On full friendly index sets of twisted product of Möbius ladders*

Guang-Yi Sun[1] Zhen-Bin Gao[2] † Sin-Min Lee[3]

[1,2] *College of Science, Harbin Engineering University, Harbin, 150001, People's Republic of China*

[3] *34803, Hollyhock Street, Union City, CA94587, USA*

Abstract. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, a vertex labeling $f : V(G) \to \mathbb{Z}_2$ induces an edge labeling $f^* : E(G) \to \mathbb{Z}_2$ defined by $f^*(x,y) = f(x) + f(y)$, for each edge $(x,y) \in E(G)$. For each $i \in \mathbb{Z}_2$, let $v_f(i) = |\{v \in V(G) : f(v) = i\}|$ and $e_f(i) = |\{e \in E(G) : f^*(e) = i\}|$. A vertex labeling f of a graph G is said to be friendly if $|v_f(1) - v_f(0)| \leq 1$. The friendly index set of the graph G, denoted by $FI(G)$, is defined as $\{|e_f(1) - e_f(0)|$: the vertex labeling f is friendly$\}$. The full friendly index set of the graph G, denoted by $FFI(G)$, is defined as $\{e_f(1) - e_f(0)$: the vertex labeling f is friendly$\}$. In this paper, we determine $FFI(G)$ and $FI(G)$ for a class of cubic graphs which are twisted product of Möbius.

Keywords: Vertex labeling, friendly labeling, cordiality, friendly index set, full friendly index set, cubic graph, Möbius ladders, twisted product.
AMS 2010 MSC: 05C78, 05C25

1. Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Let A be an abelian group. A labeling $f : V(G) \to A$ induces an edge labeling $f^* : E(G) \to A$ defined by $f^*(x,y) = f(x) + f(y)$, for each edge $(x,y) \in E(G)$. For $a \in A$, let $v_f(a) = |\{v \in V(G) : f(v) = a\}|$ and $e_f(a) = |\{e \in E(G) :$

*The work is supported by NSFC Grant No.11371109

†Corresponding author. E-mail: gaozhenbin@aliyun.com

$f^*(e) = a\}|$. G is A-cordial if there is a labeling $f : V(G) \to A$ such that for all $a, b \in A$, we have (1) $|v_f(a) - v_f(b)| \le 1$, (2) $|e_f(a) - e_f(b)| \le 1$. If $A = \mathbb{Z}_k$, we will say that G is k-cordial.

The notion of A-cordial labelings was first introduced by Hovey [11], who generalized the concept of cordial graphs of Cahit [2]. For more details of known results and open problems on cordial graphs, see [1-4, 7-9,11, 12, 14, 15] etc.

In this paper, we will exclusively focus on $A = \mathbb{Z}_2$, and drop the reference to the group. A vertex v is called a k-vertex if $f(v) = k$, $k \in \{0, 1\}$, an edge e is called a k-edge if $f^*(e) = k$, $k \in \{0, 1\}$.

In [6] the following concept was introduced.

Definition 1.1. *The friendly index set $FI(G)$ of a graph G is defined as $\{|e_f(1) - e_f(0)| :$ the vertex labeling f is friendly$\}$.*

Note that if 0 or 1 is in $FI(G)$, then G is cordial. Thus, the concept of friendly index sets could be viewed as a generalization of cordiality. Cairnie and Edwards [5] have determined the computational complexity of cordial labeling and k-cordial labeling. They proved that deciding whether a graph admits a cordial labeling is NP-complete. Even the restricted problem of deciding whether a connected graph of diameter 2 has a cordial labeling is NP-complete. Thus, in general, it is difficult to determine the friendly index sets of graphs.

The following result was established in [13]:

Theorem 1.1. *For any graph G with q edges, the friendly index set $FI(G) \subseteq \{2i : 0 \le i \le \frac{q}{2}\}$ if q is even, and $FI(G) \subseteq \{2i+1 : 0 \le i \le \frac{q-1}{2}\}$ if q is odd.*

In [14], it was shown that

Theorem 1.2. *The friendly indices of a cycle formed an arithmetic sequence:*

(i) $FI(C_{2n}) = \{4i : 0 \le i \le \frac{n}{2}\}$ *if n is even;*

$FI(C_{2n}) = \{2 + 4i : 0 \le i \le \frac{n-1}{2}\}$ *if n is odd.*

(ii) $FI(C_{2n+1}) = \{2i + 1 : 0 \le i \le n - 1\}$.

The numbers in $FI(C_n)$ for any cycle form an arithmetic progression. In [16], it is shown that for a cycle with an arbitrary non-empty set of parallel chords, the values in its friendly index set form an arithmetic progression with common difference 2. If the chords are not parallel, the numbers in the friendly index set might not form an arithmetic progression. See [17] on the friendly index sets of Möbius ladders. For more details of known results and open problems on friendly index sets, the reader can see relevant papers.

Shiu and Kwong [18] extended $FI(G)$ to $FFI(G)$.

Definition 1.2. *The full friendly index set $FFI(G)$ of a graph G is defined as $\{e_f(1) - e_f(0) :$ the vertex labeling f is friendly$\}$.*

Shiu and Kwong [18] determined $FFI(P_2 \times P_n)$. Shiu and Lee [19] determined the full friendly index sets of twisted cylinders. Shiu and Wong [20] determined the full friendly index sets of cylinder graphs. Shiu and Ho [21] determined the full friendly index sets of some permutation Petersen graphs, they also determined the full friendly index sets of slender and flat cylinder graphs [22]. Shiu and Ling [23] determined the full friendly index sets of Cartesian products of two cycles. Sinha and Kaur [24] studied the full friendly index sets of some graphs such as K_n, C_n, fans F_n, $F_{2,m}$ and $P_3 \times P_n$.

Let $(G,(x,y))$, $(H,(u,v))$ be a pair of cubic graphs with $(x,y) \in E(G)$ and $(u,v) \in E(H)$. The *twisted product* of $(G,(x,y))$, $(H,(u,v))$, denoted by $(G,(x,y))\natural(H,(u,v))$ is the graph with
$V((G,(x,y))\natural(H,(u,v))) = V(G) \cup V(H)$,
$E((G,(x,y))\natural(H,(u,v))) = (E(G)-(x,y)) \cup (E(H)-(u,v)) \cup \{(x,u),(y,v)\}$.

We have a new cubic graph $(G,(x,y))\natural(H,(u,v))$.

Notation: Let n be an integer greater than 1. The Möbius wheel is the cycle C_{2n}, with n additional edges joining diagonally opposite vertices. We will denote this graph by M_{2n}, the vertices by v_i, where $1 \le i \le 2n$. Then the edges are (v_i, v_{i+1}), where $1 \le i \le 2n-1$ and (v_{2n}, v_1) on the cycle, and the n diagonals are (v_i, v_{n+i}), where $1 \le i \le n$.

Let $V(M_{2m}) = \{u_i : 1 \le i \le 2m\}$ and $V(M_{2n}) = \{v_i : 1 \le i \le 2n\}$ for $m, n \ge 2$. Let $TM(2m,2n) = (M_{2m},(u_1,u_{2m}))\natural(M_{2n},(v_1,v_{2n}))$, which is illustrated by Figure 1. In $TM(2m,2n)$, we have $|V| = 2(m+n)$, $|E| = 3(m+n)$.

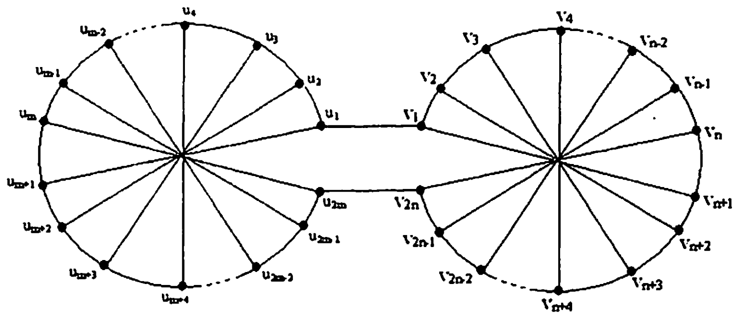

Figure.1. $TM(2m, 2n)$ graph

In this paper, we determine the full friendly index of $TM(2m, 2n)$.

2. Preliminaries

Now, we present some derived results and prove some results which will be used in the following discussions.

Lemma 2.1.([18]) *Let f be a labeling of a graph G that contains a cycle C as its subgraph. If C contains an 1-edge, then the number of 1-edges in C is a positive even number.*

An edge is called an (i, j)-edge if it is incident with an i-vertex and a j-vertex. The number of (i, j)-edges is denoted by $E_f(1, 1)$.

Lemma 2.2.([23]) *Let f be a labeling of a graph G with q edges. If the degree sum of 1-vertices is s, then $e_f(1) - e_f(0) = 2s - 4E_f(1, 1) - q$.*

Lemma 2.3.([23]) *Let f be a friendly labeling of G. If G is a regular graph of even order, then the degree sum of* 1*-vertices is equal to the size of G. Hence, $E_f(0, 0) = E_f(1, 1)$.*

Theorem 2.4. *In any friendly labeling f of $TM(2m, 2n)$, if any two vertices labels are exchanged, then $e_f(1)$ is increased by -6, -4, -2, 0, 2, 4, or 6.*

Proof Since the graph $TM(2m, 2n)$ is cubic, so, any vertex u is adjacent to three vertices u_1, u_2, and u_3.

In a friendly labeling of $TM(2m, 2n)$, suppose that the vertices u, u_1, u_2, and u_3 are labeled by x, x_1, x_2, and x_3 (x, x_1, x_2, $x_3 \in \{0, 1\}$) respectively. When we revise the label of u to $1 - x$, the number of 1-edges is increased by -3, -1, 0, 1, or 3. For any two vertices of u and v in $TM(2m, 2n)$, there will be three possibilities, which listed in Figure 2.

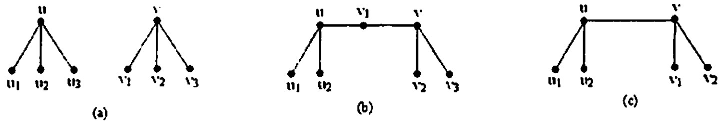

Figure.2.

Exchange the labels of u and v, $e_f(1)$ is increased by -6, -4, -2, 0, 2, 4, or 6 in (a), and -4, -2, 0, 2, or 4 in (b) and (c). Hence, if any two vertex labels are exchanged, then $e_f(1)$ is increased by -6, -4, -2, 0, 2, 4, or 6. $\square$

Theorem 2.5. *When m and n are odd, the maximum value of $e_f(1)$ is $3(m+n)$, where f is a friendly labeling of $TM(2m, 2n)$.*

Proof Note that $TM(2m, 2n)$ is bipartite with bipartition (X, Y), say, where $|X| = |Y| = m + n$. We label all vertices in X by 0 and in Y by 1. Then all edges are 1-edges under this labeling f. Hence $e_f(1) = 3(m+n)$. $\square$

Theorem 2.6. *When m and n are even, the upper bound of $e_f(1)$ is $3(m+n) - 2$, where f is a friendly labeling of $TM(2m, 2n)$.*

Proof When m, n are even, the cycles $u_1u_2 \cdots u_{m+1}u_1$ and $v_1v_2 \cdots v_{n+1}v_1$ are two edge-disjoint odd cycles. For any labeling f, $e_f(0) \geq 2$. Hence $e_f(1) \leq 3(m+n) - 2$. In the next section, we will show that this upper bound can be obtained. $\square$

Theorem 2.7. *When exactly one of m and n is odd, the upper bound of $e_f(1)$ is $3(m+n) - 2$, where f is a friendly labeling of $TM(2m, 2n)$.*

Proof Without loss of generality, we may assume m is odd and n is even. The cycle $v_1v_2 \cdots v_{n+1}v_1$ is an odd cycle. For any labeling f, there exists at least one 0-edge. Suppose $e_f(1) = 3(m+n) - 1$, then, $v_{n+2}, v_{n+3}, \cdots, v_{2n}, u_{2m}, u_{2m-1}, \ldots, u_1$ are labeled alternately by 0 and 1, thus, in v_{n+i}, where $2 \leq i \leq n$, exists one vertex (suppose v_{n+j}) such that $f(v_{n+j}) = f(v_j)$, $e_f(0) \geq 2$, contradiction. Hence, when exactly one of m and n is odd, the upper bound of $e_f(1)$ is $3(m+n) - 2$. In the next section, we will show that this upper bound can be obtained. $\square$

Theorem 2.8. *Suppose f is a friendly labeling of $TM(2m, 2n)$. The* *maximum value of* $e_f(0)$ *equals*

(1). $3(m+n)-2$ *for* $|m-n|=0$;

(2). $3(m+n)-3$ *for* $|m-n|=1$.

Proof Let f be a friendly labeling of $TM(2m,2n)$. Then $v_f(1)=v_f(0)=m+n>0$. Since $TM(2m,2n)$ is a connected, there exist adjacent vertices x and y such that $f(x)=1$ and $f(y)=0$. Thus $e_f(1)\geq 1$. Since $C=u_1u_2\cdots u_{2m}v_{2n}v_{2n-1}\cdots v_1u_1$ is a Hamiltonian cycle of $TM(2m,2n)$, by Lemma 2.1, $e_f(1)$ is a positive even number. So $e_f(1)\geq 2$ and hence $e_f(0)\leq 3(m+n)-2$.

(1). $m=n$.

Suppose $e_f(1)=2$ for some friendly labeling f of $TM(2m,2n)$. By the argument above, we know that the two 1-edges must lie in C. We claim that $f^*(u_1,v_1)=1=f^*(u_{2m},u_{2n})$.

Suppose not, without loss of generality, we may assume $f^*(v_i,v_{i+1})=1$ for some i, $1\leq i\leq 2n-1$. By symmetry, we may assume that $1\leq i\leq n$. By Lemma 2.1 and $e_f(1)=2$, the cycle $v_iv_{i+1}v_{n+i+1}v_{n+i}v_i$ contains exactly two 1-edges and other edges outside this cycle are 0-edges. So $f^*(v_{n+i},v_{n+i+1})=1$. On the other hand, the cycle $v_iv_{i+1}\cdots v_{n+i}v_i$ also contains two 1-edges. It is a contradiction.

So we conclude that if $e_f(1)=2$ for some friendly labeling f of $TM(2m,2n)$, then $f^*(u_1,v_1)=1=f^*(u_{2m},u_{2n})$. Since all other edges are 0-edges and f is friendly, $m=n$. From the proof above, it is clear that $e_f(0)=3(m+n)-2$ is attainable when $m=n$. So we have (1).

(2). $|m-n|=1$.

By Lemma 2.3, we know that $e_f(0)$ is even for any friendly labeling f. We have known that $e_f(1)\geq 2$. Since $m+n$ is odd, the number of edges $q=3(m+n)$ is odd, so, $e_f(0)\leq 3(m+n)-3$. The upper bound is attainable by labeling u_1 and v_i for $1\leq i\leq 2n$ by 0, and u_j for $2\leq j\leq 2m$ by 1. Hence we have (2). □

We shall denote the graph labeling f of $TM(2m,2n)$ by $G(a)$ in which $e_f(1)-e_f(0)=a$.

3. The full friendly index sets of $TM(2m,2n)$

In this section, we show how the values in $FFI(TM(2m,2n))$ can be obtained by exhibiting the respective of the vertices with 0 and 1.

Theorem 3.1. *When m and n are odd with $|m-n|=k$. $FFI(TM(2m,2n))$ equals*

(1). $\{-3(m+n)+4i : 1 \le i \le \frac{3(m+n)}{2}\}$ *for* $k=0$;

(2). $\{-3(m+n)+4i : 2 \le i \le \frac{3(m+n)}{2}\}$ *for* $k>0$ *and* $m+n \equiv 0,2 \pmod 4$.

Proof From Theorem 2.5, we have $G(3(m+n))$.

Define the labeling f_1 by assigning u_i, where $1 \le i \le 2m$ and v_j where $1 \le j \le 2n$ both alternately by 0 and 1. Then $v_{f_1}(1) = v_{f_1}(0)$ and all edges are 1-edges except the edges (u_1, v_1) and (u_{2m}, v_{2n}). Thus, the number of 1-edges is $3(m+n)-2$. Hence, we can obtain $G(3(m+n)-4)$.

Step 1. In $G(3(m+n))$, we exchange the labels of u_2 and u_{m+2} so that the number of 1-edges is decreased by 4. Let this new labeling be g_1. Then $e_{g_1}(1) = 3(m+n)-4$ and $e_{g_1}(0) = 4$. Successively exchange the labels of u_{2j} and u_{m+2j}, where $2 \le j \le \frac{m-1}{2}$. Each of these exchanges so that the number of 1-edges is decreased by 4 and the number of 0-edges is increased by 4. Once all the exchanges are completed, we obtain $G(3(m+n)-8j)$, where $1 \le j \le \frac{m-1}{2}$.

Step 2. In $G(3n-m+4)$, we exchange the labels of v_2 and v_{n+2} so that the number of 1-edges is decreased by 4. Let this new labeling be h_1. Then $e_{h_1}(1) = 3n+m-2$, and $e_{h_1}(0) = 2m+2$. Successively exchange the labels of v_{2j} and v_{n+2j}, where $2 \le j \le \frac{n-1}{2}$. Each of these exchanges so that the number of 1-edges is decreased by 4 and the number of 0-edges is increased by 4. Once all the exchange are completed, we obtain $G(3n-m+4-8j)$, where $1 \le j \le \frac{n-1}{2}$.

Step 3. In $G(3(m+n)-4)$, use the procedure in the proof of Steps 1 and 2 to obtain $G(3(m+n)-4-8j)$, where $1 \le j \le \frac{m+n}{2}-1$.

By Steps 1, 2 and 3, we can conclude that $\{-3(m+n)+4i : 1+\frac{(m+n)}{2} \le i \le \frac{3(m+n)}{2})\} \subseteq FFI(TM(2m,2n))$.

Case 1. $m=n$.

From Theorem 2.8, we have $G(-3(m+n)+4)$.

In $G(-3(m+n)+4)$, we exchange the labels of u_{2m} and v_1 so that four 0-edges change to 1-edges, and two 1-edges change to 0-edges. Let this new labeling be g_2. Then $e_{g_2}(1) = 4$ and $e_{g_2}(0) = 3(m+n) - 4$. Successively exchange the labels of u_{2m-j} and v_{j+1}, where $1 \leq j \leq m-2$. Each of these exchanges so that the number of 1-edges is increased by 2 and the number of 0-edges is decreased by 2. Once all the exchange are completed, we obtain $G(-3(m+n)+4j)$, where $2 \leq j \leq \frac{m+n}{2}$. Hence, $\{-3(m+n)+4i : 1 \leq i \leq \frac{m+n}{2}\} \subseteq FFI(TM(2m,2n))$.

Case 2. $|m-n| = k \geq 2$.

Without loss of generality, suppose $m - n = k$. Because m, n are odd, so, k is even. For convenience, we let $k = 2r$ for some $r \geq 1$.

Because $q = 3(m+n)$ is even, by Lemma 2.2 and 2.3, we know that $e_f(1)$ is even for any friendly labeling f. Hence, $e_f(1) \geq 4$.

Define the labeling f_3 by assigning u_i and u_{m+j}, where $1 \leq j \leq m-r$ by 0 and the other vertices by 1. Then all edges are labeled by 0 except the edges (u_1, v_1), (u_{m-r}, u_{m-r+1}), (u_m, u_{m+1}) and (u_{2m-r}, u_{2m-r+1}). Hence, $e_{f_3}(1) = 4$ and $e_{f_3}(0) = 3(m+n) - 4$. We obtain $G(-3(m+n)+8)$.

In $G(-3(m+n)+8)$, we exchange the labels of u_{j+1} and $u_{2m-r+1+j}$ where $0 \leq j \leq r-1$. Each of these exchanges so that the number of 1-edges is increased by 2 and the number of 0-edges is decreased by 2. Once all the exchange are completed, we obtain $G(-3(m+n)+12+4j)$, where $0 \leq j \leq r-1$.

In $G(-3(m+n)+8+4r)$, successively exchange the labels of u_{j+r} and v_{2n-j+1}, where $1 \leq j \leq n-2$. Each of these exchanges so that the number of 1-edges is increased by 2 and the number of 0-edges is decreased by 2. Once all the exchange are completed, we obtain $G(-3(m+n)+8+4r+4j)$, where $1 \leq j \leq n-2$. Since $m - n = k = 2r$, $G(-3(m+n)+4r+4n) = G(-m-n)$.

Combining with the result obtained after Step 3, we have $\{-3(m+n)+4i\colon 2 \leq i \leq \frac{3(m+n)}{2}\} \subseteq FFI(TM(2m,2n))$.

For both two cases, by Lemma 2.2, Theorems 2.5 and 2.8, we have the result. This completes the proof. □

Theorem 3.2. *When m and n are even with $|m-n| = k$. $FFI(TM(2m,2n))$ equals*

(1). $\{-3(m+n)+4i\colon 1 \leq i \leq \frac{3(m+n)}{2} - 1\}$ *for* $k = 0$;

(2). $\{-3(m+n)+4i\colon 2 \leq i \leq \frac{3(m+n)}{2} - 1\}$ *for* $k > 0$ *and* $m + n \equiv 0, 2$

(mod 4).

Proof Define the labeling f_1 by assigning u_1, u_2, $\cdots$, u_m, v_{n+1}, v_{n+2}, ..., v_{2n} alternately by 0 and 1; u_{m+1}, u_{m+2}, ..., u_{2m}, v_1, v_2, $\cdots$, v_n alternately by 1 and 0. Then $v_{f_1}(1) = v_{f_1}(0)$, all edge are 1-edges except edges (u_m, u_{m+1}) and (v_n, v_{n+1}). Hence, $e_{f_1}(1) = 3(m+n)-2$, $e_{f_1}(0) = 2$. so, we obtain $G(3(m+n)-4)$.

Define the labeling f_2 by assigning u_1, u_2, ..., u_m, and v_1, v_2, ..., v_n both alternately by 0 and 1; u_{m+1}, u_{m+2}, $\cdots$, u_{2m}, and v_{n+1}, v_{n+2}, ..., v_{2n} both alternately by 1 and 0. Then $v_{f_2}(1) = v_{f_2}(0)$, all edges are 1-edges except the edges (u_m, u_{m+1}), (v_n, v_{n+1}), (u_1, v_1) and (u_{2m}, v_{2n}), so, $e_{f_2}(1) = 3(m+n)-4$,$e_{f_2}(0) = 4$. Hence, we obtain $G(3(m+n)-8)$.

Step 1. In $G(3(m+n)-4)$, we exchange the labels of u_{2j} and u_{m+2j} where $1 \leq j \leq \frac{m-2}{2}$ successively. After the manner of the argument of the Step 1 of Theorem 3.1, once all the exchanges are completed, we obtain $G(3(m+n)-4-8j)$, where $1 \leq j \leq \frac{m-2}{2}$.

Step 2. In $G(3n-m+4)$, we exchange the labels at v_{2j} and v_{n+2j}, where $1 \leq i \leq \frac{n-2}{2}$ successively. After the manner of the argument of the Step 1 of Theorem 3.1, once all the exchanges are completed, we obtain $G(3n - m+4-8j)$, where $1 \leq j \leq \frac{n-2}{2}$.

Step 3. In $G(3(m+n)-8)$, use the procedure in the proof of Steps 1 and 2 to obtain $G(3(m+n)-8-8j)$, where $1 \leq j \leq \frac{(m+n)}{2} - 2$.

By Steps 1, 2 and 3, we can conclude that $\{-3(m+n)+4+4i : 1 + \frac{(m+n)}{2} \leq i \leq \frac{3(m+n)}{2} - 2)\} \subseteq FFI(TM(2m, 2n))$.

Case 1. $m = n$.

From Theorem 2.6, we have $G(-3(m+n)+4)$.

In $G(-3(m+n)+4)$, we exchange the labels of u_{2m} and v_1 so that four 0-edges change to 1-edges and two 1-edges change to 0-edges. Let this new labeling be g_2. Then $e_{g_2}(1) = 4$ and $e_{g_2}(0) = 3(m+n)-4$. Successively exchange the labels of u_{2m-j} and v_{j+1}, where $1 \leq j \leq m-1$. Each of these exchanges so that the number of 1-edges is increased by 2 and the number of 0-edges is decreased by 2. Once all the exchanges are completed, we obtain $G(-3(m+n)+4+4j)$, where $1 \leq j \leq \frac{(m+n)}{2}$. Hence, $\{-3(m+n)+4i$: $1 \leq i \leq \frac{(m+n)}{2} + 1\} \subseteq FFI(TM(2m, 2n))$.

Case 2. $|m-n| = k > 0$.

Without loss of generality, suppose $m-n=k$. For convenience, we let $k=2r$ for some $r \geq 1$.

After the manner of the argument of the case 2 of Theorem 3.1, we know that $e_f(1) \geq 4$ for any friendly labeling f.

Define the labeling f_3 by assigning u_j and u_{m+j} where $1 \leq j \leq m-r$ by 0 and the other vertices by 1. Then all edges are labeled by 0 except the edges (u_1, v_1), (u_{m-r}, u_{m-r+1}), (u_m, u_{m+1}) and (u_{2m-r}, u_{2m-r+1}). Hence, $e_{f_4}(1)=4$, $e_{f_4}(0)=3(m+n)-4$. We obtain $G(-3(m+n)+8)$.

In $G(-3(m+n)+8)$, we exchange the labels of u_{j+1} and $u_{2m-r+1+j}$, where $0 \leq j \leq r-1$. Each of these exchanges so that the number of 1-edges is increased by 2 and the number of 0-edges is decreased by 2. Once all these exchanges are completed, we obtain $G(-3(m+n)+12+4j)$, where $0 \leq j \leq r-1$.

In $G(-3(m+n)+8+4r)$, successively exchange the labels of u_{r+j} and v_{2n-j+1}, where $1 \leq j \leq n-1$. Each of these exchanges so that the number of 1-edges is increased by 2 and the number of 0-edges is decreased by 2. Once all the exchanges are completed, we obtain $G(-3(m+n)+8+4r+4j)$, where $1 \leq j \leq n-2$. Since $m-n=k=2r$, $G(-3(m+n)+4+4r+4n)$ $= G(-m-n+4)$.

Combining with the result obtain after Step 3, we have $\{-3(m+n)+4i\colon 2 \leq i \leq \frac{3(m+n)}{2}-1\} \subseteq FFI(TM(2m, 2n))$.

For both two cases, by Lemma 2.2, Theorems 2.6 and 2.8, we have the result. This completes the proof. □

Theorem 3.3. *When exactly one of m and n is odd with $|m-n|=k$. $FFI(TM(2m,2n))$ equals*

(1). $\{-3(m+n)+2+4i\colon 1 \leq i \leq \frac{3(m+n-1)}{2}\}$ *for* $|m-n|=1$;

(2). $\{-3(m+n)+6+4i\colon 1 \leq i \leq \frac{3(m+n)-5}{2}\}$ *for* $|m-n|=k>1$.

Proof Without loss of generality, we may assume m is odd and n is even.

Define the labeling f_1 by assigning u_1, u_2, $\ldots$, u_{2m}, v_{2n}, v_{2n-1}, $\ldots$, v_{n+1}, and v_1, v_2, $\ldots$, v_n are alternately by 0 and 1. Then all edge are labeled by 1 except the edges (u_1, v_1) and (v_n, v_{n+1}). Hence, $e_{f_1}(1) = 3(m+n)-2$, $e_{f_1}(0)=2$. So, we have $G(3(m+n)-4)$.

In $G(3(m+n)-4)$, we exchange the labels at v_n and v_{2n} so that the labels of the edges (u_{2m}, v_{2n}), (v_n, v_{n-1}) and (v_{2n-1}, v_n) are changed by 0

and the label of (v_n, v_{n+1}) is changed by 1. Let this new labeling be f_2. Then $e_{f_2}(1) = 3(m+n) - 4$, $e_{f_2}(0) = 4$. Hence, we get $G(3(m+n)-8)$.
Step 1. In $G(3(m+n)-4)$, we exchange the labels of u_2 and u_{m+2} so that four 1-edges change to 0-edges. Let this new labeling be g_1. Then $e_{g_1}(1) = 3(m+n) - 6$ and $e_{g_1}(0) = 6$. Successively exchange the labels of u_{2j} and u_{m+2j}, where $2 \le j \le \frac{m-1}{2}$. Each of these exchanges so that the number of 1-edges is decreased by 4 and the number of 0-edges is increased by 4. Once all these exchanges are completed, we obtain $G(3(m+n)-4-8j)$, where $1 \le j \le \frac{m-1}{2}$.
Step 2. In $G(3n-m)$, successively exchange the labels of v_{2j} and v_{n+2j}, where $1 \le j \le \frac{n-2}{2}$. Each of these exchanges so that the number of 1-edges is decreased by 4 and the number of 0-edges is increased by 4. Once all these exchanges are completed, we obtain $G(3n-m-8j)$, where $1 \le j \le \frac{n-2}{2}$.
Step 3. In $G(3(m+n)-8)$, use the procedure in the proof of Steps 1 and 2 to obtain $G(3(m+n)-8j)$, where $2 \le j \le \frac{3(m+n)-1}{2}$.

By Steps 1, 2 and 3, we can conclude that $\{-3(m+n)+2+4i : \frac{(m+n)+1}{2} \le i \le \frac{3(m+n-1)}{2})\} \subseteq FFI(TM(2m, 2n))$.

Suppose $m - n = k > 0$. For convenience, we let $k = 2r - 1$ for some $r \ge 1$.

Case 1. $r = 1$.

From Theorem 2.8, we have $G(-3(m+n)+6)$.

In $G(-3(m+n)+6)$, successively exchange the labels of u_{2m-j} and v_j, where $1 \le j \le n-1$. Each of these exchanges so that the number of 1-edges is increased by 2 and the number of 0-edges is decreased by 2. Once all the exchanges are completed, we obtain $G(-3(m+n)+6+4j)$, where $1 \le j \le n-1$. Since $m - n = 1$, $G(-3m+n+2) = G(-m-n)$. Hence, $\{-3(m+n)+6+4i\text{: } 0 \le i \le \frac{m+n-3}{2}\} \subseteq FFI(TM(2m, 2n))$.

Case 2. $r > 1$.

After the manner of the argument of the case 2 of Theorem 3.1, we know that $e_f(1) \ge 5$ for any friendly labeling f.

Define the labeling f_3 by assigning $u_j, u_2, \ldots, u_{m-r+1}, u_{m+1}, u_{m+2}, \ldots,$ u_{2m-r} by 0 and the other vertices by 1. Then all edges are labeled by 0 except the edges (u_1, v_1), (u_{m-r+1}, u_{m-r+2}), (u_{m-r+1}, u_{2m-r+1}), (u_m, u_{m+1}) and (u_{2m-r}, u_{2m-r+1}). Hence, we obtain $G(-3(m+n)+10)$.

Subcase 2.1. $r = 2$.

In $G(-3(m+n)+10)$, we exchange the vertex labels of u_{2m-2} and v_1 so that four 0-edges change to four 1-edges and two 1-edges change to two 0-edges, thereby the number of 1-edges is increased by 2 and the number of 0-edges is decreased by 2. Successively exchange the labels of u_{2m-2-j} and v_{j+1}, where $1 \leq j \leq n-2$. Each of these exchanges so that the number of 1-edges is increased by 2 and the number of 0-edges is decreased by 2. Once all the exchanges are completed, we obtain $G(-3(m+n)+10+4j)$, where $1 \leq j \leq \frac{m+n-5}{2}$.

Subcase 2.2. $r > 2$.

In $G(-3(m+n)+10)$, successively exchange the labels of $u_{m-r+2+i}$ and u_{2m-r-i}, where $0 \leq j \leq r-3$. Each of these exchanges so that the number of 1-edges is increased by 2 and the number of 0-edges is decreased by 2. Once all the exchanges are completed, we obtain $G(-3(m+n)+10+4j)$, where $1 \leq j \leq r-2$.

In $G(-3(m+n)+2+4r)$, successively exchange the labels of $u_{2m-k+1-j}$ and v_{j+1}, where $1 \leq j \leq n-2$. Each of these exchanges so that the number of 1-edges is increased by 2 and the number of 0-edges is decreased by 2. Once all the exchanges are completed, we obtain $G(-3(m+n)+2+4r+4j)$, where $1 \leq j \leq n-2$.

Since $2r-1 = k = m-n$, so, $G(-3(m+n)-2+4r+4n) = G(-m-n)$. Hence, $\{-3(m+n)+6+4i\colon 1 \leq i \leq \frac{m+n-3}{2}\} \subseteq FFI(TM(2m,2n))$.

For $m < n$, after the manner of the above discussions, we have the same results.

By Lemma 2.2, Theorems 2.7 and 2.8 and the argument above, we have the result. This completes the proof. $\square$

Base on the above results, we have

Corollary 3.4. *When m, n are odd,*

1. $m+n \equiv 0 \pmod 4$, *$TM(2m,2n)$ is cordial, and* $FI(TM(2m,2n)) = \{4i\colon 0 \leq i \leq \frac{3(m+n)}{4}\}$;
2. $m+n \equiv 2 \pmod 4$, *$TM(2m,2n)$ is not cordial, and* $FI(TM(2m,2n)) = \{2+4i\colon 0 \leq i \leq \frac{3(m+n)-2}{4}\}$.

Corollary 3.5. *When m, n are even,*

1. $m+n \equiv 0 \pmod 4$, $TM(2m,2n)$ *is cordial, and* $FI(TM(2m,2n)) = \{4i\colon 0 \le i \le \frac{3(m+n)}{4} - 1\}$;
2. $m+n \equiv 2 \pmod 4$, $TM(2m,2n)$ *is not cordial, and* $FI(TM(2m,2n)) = \{2+4i\colon 0 \le i \le \frac{3(m+n)-6}{4}\}$.

Corollary 3.6. *When exactly one of* m *and* n *is odd with* $|m-n| = k$. $(M_{2m}, e)\between(M_{2n}, f)$ *is cordial,*
1. $|m-n| = 1$, $FI(TM(2m,2n)) = \{2i-1\colon 1 \le i \le \frac{3(m+n-1)}{2}\}$;
2. $|m-n| = k > 1$, $FI(TM(2m,2n)) = \{2i-1\colon 1 \le i \le \frac{3(m+n)-7}{2}\} \cup \{3(m+n)-4\}$.

Acknowledgement We sincerely appreciate Prof. H.K. Ng and Prof. Harris Kwong carefully proofreading the first draft of the paper and making valuable suggestions—these efforts reduce the errors and simplifying the proof.

The authors would like to thank the anonymous referees very much for the valuable suggestions, corrections and comments, which result in a great improvement of the original manuscript.

References

[1] M. Benson and S-M. Lee. On cordialness of regular windmill graphs, Congr. Numer., 68 (1989) 49-58.

[2] I. Cahit. Cordial graphs: a weaker version of graceful and harmonious graphs, Ars Combin., 23 (1987) 201-207.

[3] I. Cahit. On cordial and 3-equitable graphs, Utilitas Mathematica, 37 (1990) 189-198.

[4] I. Cahit. Recent results and open problems on cordial graphs, Contemporary Methods in Graph Theory, 209-230, Bibligraphisches Inst., Mannheim, 1990.

[5] N. Cairnie and K. Edwards. The computational complexity of cordial and equitable labelling, Discrete Math., 216 (2000) 29-34.

[6] G. Chartrand. S-M. Lee and P. Zhang. Uniformly cordial graphs, Discrete Math. 306 (2006) 726-737.

[7] A. Elumalai. On graceful, cordial and elegant labelings of cycle related and other graphs, Ph.D. dissertation of Anna University, 2004, Chennai, India.

[8] Y.S. Ho, S-M. Lee and S.S. Shee. Cordial labellings of the Cartesian

product and composition of graphs, Ars Combin., 29 (1990) 169-180.
[9] Y.S. Ho, S-M. Lee and S.S. Shee. Cordial labellings of unicyclic graphs and generalized Petersen graphs. Congr. Numer., 68 (1989) 109-122.
[10] Y.S. Ho. S-M. Lee and H.K. Ng. On friendly index sets of root-unions of stars by cycles, Journal of Combinatorial Mathematics and Combinatorial Computing. 62 (2007) 97-120.
[11] M. Hovey. A-cordial graphs, Discrete Math., 93 (1991) 183-194.
[12] S. Kuo, G.J. Chang and H. Kwong. Cordial labeling of mK_n, Discrete Math., 169 (1997) 1-3.
[13] H. Kwong and S-M. Lee. On friendly index sets of generalized books, J. Combin. Math. Combin. Comput. 66 (2008) 43-58.
[14] H. Kwong, S-M. Lee and H.K. Ng. On friendly index sets of 2-regular graphs, Discrete Mathematics. 308 (2008) 5522-5532.
[15] H. Kwong, S-M. Lee and H.K. Ng. On product-cordial index sets and friendly index sets of 2-regular graphs and generalized wheels, Acta Mathematica Sinica, English Series, 28 (2012) 661-674.
[16] S-M. Lee and H.K. Ng. On friendly index sets of cycles with parallel chords, Ars Combinatoria 97A (2010) 253-267.
[17] S-M. Lee and H.K. Ng. On friendly index sets of Prisms and Möbius ladder, Journal of Combinatorial Mathematics and Combinatorial Computing, 90 (2014) 59-74.
[18] W.C. Shiu and H. Kwong. Full friendly index sets of $P_2 \times P_n$, Discrete Math., 308 (2008) 3688-3693.
[19] W.C. Shiu and S-M. Lee. Full friendly index sets and full product-cordial index sets of twisted cylinders, J. Combin. Number Theory, 3(3) (2012) 209-216.
[20] W.C. Shiu and F.S. Wong. Full friendly index sets of cylinder graphs, Australasian Journal of Combinatorics, 52 (2012) 141-162.
[21] W.C. Shiu and M.-H. Ho. Full friendly index sets and full product-cordial index sets of some permutation Petersen graphs, J. Combin. Number Theory, 5(3) (2013) 227-244.
[22] W.C. Shiu and M.-H. Ho. Full friendly index sets of slender and flat cylinder graphs, Transactions on Combinatorics, 2(4) (2013) 63-80.
[23] W.C. Shiu and M.-H. Ling. Full friendly index sets of Cartesian prod-

ucts of two cycles, Acta Math. Sinica, 26 (2010) 1233-1244.
[24] D. Sinha and J. Kaur. Full friendly index set–I, Discrete Applied Mathematics, 161 (2013) 1262-1274.