomorphic. A digraph G is *$\{-k\}$-reconstructible* if every graph G' which is $\{-k\}$-hypomorphic to G is isomorphic to G. When G and G' are $\{p\}$-hypomorphic, for every $p \leq k$, we say that G and G' are *$(\leq k)$-hypomorphic.* When G and G' are $\{k\}$-hypomorphic and $\{l\}$-hypomorphic, we say that G and G' are $\{k, l\}$-hypomorphic. In the same way, we introduce the notion of *$(\leq k)$-reconstruction.*

The following notion is due to G. Lopez [4]. Let $G = (V, A)$ and $G' = (V, A')$ be two (≤ 2)-hypomorphic digraphs. Denote $D_{G,G'}$ the binary re-

lation on V such that : for $x \in V$, $xD_{G,G'}x$; and for $x \neq y \in V$, $xD_{G,G'}y$ if there exists a sequence $x_0 = x, ..., x_n = y$ of elements of V satisfying $(x_i, x_{i+1}) \in A$ if and only if $(x_i, x_{i+1}) \notin A'$, for all i, $0 \leq i \leq n-1$. The relation $D_{G,G'}$ is an equivalence relation called *the difference relation*, its classes are called *difference classes.*
The main result of this paper is :

Theorem 1.1. *Let $k \geq 1$ be an integer. If G and G' are two $\{4, -k\}$-hypomorphic digraphs on at least $k+6$ vertices and C is a difference class of $D_{G,G'}$, then :*

1. *The subdigraphs $G'[C-A]$ and $G[C-A]$ are isomorphic for all subset A of V of at most k vertices.*
2. *$G'[C]$ and $G[C]$ are $\{-h\}$-hypomorphic for all integer $h \in \{1, 2, \ldots, k\}$.*
3. *$G'[C] \simeq G[C]$.*
4. *$G' \simeq G$.*

In particular for the cases $k = 1$ and $k = 4$, we obtain the following result of G. Lopez and C. Rauzy [7].

Theorem 1.2. *[7]*

1. *The digraphs on at least 7 vertices are $\{4, -1\}$-reconstructible, (i.e : if G and G' are $\{4, -1\}$-hypomorphic then G' and G are isomorphic).*
2. *The digraphs on at least 10 vertices are $\{-4\}$-reconstructible.*

An application of the Theorem 1.1 is the following :

Theorem 1.3. *Two $\{-4\}$-hypomorphic digraphs on at least 10 vertices, are hereditary isomorphic.*

In Theorem 1.3, the particular case of tournaments was obtained by Y. Boudabbous [2].

Theorem 1.4. *[2] Two $\{-4\}$-hypomorphic tournaments, on at least 10 vertices, are hereditary isomorphic.*

2 Some particular digraphs

Consider the following digraphs : $F_3 = (\{0,1,2\}, \{(1,0),(0,2),(2,0)\})$, $P_f = (\{0,1,2\}, \{(1,0),(2,0),(1,2),(2,1)\})$, $P_v = (\{0,1,2\}, \{(1,0),(2,0)\})$. A digraph isomorphic either to F_3 or to F_3^* is called a *flag*. A *full* (resp. *void*) *peak* is a digraph isomorphic to P_f or its dual (resp. to P_v or its dual). A *peak* is either a full pick or a void pick (see Figure 1).

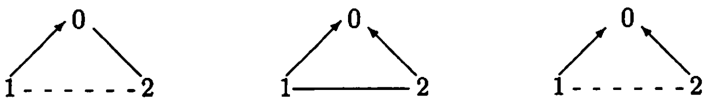

FIGURE 1 – Flag, full peak, void peak.

Let $D_4 = (\{0,1,2,3\},\{(0,2),(2,1),(1,0),(0,3),(1,3),(2,3)\})$. A *diamond* is a digraph isomorphic to D_4 or to D_4^* (see Figure 2). Given an integer $n \geq 2$, the digraph C_n is defined on $\{0,1,\ldots,n-1\}$ by $i \longrightarrow_{C_n} j$ if and only if $j = i+1$ for each $i \in \{0,1,\ldots,n-2\}$, and all its other pairs are neutral and having the same type. A digraph isomorphic to C_n is called a *n-consecutivity* or simply a consecutivity. Clearly, a n-consecutivity has exactly $(n-1)$ directed pairs and its neutral pairs are all void pairs or all full pairs. A *n-cycle* or cycle is any digraph isomorphic to one of the digraphs gotten from C_n by replacing the neutral pair $\{0, n-1\}$ by $(n-1) \longrightarrow 0$, clearly every 3-*cycle* is iomorphic to the tournament $(\{0,1,2\},\{(1,0),(0,2),(2,1)\})$.

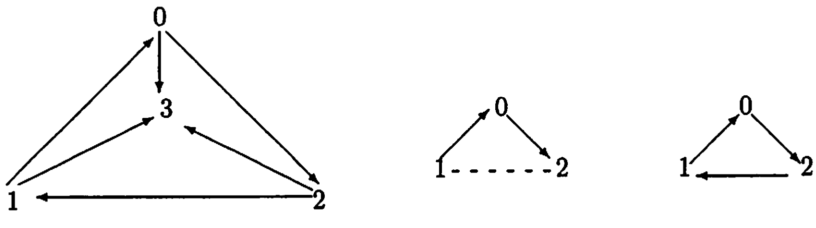

FIGURE 2 – Diamond, 3-consecutivity, 3-cycle.

A *n-near-chain* or a near-chain is every digraph obtained from a n-chain by replacing the directed pair formed by its extremities by a neutral pair.

Clearly, a 3-near-chain is a 3-consecutivity.
Given a digraph $G = (V, A)$, a subset I of V is an *interval* of G if for $a, b \in I$ and $x \in V - I$, $(a, x) \in A$ if and only if $(b, x) \in A$, and the same for (x, a) and (x, b). For instance, $\emptyset$, V and $\{x\}$, where $x \in V$, are intervals of G, called *trivial intervals*. A digraph is *indecomposable* if all its intervals are trivial, otherwise it is *decomposable*.
Given a digraph $S = (\{0, 1, \dots, m-1\}, A)$, where $m \geq 1$ is an integer, for $i \in \{0, 1, \dots, m-1\}$ we associate a digraph $G_i = (V_i, A_i)$, with $|V_i| \geq 1$, such that the V_i's are mutually disjoint. The S-*sum* of the G_i's is the digraph denoted by $S(G_0, G_1, \dots, G_{m-1})$ and defined on the union of the V_i's as follows : given $x \in V_i$ and $y \in V_j$, where $i, j \in \{0, 1, \dots, m-1\}$, (x, y) is an arc of $S(G_0, G_1, \dots, G_{m-1})$ if either $i = j$ and $(x, y) \in A_i$ or $i \neq j$ and $(i, j) \in A$. This digraph replaces each vertex i of S by G_i. We say that the vertex i of S is *dilated by* G_i.
Given an integer $n \geq 1$, we define the family S_n of digraphs on the $2n$ vertices $t_1, t_2, \dots, t_{2n}$ by $t_i \longrightarrow_{S_n} t_{i+k}$ for $i \in \{1, 2, \dots, 2n\}$ and $k \in \{1, 2, \dots, n-1\}$, and the pairs $\{t_i, t_{i+n}\}$ are neutral for every $i \in \{1, 2, \dots, n\}$, these integers are here considered modulo 2n. Next, we introduce a particular family $\mathcal{E}(S_n)$ of extensions of the digraphs family S_n. An element δ_n of this family is obtained from an element of S_n by adding mutually disjoint sets $s_1, s_2, \dots, s_{2n}$ (the set s_i is called a *sector* and it could be empty) to the vertex set $\{t_1, t_2, \dots, t_{2n}\}$ of S_n satisfying the following conditions :

(i) δ_n does not embed diamonds.

(ii) For all $i \in \{1, 2, \dots, 2n\}$, the subdigraph $\delta_n[s_i \cup \{t_i, t_{i+1}\}]$ is a finite chain with t_i as first element and t_{i+1} as last element.

(iii) For all i, j, if $s_i \cup s_j$ is non-empty, then $\delta_n(s_i \cup s_j)$ is a tournament.

(iv) For all i, for all x in $s_i \cup \{t_i\}$ and for every y in s_{i+j}, where $j \in \{1, 2, \dots, n-1\}$, we have $x \longrightarrow_{\delta_n} y$.

(v) For all i, for all $j \in \{n, n+1, \dots, 2n-1\}$ and for all y in s_{i+j}, we have $y \longrightarrow_{\delta_n} t_i$.

Let h be an non-negative integer. The integers below are considered modulo $2h+1$. The tournament T_{2h+1} defined on $\{0, 1, \dots, 2h\}$ (see Figure 3) by $T_{2h+1}[\{0, 1, \dots, h\}]$ is the usual total order on $\{0, 1, \dots, h\}, T_{2h+1}[\{h+1, \dots, 2h\}]$ is also the usual order on $\{h+1, h+2, \dots, 2h\}$ however $\{i+1, i+2, \dots\dots, h\} \longrightarrow_{T_{2h+1}} i+h+1 \longrightarrow_{T_{2h+1}} \{0, 1, \dots, i\}$ for every $i \in \{0, 1, \dots, h-1\}$. A digraph G is said to be an element of $D(T_{2h+1})$ if G is obtained by dilating each vertex of T_{2h+1} by a finite chain p_i, then $G = T_{2h+1}(p_0, p_1, \dots, p_{2h})$. We recall that T_{2h+1} is indecomposable and $D(T_{2h+1})$ is the class of finite tournaments without diamond [6].

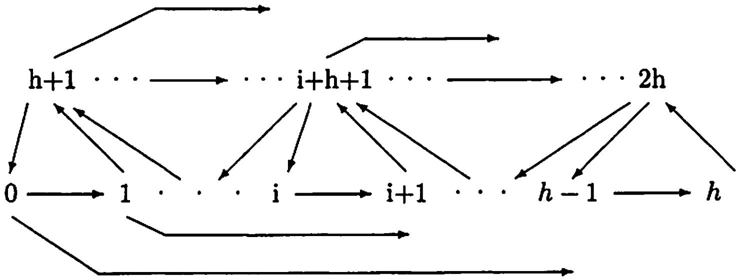

FIGURE 3 – T_{2h+1}.

3 Combinatorial Lemma of M. Pouzet

A well-known lemma in combinatorics is the following famous result of M. Pouzet [8].

Lemma 3.1. *(Combinatorial lemma of M. Pouzet [8]) Let p and r be non-negative integers, V be a set of size $v \geq p+r$ elements, U and U' be sets of subsets P of p elements of V. If for every subset Q of $p+r$ elements of V, the number of elements of U which are contained in Q is equal to the number of elements of U' which are contained in Q, then for every finite subsets P' and Q' of V, such that P' is contained in Q' and $Q' \setminus P'$ has at least $p+r$ elements, the number of elements of U which contain P' and are contained in Q' is equal to the number of elements of U' which contain P' and are contained in Q'.*
In particular if $|V| \geq 2p+r$, we have $U = U'$.

To continue, some notations are needed :

Notation 3.2. *Given a digraph G=(V,A), a subset A of V and a digraph H, denote : $\mathcal{N}(G,H,A) = \{P \subset V \ : \ A \subset P,\ G[P] \simeq H\}$ and $n(G,H,A) = |\mathcal{N}(G,H,A)|$. Moreover if $|V(H)| = |A|+1$, we set $N(G,H,A) = \{x \in V - A \ : \ G[A \cup \{x\}] \simeq H\}$.*
Note that $x \in N(G,H,A)$ if and only if $A \cup \{x\} \in \mathcal{N}(G,H,A)$, so $|N(G,H,A)| = n(G,H,A)$.

From Lemma 3.1, follows the following results :

Lemma 3.3. [8, 9] *Let $k \geq 1$ be an integer, G and G' be two digraphs defined on the same vertex set V of v elements. If G and G' are $\{k\}$-hypomorphic, then G and G' are $(\leq min(k, v-k))$-hypomorphic.*

From Lemma 3.3, follows immediately this result.

Corollary 3.4. *For an integer $k \geq 1$, two digraphs $\{-k\}$-hypomorphic on at least $2k$ vertices are $(\leq k)$-hypomorphic.*

Proposition 3.5. *Let $k \geq 1$ be an integer, G and G' be two $\{-k\}$-hypomorphic digraphs defined on the same vertex set V of cardinality v, and H be a digraph verifying $|V(H)| \leq v-k$. Then $n(G, H, A) = n(G', H, A)$ for each subset A of V with $|A| \leq k$.*

Proof. We apply Lemma 3.1 to the sets :

$$U = \{P \subset V \ : G[P] \simeq H\} \text{ and } U' = \{P \subset V \ : G'[P] \simeq H\}$$

We conclude considering $p = |V(H)|$, $r = v - k - p$, $P' = A$ and $Q' = V$. □

4 Proof of Theorem 1.1

The proof uses the following results.

Lemma 4.1. [6] *Given two (≤ 4)-hypomorphic digraphs G and G', and C an equivalence class of $D_{G,G'}$, then :*

1. *C is an interval of G and G', moreover if C' is an another class of $D_{G,G'}$, we have either $C \longrightarrow C'$ or $C' \longrightarrow C$ or $C \longleftrightarrow C'$ or $C \ldots C'$ in G and G'.*
2. *If $G'[C]$ and $G[C]$ are isomorphic for each equivalence class C of $D_{G,G'}$, then G and G' are isomorphic.*
3. *Neither peaks nor flags and no diamonds are embeddable in the sub-digraphs $G[C]$ and $G'[C]$.*
4. *Every 3-consecutivity (resp. 3-cycle) in $G[C]$ is reversed in $G'[C]$.*
5. *If $G[C]$ is an element of $\mathcal{E}(S_n)$ or $G[C]$ embeds a 3-cycle, then there is no adjacent neutral pairs in $G[C]$.*

Lemma 4.2. [6] *Given two (≤ 4)-hypomorphic digraphs G and G', and a difference class C of $D_{G,G'}$.*

1. *If $G[C]$ is a tournament, then there exists an integer $h \geq 0$ satisfying that $G[C]$ is an element of $D(T_{2h+1})$.*
2. *If $G[C]$ has not any 3-cycle, then $G[C]$ is either a chain or a near-chain or a consecutivity or a cycle.*

3. *If $G[C]$ has a 3-cycle and $G[C]$ is not a tournament, then there exists an integer $n \geq 1$ such that $G[C]$ is an element of $\mathcal{E}(S_n)$.*

From Lemmas 4.1 and 4.2, follow immediately this result :

Corollary 4.3. *Let G and G' be two (≤ 4)-hypomorphic digraphs and C be a difference class of $D_{G,G'}$.*

1. *If $G[C]$ is neither a tournament without diamonds nor an element of $\mathcal{E}(S_n)$, then $G'[C]$ and $G[C]$ are hereditary isomorphic.*
2. *Whenever $G[C]$ is either a tournament without diamonds or an element of $\mathcal{E}(S_n)$, then $G'[C] \simeq G^*[C]$.*

Proposition 4.4. *[1, 3] Given a digraph G on at least two vertices, consider the digraph R (resp. R') obtained by dilating a vertex i_0 of G by a digraph H (resp. H'). If R and R' are isomorphic, then H and H' are isomorphic too.*

The next result is a consequence of Proposition 4.4.

Corollary 4.5. *Let $p \geq 1$ be an integer, M and M' be two digraphs defined on the same vertex set $\{0, 1, \ldots, p\}$, f be an isomorphism from M into M' and H (resp. H') be a digraph. Given $i_0 \in \{0, \ldots, p\}$, and a digraph R (resp. R') obtained from M (resp. M') by dilating the vertex i_0 (resp. $f(i_0)$) by H (resp. H'), then $R' \simeq R$ if and only if $H' \simeq H$.*

Proposition 4.6. *Given $G = (V, A)$, $G' = (V, A')$ two $\{4, -1\}$-hypomorphic digraphs on at least 7 vertices. For each difference class C of $D_{G,G'}$, we have*

1. *$G'[C]$ and $G[C]$ are isomorphic.*
2. *$G'[C - x]$ and $G[C - x]$ are isomorphic for every x in V.*

Proof. If G has one difference class, we apply directly Theorem 1.2. Otherwise, if $G[C]$ is neither a tournament without diamonds nor an element of $\mathcal{E}(S_n)$, by Corollary 4.3, $G'[C]$ and $G[C]$ are hereditary isomorphic. To continue, we assume that $G[C]$ is either a tournament without diamonds or an element of $\mathcal{E}(S_n)$.

1. Let $b \in C$ and $H = G[C]$. From Lemma 4.1, H has neither peaks nor diamonds and its eventual neutral pairs are disjoint. Evidently, C is an element of $\mathcal{N}(G, H, \{b\}) = \{P \subset V \ : \ b \in P \text{ et } G[P] \simeq H\}$. Consider $P \subset V$ such that $P \neq C$. We claim that $P \in \mathcal{N}(G, H, \{b\})$ if and only if $P \in \mathcal{N}(G', H, \{b\})$. Let C' be a difference class of $D_{G,G'}$ such that $P \cap C' \neq \emptyset$, and $x \in P - C'$. By Lemma 4.1, $C' \cap P$ is an interval of $G[P]$ and $G'[P]$.
   - If $x \longleftrightarrow_G C'$ or $x \ldots_G C'$, as $G(P) \simeq H$ and its eventual neutral pairs are disjoint in H, then $|P \cap C'| = 1$, so $G'(P \cap C') \simeq G(P \cap C')$.

- If $x \longrightarrow_G C'$ or $C' \longrightarrow_G x$, as $G(P) \simeq H$ and H does not embed any peak, then $G(P \cap C')$ is a tournament. In an other hand, $G[P]$ does not embed any diamond, then $G(P \cap C')$ is a chain. Thus, $G'(P \cap C') \simeq G(P \cap C')$.

From the two cases above, we have $G'[P] \simeq G[P]$, so $P \in \mathcal{N}(G, H, \{b\})$ if and only if $P \in \mathcal{N}(G', H, \{b\})$.
According to Proposition 3.5, $n(G', H, \{b\}) = n(G, H, \{b\})$ and then $C \in \mathcal{N}(G', H, \{b\})$. Therefore, $G'[C]$ and $G[C]$ are isomorphic.

2. Consider $i_0 \in C$. The digraphs M and M' are defined as follows : $M = G[(V-C)\cup\{i_0\}]$ and $M' = G'[(V-C)\cup\{i_0\}]$. As any difference class C' of $D_{G,G'}$ distinct from C is a difference class of $D_{M,M'}$, by the first assertion $M'[C'] \simeq M[C']$. Then, there exists an isomorphism f from M onto M' satisfying that $f(i_0) = i_0$. Let $H = G[C - x]$ and $H' = G'[C - x]$. Clearly, the digraph $R = G - x$ (resp. $R' = G' - x$) is obtained from M (resp. M') by dilating the vertex i_0 by H (resp. H'). Since $G - x$ and $G' - x$ are isomorphic, then by Corollary 4.5, $G[C - x]$ and $G'[C - x]$ are isomorphic.

□

Proof of Theorem 1.1. The assertions 2 and 3 are consequence of the first one. The assertion 4 follows from 3, and 2 of Lemma 4.1. We will do the proof of the first assertion by induction on k. If $k = 1$, we conclude by Proposition 4.6. For $k \geq 1$, assume that G and G' are $\{4, -(k+1)\}$-hypomorphic, consider C a difference class of $D_{G,G'}$. Denote by V the vertex set of G and G'. We will prove that $G'[C-A] \simeq G[C-A]$, for all subset A of V with $|A| \leq k+1$. If $G[C]$ is neither a tournament without diamonds nor an element of $\mathcal{E}(S_n)$ then, by Corollary 4.3, $G'[C]$ and $G[C]$ are hereditary isomorphic, so $G'[C-A]$ and $G[C-A]$ are isomorphic. Now, we may assume that $G[C]$ is a tournament without diamonds or an element of $\mathcal{E}(S_n)$.
Remark If $x \in C$, then $C - x$ is a chain or a neutral pair or a difference class of $D_{G-x,G'-x}$.
Indeed, since $G[C]$ is a tournament without diamonds or an element of $\mathcal{E}(S_n)$, $G[C - x]$ is a chain or a neutral pair or admits a 3-cycle or a 3-consecutivity. From Lemma 4.1 every 3-cycle and 3-consecutivity in $G[C]$ is reversed in $G'[C]$, then $C - x$ is a difference class of $D_{G-x,G'-x}$.

1. If $V = C$, we have $G - x$ and $G' - x$ are $\{4, -k\}$-hypomorphic. From induction hypothesis $G' - x \simeq G - x$, so G and G' are $\{4, -1\}$-hypomorphic. Hence, Theorem 1.2 implies that $G' \simeq G$. From Remark above we may suppose that $V - x$ is a difference class of $D_{G-x,G'-x}$. From the induction hypothesis, $G'[V - B \cup \{x\}] \simeq G[V - B \cup \{x\}]$ for $B \subset V$ with $B| \leq k$, so $G'[C - A] \simeq G[C - A]$ for all subset A of V with $|A| \leq k+1$.
2. If $V \neq C$, let $y \in V - C$, we have $G - y$ and $G' - y$ are $\{4, -k\}$-

hypomorphic and C is a difference class of $D_{G-y,G'-y}$. From the induction hypothesis, $G'[C] \simeq G[C]$. Let $x \in C$, we have $G - x$ and $G' - x$ are $\{4, -k\}$-hypomorphic. From Remark above, we assume $C - x$ is a difference class of $D_{G-x,G'-x}$. From the induction hypothesis, $G'[C - B \cup \{x\}] \simeq G[C - B \cup \{x\}]$ for all subset $B \subset V$ with $|B| \leq k$. So, $G'[C - A] \simeq G[C - A]$ for all subset A of V with $|A| \leq k + 1$.

□

5 Proof of Theorem 1.3

We introduce the three non self-dual subdigraphs, of cardinality five, gotten from S_n. We denote by A_5 and B_5 elements of $\mathcal{E}(S_1)$ such that $s_1 = \{1, 2\}$, $s_2 = \{3\}$ are the only sectors of A_5 also B_5, $s_2 \longrightarrow s_1$ and the neutral pair of A_5 (resp. B_5) is full (resp. is empty). We call C_5 every element of $\mathcal{E}(S_2)$ such that the cardinality of the sector s_1 is 1 and its two neutral pairs are full for $i = 1$ and void for $i = 2$, (see Figure 4).

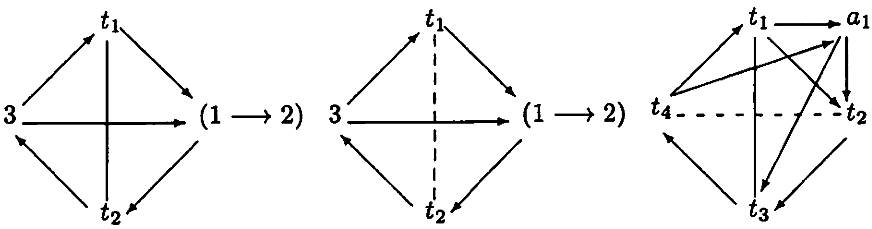

FIGURE 4 – A_5, B_5 and C_5.

Let us recall the following result due to G. Lopez which is very useful for the proof.

Theorem 5.1. *[4, 5] All digraphs are* (≤ 6)*-reconstructible.*

From Theorem 5.1 and Lemma 3.3 follows immediatly this result

Corollary 5.2.

1. *Every two* (≤ 6)*-hypomorphic digraphs are hereditary isomorphic.*
2. *For all integer* $k \geq 6$*, every two* $\{-k\}$*-hypomorphic digraphs on at least* $k + 6$ *vertices are hereditary isomorphic.*

3. *If $G'[C]$ and $G[C]$ are (≤ 6)-hypomorphic and C is an interval of G and G' for each equivalence class C of $D_{G,G'}$, then G and G' are hereditary isomorphic.*

Proposition 5.3. *Given two $\{-4\}$-hypomorphic digraphs G and G' defined on the same vertex set on at least 9 vertices. Let C be a difference class of $D_{G,G'}$. If $G[C]$ is not a tournament without diamonds, then $G'[C]$ and $G[C]$ are hereditary isomorphic.*

Proof. From Corollary 3.4, G and G' are (≤ 4)-hypomorphic, so $G[C]$ and $G'[C]$ are (≤ 4)-hypomorphic. From Corollary 4.3, we may suppose that there exists an integer $n \geq 1$ such that $G[C]$ is an element of $\mathcal{E}(S_n)$. Proposition 3.5 is often used in this proof by choosing particular subset X of $V(G)$ such that $G[X] \simeq A_5$ or B_5 or C_5 or sometimes their duals. As every 3-consecutivity and even 3-cycle in $G[C]$ is reversed in $G'[C]$ (Lemma 4.1), $G'[X] \simeq G^*[X]$. According to n, we have the next claims :

Claim 5.4. *If $n \geq 3$, then $G[C] \in S_n$.*

Proof. By contradiction, we may assume that there exists an $i \in \{1, \ldots, 2n\}$ such that $s_i \neq \emptyset$. Without loss of generality, suppose that $s_1 \neq \emptyset$. Let $b \in s_1$. Set $A = \{t_{3+n}, t_{2+n}, t_{1+n}, b\}$ and H the subdigraph induced by G on $A \cup \{t_1\}$. Notice that $H \simeq A_5^*$ or B_5^*. The only sectors of H are $\{b\}$ and $\{t_{2+n}, t_{3+n}\}$. As every 3-consecutivity and even 3-cycle in $G[C]$ is reversed in $G'[C]$ (Lemma 4.1), $G'[A \cup \{t_1\}]$ is isomorphic to H^*. It is clear that $N(G, H, A)$ (resp. $N(G', H, A)$) is a subset of $\{t_1, t_2, t_3\}$, $t_1 \in N(G, H, A)$ and $t_1 \notin N(G', H, A)$. As $\{b, t_{3+n}\}$ is an interval of the induced subdigraphs of G and G' on $\{b, t_2, t_{2+n}, t_{3+n}\}$, and not an interval of $G[A \cup \{t_2\}]$ and $G'[A \cup \{t_2\}]$, t_2 is neither an element of $N(G, H, A)$ nor of $N(G', H, A)$. Likewise, the vertex t_3 is neither an element of $N(G, H, A)$ nor of $N(G', H, A)$. Thus, $n(G, H, A) = 1$ and $n(G', H, A) = 0$; which contradicts Proposition 3.5. □

Claim 5.5. *If $n \geq 3$, then $n = 3$, so $G[C] \in S_3$ and its neutral pairs have the same type. So, $G'[C]$ and $G[C]$ are hereditary isomorphic.*

Proof. Firstly, suppose by contradiction that $n \geq 4$. By Claim 5.4, the sector s_i is empty, for every $i \in \{1, \ldots, 2n\}$. We consider $A_1 = \{t_1, t_2, t_3, t_{1+n}\}$ and $H_1 = G[A_1 \cup \{t_{4+n}\}]$. Notice that $H_1 \simeq A_5$ or B_5. The only sectors of H_1 are $\{t_2, t_3\}$ and $\{t_{4+n}\}$.
By definition of S_n, $N(G, H_1, A_1) = \{t_{i+n} \, ; 4 \leq i \leq n\}$ and $N(G', H_1, A_1) = \emptyset$. This implies that $n(G, H_1, A_1) = n - 3 \geq 1$ and $n(G', H_1, A_1) = 0$; which contradicts Proposition 3.5.
Now $G[C] \in S_3$. Without loss of generality, we assume that all the neutral pairs of S_3 are void except that of $\{t_3, t_6\}$. For $A_2 = \{t_1, t_2, t_3, t_4\}$

and $H_2 = G[A_2 \cup \{t_6\}]$, we have $H_2 \simeq C_5$, $N(G, H_2, A_2) = \{t_6\}$ and $N(G', H_2, A_2) = \emptyset$. Thus, $n(G, H_2, A_2) \neq n(G', H_2, A_2)$; which contradicts Proposition 3.5. So, all neutral pairs of $G[C]$ are of the same type. We recall that every 3-consecutivity and 3-cycle in $G[C]$ is reversed in $G'[C]$ (Lemma 4.1). Hence $G[C]$ and $G'[C]$ are hereditary isomorphic. □

Claim 5.6. *If $n = 2$, then $G[C] \in S_2$ or $G[C] \in \mathcal{E}(S_2)$ and its two neutral pairs have the same type and its sectors are empty except one of cardinality 1. So, $G[C]$ and $G'[C]$ are hereditary isomorphic.*

Proof. Assume by contradiction that there exists an $i \in \{1,2,3,4\}$ such that $|s_i| \geq 2$. Without loss of generality, let $b \neq c \in s_1$. $A_1 = \{t_1, b, c, t_4\}$ and $H_1 = G[A_1 \cup \{t_3\}]$. Clearly, $H_1 \simeq A_5$ or B_5, $\{b, c\}$ and $\{t_4\}$ are the only sectors of H_1, $N(G, H_1, A_1)$ and (resp. $N(G', H_1, A_1)$) is a subset of $\{t_2, t_3\}$. From Lemma 4.1, $G'[A_1 \cup \{t_3\}] \simeq H_1^*$ then $t_3 \in N(G, H_1, A_1)$, but $t_3 \notin N(G', H_1, A_1)$. Since $\{t_1, b, c\}$ is an interval of the induced subdigraphs of G and G' on $\{t_4, t_1, b, c, t_2\}$, t_2 is neither an element of $N(G, H_1, A_1)$ nor of $N(G', H_1, A_1)$. So, $n(G, H_1, A_1) = 1$ and $n(G', H_1, A_1) = 0$; which contradicts Proposition 3.5. Thus, if it exists an $i \in \{1,2,3,4\}$ such that $s_i \neq \emptyset$, then s_i is a singleton. We more suppose that there is at least two non-empty sections. If those two sectors are consecutive; for instance, let $s_1 = \{b_1\}$ and $s_2 = \{b_2\}$. Let $A_2 = \{b_1, t_2, b_2, t_3\}$, and $H_2 = G[A_2 \cup \{t_4\}]$. Notice that $H_2 \simeq A_5$ or B_5, $\{b_2, t_3\}$ and $\{b_1\}$ are the only sectors of H_2 and $N(G, H_2, A_2)$ (resp. $N(G', H_2, A_2)$) is a subset of $\{t_1, t_4\}$. Clearly, $t_4 \in N(G, H_2, A_2)$ but $t_4 \notin N(G', H_2, A_2)$. As $\{b_1, b_2, t_2\}$ is an interval of the induced subdigraphs of G and G' on $\{t_3, b_1, b_2, t_2, t_1\}$, t_1 is neither an element of $N(G, H_2, A_2)$ nor of $N(G', H_2, A_2)$. Thus, $n(G, H_2, A_2) = 1$ but $n(G', H_2, A_2) = 0$; which is absurd. Therefore, there exists exactly two non-consecutive sectors which are singletons. Without loss of generality, we may suppose that $s_1 = \{c_1\}$, $s_3 = \{c_3\}$ and $c_1 \longrightarrow_G c_3$. Consider $A_3 = \{c_1, c_3, t_3, t_4\}$ and $H_3 = G[A_3 \cup \{t_2\}]$. We have $H_3 \simeq A_5$ or B_5, $\{t_3, c_3\}$ and $\{c_1\}$ are the only sectors of H_3. $N(G, H_3, A_3)$ (resp. $N(G', H_3, A_3)$) is a subset of $\{t_1, t_2\}$. We have $t_2 \in N(G, H_3, A_3)$ but $t_2 \notin N(G', H_3, A_3)$. Since the set $\{c_3, t_4\}$ is an interval of the induced subdigraphs of G and G' on $\{t_1, t_3, c_3, t_4\}$ and not an interval of the induced subdigraphs of G and G' on $A_3 \cup \{t_1\}$, t_1 is neither an element of $N(G, H_3, A_3)$ nor of $N(G', H_3, A_3)$. So, we have $n(G, H_3, A_3) = 1$ which is not equal to $n(G', H_3, A_3) = 0$. Consequently, $G[C]$ has three empty sectors and the forth sector is a singleton. To conclude, it suffices to show that if $G[C]$ is an element of $\mathcal{E}(S_2)$, its two neutral pairs have the same type and its sectors are empty except one of cardinality 1. By contradiction, we may assume that $t_1 \ldots t_3$ and $t_2 \longleftrightarrow t_4$ in G and G'. For $A_4 = C - \{t_4\}$ and $H_4 = G[C]$, $n(G, H_4, A_4) = 1$ and $n(G', H_4, A_4) = 0$; which contradicts Proposition 3.5. □

Claim 5.7. *If $n = 1$, then $G[C]$ is either a near-chain, or an element* *of $\mathcal{E}(S_1)$ on 5 vertices with sectors $s_1 = \{b_1, c_1\}$ and $s_2 = \{b_2\}$ such that $G[\{b_1, b_2, c_1\}]$ is a 3-cycle, or an element of $\mathcal{E}(S_1)$ on 4 vertices. So, $G[C]$ and $G'[C]$ are hereditary isomorphic.*

Proof. If $|s_i| \geq 3$, we will prove that $s_j = \emptyset$ for $j \neq i \in \{1, 2\}$. By contradiction, we may assume that for instance there exists $b_1, c_1, d_1 \in s_1$ and $b_2 \in s_2$ such that $b_2 \longrightarrow_G \{b_1, c_1\}$. Set $A_1 = \{t_1, b_1, b_2, c_1\}$ and $H_1 = G[A_1 \cup \{t_2\}]$. Notice that $H_1 \simeq A_5$ or B_5, $\{b_1, c_1\}$ and $\{b_2\}$ are the only sectors of H_1. It is obvious that $N(G, H_1, A_1) = \{t_2\}$ and $N(G', H_1, A_1) = \emptyset$. Consequently, we get $n(G, H_1, A_1) = 1$ and $n(G', H_1, A_1) = 0$; which is absurd by Proposition 3.5. Therefore, $G[C]$ is a near-chain or $|s_i| \in \{0, 1, 2\}$. If $|C| = 5$, we suppose for instance that $s_1 = \{b_1, c_1\}$ and $s_2 = \{b_2\}$. It is sufficient to verify that the subdigraph $G[\{b_1, b_2, c_1\}]$ is a 3-cycle. By contradiction, assume that $G[\{b_1, b_2, c_1\}]$ is a 3-chain. If $\{b_1, c_1\}$ is an interval of G, for $A_2 = \{t_1, b_1, b_2, c_1\}$ and $H_2 = G[A_2 \cup \{t_2\}]$, $n(G, H_2, A_2) = 1$ and $n(G', H_2, A_2) = 0$. Therefore, $\{b_1, c_1\}$ is not an interval of G. Without loss of generality, suppose that $b_1 \longrightarrow_G c_1$. If $b_1 \longrightarrow_G b_2$, necessary $b_2 \longrightarrow_G c_1$. We get $\{b_1, b_2, t_1\} \longrightarrow_G c_1$ and $G[\{b_1, b_2, t_1\}]$ is a 3-cycle. Consequently, $G[\{b_1, b_2, t_1, c_1\}]$ is a diamond; which contradicts Lemma 4.1. Thus, $b_2 \longrightarrow_G b_1$ so $c_1 \longrightarrow_G b_2$. We get $G[\{b_1, b_2, c_1\}]$ is a 3-cycle. If $|C| = 6$, then $s_1 = \{b_1, c_1\}$ and $s_2 = \{b_2, c_2\}$. Let $b_1 \longrightarrow_G c_1$ and $b_2 \longrightarrow_G c_2$. From the previous case, to forbid chains, we have $b_2 \longrightarrow_G b_1$ then $G[\{b_1, b_2, c_2\}]$ becomes a chain; which is absurd. □

Let us recall the following result of Y. Boudabbous [2].

Lemma 5.8. *[2] Let T and T' be two (≤ 4)-hypomorphic tournaments on at least 6 vertices. Then, T and T' are (≤ 5)-hypomorphic.*

Proof of Theorem 1.3. We shall denote the vertex set of G by V. Let C be a difference class of $D_{G,G'}$. From Proposition 5.3, we may assume that $G[C]$ is a tournament without diamonds. From Corollary 3.4, G and G' are (≤ 4)-hypomorphic. In addition, Lemma 5.8 proves that $G[C]$ and $G'[C]$ are (≤ 5)-hypomorphic. Using Corollary 5.2, it suffices to prove that $G[C]$ and $G'[C]$ are $\{6\}$-hypomorphic. We distinguish five cases :
Case 1 : $|C| \geq 10$. Let $X \subseteq C$ such that $|X| = 4$. By Theorem 1.1, $G[C - X]$ and $G'[C - X]$ are isomorphic. Thus, $G[C]$ and $G'[C]$ are $\{-4\}$-hypomorphic. We may conclude by Theorem 1.4.
Case 2 : $|C| = 9$. Let $X \subseteq C$ such that $|X| = 3$. By Theorem 1.1, $G[C - X]$ and $G'[C - X]$ are isomorphic. which implies that $G[C]$ and $G'[C]$ are $\{6\}$-hypomorphic.
Case 3 : $|C| = 8$. From Theorem 1.1, for $x \neq y \in C$, $G'[C - \{x, y\}]$ and $G[C - \{x, y\}]$ are isomorphic then $G[C]$ and $G'[C]$ are $\{6\}$-hypomorphic.
Case 4 : $|C| = 7$. Using Theorem 1.1, for $x \in C$, $G'[C - \{x\}]$ and $G[C - \{x\}]$

are isomorphic then $G[C]$ and $G'[C]$ are $\{6\}$-hypomorphic.
Case 5 : $|C| \leq 6$. From Theorem 1.1, $G'[C]$ and $G[C]$ are isomorphic then $G[C]$ and $G'[C]$ are $\{6\}$-hypomorphic. □

6 $\{-k\}$-hypomorphic digraphs

Theorem 1.3 is done with $k = 4$, we will generalize it for $k \geq 4$.

Corollary 6.1. *Given an integer $k \geq 4$, let $G = (V, A)$ and $G' = (V, A')$ be two $\{-k\}$-hypomorphic digraphs with at least k+6 vertices. Then, G and G' are hereditary isomorphic.*

Proof.
- If $k = 4$, we apply simply Theorem 1.3 to G and G'.
- If $k \geq 6$, by Corollary 5.2, G and G' are hereditary isomorphic.
- If $k = 5$, by Corollary 3.4, G and G' are (≤ 5)-hypomorphic. Given a subset A of V on at most 6 vertices, we shall prove that $G[A] \simeq G'[A]$. Let x be an element of $V - A$. Then, $G' - x$ and $G - x$ are two $\{-4\}$-hypomorphic digraphs with at least 10 vertices. Using Theorem 1.3, $G - x$ and $G' - x$ are hereditary isomorphic, in particular, $G[A]$ and $G'[A]$ are isomorphic, hence G and G' are (≤ 6)-hypomorphic, so by Corollary 5.2, G and G' are hereditary isomorphic.

□

In Corollary 6.1, the value $k+6$ is optimal. For example, the tournament T (see Figure 5) and its dual are $\{-k\}$-hypomorphic, where T is the dilating of a 3-cycle by a 2-chain and a $(k+2)$-chain.

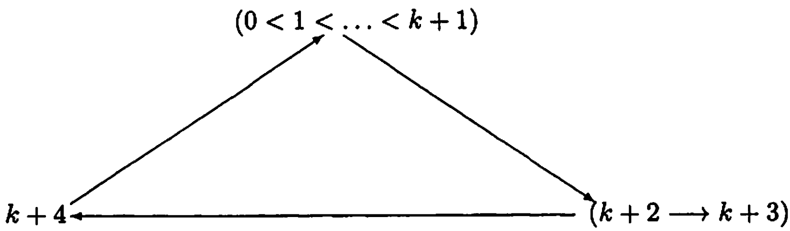

FIGURE 5 – The dilating of a 3-cycle by a 2-chain and a $(k+2)$-chain.

Obviously, T and T^* are not (≤ 6)-hypomorphic (the restrictions of T and T^* on the set $\{0, 1, 2, k+2, k+3, k+4\}$ are not isomorphic).

Corollary 6.1 shows that two $\{-k\}$-hypomorphic digraphs on at least $k+6$ vertices are hereditary isomorphic for all $k \geq 4$. What would happen for $k \in \{1,2,3\}$? The answer is negative for $k \in \{1,2\}$, see Proposition 6.3, however for $k=3$ we conjecture this :

Conjecture 6.2. *All $\{4,-3\}$-hypomorphic digraphs are hereditary isomorphic.*

Proposition 6.3. *Given $h \geq 4$ and the tournaments $T = T_{2h+1}$ and $T' = T^*_{2h+1}$.*

1. *T and T' are $\{4,-1\}$-hypomorphic.*
2. *T and T' are $\{4,-2\}$-hypomorphic.*
3. *T and T' are not $\{-3\}$-hypomorphic.*
4. *T and T' are not (≤ 6)-hypomorphic.*

Proof. Given an integer h, all the considered integer are modulo $2h+1$. We start with two simple observations. Firstly, it is well-known that T_{2h+1} and its dual are (≤ 4)-hypomorphic. Secondly, for $h \geq 2$, since in T_{2h+1} every vertex i dominates its h successive vertices and dominated by the other vertices that is $\{i+h+1, i+h+2, \ldots, i-1\} \longrightarrow_{T_{2h+1}} i$ and $i \longrightarrow_{T_{2h+1}} \{i+1, i+2, \ldots, i+h\}$ for every $i \in \{0,1,\ldots,2h\}$, T_{2h+1} is self-dual. It suffice to consider the isomorphism f such that $f(i) = i$ and $f(i+j) = i-j$ for every $j \in \{1,2,\ldots,h\}$.

1. $T_{2h+1} - i$ admits a single non trivial interval $I_i = \{i+h, i+h+1\}$. Besides, we have $T_{2h+1} - \{i, i+h\}$, $T_{2h+1} - \{i, i+h+1\}$ and T_{2h-1} are isomorphic. As T_{2h-1} is self-dual, we get an isomorphism from $T-i$ to $T'-i$ which is denoted by g such that $g(I_i) = I_i$. Therefore T and T' are $\{4,-1\}$-hypomorphic.
2. $T_{2h+1} - \{i, i+1\}$ admits a single non-trivial interval $I_i = \{i+h, i+h+1, i+h+2\}$. Consider the distinct elements x, y of I_i. Evidently, $T_{2h+1} - \{i, i+1, x, y\} \simeq T_{2h-3}$. Thus there is an isomorphism h from $T_{2h+1} - \{i, i+1\}$ to $T^*_{2h+1} - \{i, i+1\}$ such that $h(I_i) = I_i$. Lastly, it exists $i \neq j \in \{0,1,\ldots,2h\}$ such that $T_{2h+1} - \{i,j\}$ has exactly two non trivial intervals $I_i = \{i+h, i+h+1\}$ and $I_j = \{j+h, j+h+1\}$. Clearly there is an isomorphism k from $T_{2h+1} - \{i,j\}$ to $T^*_{2h+1} - \{i,j\}$ such that $k(I_i) = I_j$ and $k(I_j) = I_i$. Therefore T and T' are $\{4,-2\}$-hypomorphic.
3. It is sufficient to remove $\{0,1,3\}$, we obtain that $I_0 = \{h, h+1, h+2\}$ and $I_3 = \{h+3, h+4\}$ the two non-trivial intervals of $T_{2h+1} - \{0,1,3\}$. Then, $T - \{0,1,3\}$ and $T' - \{0,1,3\}$ are not isomorphic because $|I_0| \neq |I_3|$. Therefore T and T' are not $\{-3\}$-hypomorphic.

4. $T[0,1,2,3,4,3+h]$ and even $T'[0,1,2,3,4,3+h]$ are gotten from a 3-cycle, with one of its vertices is $3+h$, by dilating the other vertices by $\{0,1,2\}$ and $\{3,4\}$. Obviously, $T[0,1,2,3,4,3+h]$ and $T'[0,1,2,3,4,3+h]$ are not isomorphic. Therefore, T and T' are not (≤ 6)-hypomorphic.

□

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