Turán-type problem for a cycle of length 6 in bipartite Eulerian digraph

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Abstract

We prove the following Turán-Type result: If there are more than $9mn/16$ edges in a simple and bipartite Eulerian digraph with vertex partition size m and n, then the graph contains a directed cycle of length 4 or 6. By using this result, we improve an upper bound for the diameter of interchange graphs.
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1 Introduction

Interchange graphs are defined by Brualdi [1] as follows: Let $R = (r_1, r_2, \cdots, r_m)$ and $S = (s_1, \cdots, s_n)$ be non-negative integer vectors. Let $A(R, S)$ be a set of all $m \times n$ matrices $A = (a_{ij})$ satisfying

$$a_{ij} = 0 \text{ or } 1 \text{ for } i = 1, \cdots, m \text{ and } j = 1, \cdots, n;$$

$$\sum_{j=1}^{n} a_{ij} = r_i \text{ for } i = 1, \cdots, m;$$

and

$$\sum_{i=1}^{m} a_{ij} = s_j \text{ for } j = 1, \cdots n.$$

We define the interchange in $A(R, S)$ as the transformation which replaces the $2\times$ 2 submatrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ with $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or vise versa. Obviously, an interchange does not change the row and column sum vectors of a matrix, so an interchange transforms a matrix of $A(R, S)$ into another matrix of

$A(R,S)$. If the matrix $A \in A(R,S)$ can be obtained by exactly one interchange from $B \in A(R,S)$, we define an edge between A and B. Let E be a set of all edges defined like above and let V be $A(R,S)$. The graph $G(R,S) = (V,E)$ is called interchange graph. Ryser [2] and Gale [3] independently proved all of the interchange graphs are connected.

For $A, B \in A(R,S)$, let $i(A,B)$ be the distance from A to B in $G(R,S)$, that is, $i(A,B)$ is the minimum number of interchanges we need to transform A into B.

The diameter of the interchange graph is defined as $D(R,S) := \max\{ i(A,B) \,|A, B \in G(R,S)\}$. The following conjecture is still an open problem.

Conjecture 1 (Brualdi, 1980, [1])

$$D(R,S) \leq \frac{mn}{4} = 0.25mn.$$

Where m and n are the sizes of the matrix we consider.

Shen and Yuster [4] proved $D(R,S) \leq 5/12mn \approx 0.4167mn$. Their method is to use a distance formula, the explanation of which requires several notations. Let $d(A,B)$ be the number of different entries of A and B. Let Γ be the bipartite digraph defined by $A - B = (c_{ij})$ as follows: A partition sets of Γ are defined as $X = \{x_1, \cdots, x_m\}$, $Y = \{y_1, \cdots, y_n\}$ where m, n are row and column sizes of matrices A and B. Directed edges of Γ, (x_i, y_j) and (y_j, x_i) are defined according to $c_{ij} = 1$ or -1 respectively (namely, Γ is a digraph whose adjacency matrix is $A - B$). A digraph $G = (V,E)$ is called Eulerian if for each vertex of V, the indegree of the vertex is equal to its outdegree. From the definition, Γ is a simple and Eulerian graph. Let

$$q(A,B) = \max\{n | \Gamma \text{ can be divided into } n \text{ parts of edge-disjoint directed cycles}\}.$$

Brualdi's distance formula is the following:

Theorem 1 (Brualdi, 1980, [1]) We have

$$i(A,B) = \frac{d(A,B)}{2} - q(A,B).$$

From the formula, we know the following Turán type question is worth considering.

Question For a simple and bipartite Eulerian digraph, how many edges do we need to guarantee the existence of small directed cycle in the graph, regardless of the edge arrangement of the graphs?

Let α be the edge density of Γ, that is, $\alpha = \#E/mn$ where m, n are sizes of partition sets of Γ.

Lemma 1 (Shen and Yuster, 2002, [4])

$$\text{If } \alpha > \frac{2}{3} \approx 0.66, \text{then } \Gamma \text{ has a cycle of length 4.}$$

By using Lemma 1, they proved $D(R, S) \leq 5mn/12 \approx 0.4167mn$. Similar to their method, we get the following lemma, which will be proved in the next section.

Lemma 2 If $\alpha > \frac{9}{16} \approx 0.56$, then Γ has a cycle of length 4 or 6 .

By Lemmas 1 and 2, we can get the following upper bound for the interchange graphs.

Theorem 2 We have the inequality

$$D(R, S) \leq \frac{115}{288}mn \approx 0.3994mn.$$

Proof of Theorem 2. Let A, B be elements in $A(R, S)$ and Γ an bipartite digraph whose adjacency matrix is $A - B$. From Lemma 1, Γ has at least $\left\lceil \frac{d(A,B) - 2mn/3}{4} \right\rceil = \left\lceil \frac{d}{4} - \frac{mn}{6} \right\rceil$ arc-disjoint cycles of length 4. Let Γ' be a subgraph obtained from Γ by deleting edges of the cycles. Put $E = \left\lceil \frac{d}{4} - \frac{mn}{6} \right\rceil$. From Lemma 2, there are at least $\left\lceil \frac{d(A,B) - E \cdot 4 - \frac{9}{16}mn}{6} \right\rceil$ cycles of length 4 or 6 in Γ'. So, we have

$$\begin{aligned}
q(A, B) &\geq E + \left\lceil \frac{d(A,B) - E \cdot 4 - \frac{9}{16}mn}{6} \right\rceil \\
&\geq E + \frac{d(A,B)}{6} - \frac{2}{3}E - \frac{3}{32}mn \\
&= \frac{1}{3}\left(\frac{d(A,B)}{4} - \frac{mn}{6}\right) + \frac{d(A,B)}{6} - \frac{3}{32}mn \\
&= \frac{d(A,B)}{4} - \frac{43}{288}mn.
\end{aligned}$$

This together with Theorem 1 gives

$$\begin{aligned} i(A,B) &\le \frac{d(A,B)}{2} - \frac{d(A,B)}{4} + \frac{43}{288}mn \\ &= \frac{d(A,B)}{4} + \frac{43}{288}mn \\ &\le \frac{mn}{4} + \frac{43}{288}mn = \frac{115}{288}mn. \end{aligned}$$

Hence we proved Theorem 2. □

2 Proof of Lemma 2

Proof of Lemma 2. Let $\Gamma = (V, E)$ be a simple and bipartite Eulerian digraph which does not contain any cycle of length 4 or 6. Let X and Y be partition sets of V and d_v^+ out degree of $v \in V$. We can assume all of the outdegrees of the vertices are positive. For each $v \in V$, we define ρ_v as

$$\rho_v := \begin{cases} \dfrac{d_v^+}{n} & (v \in X), \\ \dfrac{d_v^+}{m} & (v \in Y). \end{cases}$$

Let $\rho = \max\{\rho_v | v \in V\}$ and we choose a v^* such that $\rho = \rho_{v^*}$. Without loss of generality, we can assume $v^* \in X$. And let $Y^+, Y^-, X^+, X^-, Y^{++}, Y^{--}$ be the subsets of V such that:

$$\begin{aligned} Y^+ &= \{y \in Y | (v^*, y) \in E\}, \\ Y^- &= \{y \in Y | (y, v^*) \in E\}, \\ X^+ &= \{x \in X | \exists y \in Y^+, (y, x) \in E\}, \\ X^- &= \{x \in X | \exists y \in Y^-, (x, y) \in E\} \cup \{v^*\}, \\ Y^{++} &= \{y \in Y \setminus (Y^+ \cup Y^-) | \exists x \in X, (y, x) \in E, (x, z) \in E \Rightarrow z \notin Y^-\}, \\ Y^{--} &= \{y \in Y \setminus (Y^+ \cup Y^-) | \exists x \in X, (x, y) \in E, (z, x) \in E \Rightarrow z \notin Y^+\}. \end{aligned}$$

Let $\alpha^- := \#X^-/m$, $\alpha^+ := \#X^+/m$, $\beta := \#Y^{--}/n$, $\beta' := \#Y^{++}/n$ and order the vertices of each sets such that $Y^+ = \{y_1, y_2, \cdots, y_{\rho n}\}$, $Y^- = \{y_{\rho n+1}, \cdots, y_{2\rho n}\}$, $X^- = \{x_1 = v^*, x_2, \cdots, x_{\alpha - m}\}$, $X^+ = \{x'_1, x'_2, \cdots, x'_{\alpha + m}\}$, $Y^{--} = \{y'_1, y'_2, \cdots, y'_{\beta n}\}$, $Y^{++} = \{y''_1, \cdots, y''_{\beta' n}\}$. $X \setminus (X^- \cup X^+) = \{x_{\alpha - m+1}, x_{\alpha - m+2}, \cdots, x_{n-\alpha+m}\}$. And we define $x_{m-\alpha-m+i} = x'_i$, $y_{2\rho n+j} = y'_j$, $y_{2\rho n+\beta n+j} = y''_j$. The set Y must satisfy $Y = Y^+ \cup Y^- \cup Y^{++} \cup Y^{--}$(if there exists $y \in Y \setminus \{Y^+ \cup Y^- \cup Y^{++} \cup Y^{--}\}$ then there is a cycle of length 6).

We set adjacency matrix M of Γ with m rows and n columns as fallows. Let $M = (a_{ij})$, and for each $x_i \in X$, and $y_j \in Y$,

$$a_{ij} = \begin{cases} 1 & (x_i, y_j) \in E, \\ -1 & (y_j, x_i) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

We can divide M into twelve sub-matrices $M_1, \cdots, M_{12}$ corresponding to subsets of vertex sets (Figure 1). Denote by M_1 the sub-matrix consisting of the rows $i = 1, \cdots, \alpha^- m$ and columns $j = 1, \cdots, \rho n$. Denote by M_2 the sub matrix consisting of rows $i = \alpha^- m + 1, \cdots, m - \alpha^+ m$ and columns $j = 1, \cdots, \rho n$. The matrices $M_3, \cdots, M_{12}$ are also defined similarly so we omit to state their definitions. Because the graph does not contain a cycle

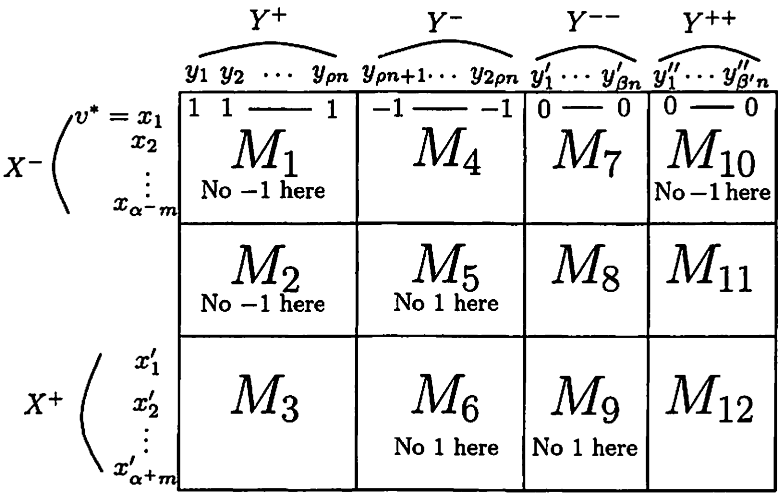

Figure 1: Adjacency matrix of Γ

of length 4, there is no -1 in M_1. Similarly, there is no 1 in M_6. By the definitions of X^+ and X^-, there is no -1 (resp.1) in M_2 (resp.M_5). And by the definitions of Y^{--}, Y^{++}, there is no 1 (resp.-1) in M_9 (resp.M_{10}).

For each $i = 1, 2, \cdots, 12$ and $j = -1, 0, 1$, let f be

$$f(i; j) := \frac{1}{mn}(\text{number of } j's \text{ in } M_i).$$

For simplicity, we use the following notations. For each $j = -1, 0, 1$ and a subset $\{a_1, \cdots, a_k\} \subset \{1, 2, \cdots, 12\}$, we define

$$f(a_1, a_2, \cdots, a_k; j) = \sum_{i=1}^{k} f(a_i, j),$$

and for each $i = 1, 2, \cdots, 12$ and a subset $\{b_1, \cdots, b_l\} \subset \{-1, 0, 1\}$,

$$f(i; b_1, b_2 \cdots, b_l) = \sum_{j=1}^{l} f(i, b_j).$$

From the definition of ρ, we have for each $y \in Y$,

$$(\text{number of 1 in } y \text{ column part of } M) = d_y^+ \leq \rho m. \tag{1}$$

Similarly, it holds

$$(\text{number of 1 in } x \text{ row part of } M) = d_x^+ \leq \rho n \tag{2}$$

for each $x \in X$. Since α means the edge density, we obtain

$$\begin{aligned} 1 - \alpha &= \frac{(\text{number of 1 in } M)}{mn} = \sum_{i=1}^{12} f(i, 0) \\ &\geq f(1, 2, 6; 0) + f(5; 0) + f(9, 12; 0) + f(7, 8, 10, 11; 0). \end{aligned} \tag{3}$$

From $f(1, 2; 0) + f(1, 2, ; 1) = \rho(1 - \alpha^+), f(1, 2, 3; 1) = f(3; -1)$ and $f(6; 0, -1) = \rho\alpha^+$, we have

$$\begin{aligned} f(1, 2, 6; 0) &= f(1, 2; 0) + f(6; 0) \\ &= \rho(1 - \alpha^+) - f(1, 2; 1) + f(6; 0) \\ &= \rho(1 - \alpha^+) + f(3; 1) - f(3; -1) + f(6; 0) \\ &= \rho - f(3; -1) + f(3; 1) - f(6; -1). \end{aligned} \tag{4}$$

By $f(9, 12; 0) = (1 - 2\rho)\alpha^+ - f(9; -1) - f(12; 1, -1)$, $f(3, 6, 9, 12; -1) = f(3, 12; 1)$ and (4), we have

$$\begin{aligned} f(1, 2, 6; 0) &+ f(9, 12; 0) \\ &= \rho - f(3, 6, 9, 12; -1) + f(3; 1) + (1 - 2\rho)\alpha^+ - f(12; 1) \\ &= \rho - f(3, 12; 1) + f(3; 1) + (1 - 2\rho)\alpha^+ - f(12; 1) \\ &= \rho + (1 - 2\rho)\alpha^+ - 2f(12; 1). \end{aligned} \tag{5}$$

Furthermore, by $f(7,8,10,11;0) = (1-2\rho)(1-\alpha^+) - f(7,8,10,11;1) - f(7,8,11;-1)$ and (5), we obtain

$$\begin{aligned} f(1,2,6;0) &+ f(9,12;0) + f(7,8,10,11;0) \\ &= \rho + (1-2\rho) - f(7,8,10,11,12;1) - f(12;1) - (7,8,11;-1) \end{aligned}$$

From (1), we know that the inequality $f(7,8,10,11,12;1) \le \rho(1-2\rho)$ holds, from which we deduce

$$\begin{aligned} f(1,2,6;0) &+ f(9,12;0) + f(7,8,10,11;0) \\ &\ge \rho + (1-2\rho) - \rho(1-2\rho) - f(12;1) - f(7,8,11;-1) \\ &= 2\rho^2 - 2\rho + 1 - f(12;1) - f(7,8,11;-1). \end{aligned} \tag{6}$$

Combining (3) and (6), we get

$$1-\alpha \ge 2\rho^2 - 2\rho + 1 + \{f(5;0) - f(8,11;-1)\} - f(7;-1) - f(12;1). \tag{7}$$

On the other hand, from the fact that Γ is Eulerian and inequality (2), we know

$$\begin{aligned} f(5;0) - f(8,11;-1) &= \rho(1-\alpha^+ - \alpha^-) - f(5,8,11;-1) \\ &\ge \rho(1-\alpha^+ - \alpha^-) - \rho(1-\alpha^+ - \alpha^-) = 0. \end{aligned} \tag{8}$$

Again, from (1), we have

$$f(7;-1) \le f(7,8,9;-1) \le \rho\beta, \tag{9}$$

$$f(12;1) \le f(10,11,12;1) \le \rho\beta'. \tag{10}$$

By $\beta + \beta' = 1 - 2\rho$ and (7) – (10), we have

$$\begin{aligned} 1-\alpha &\ge 2\rho^2 - 2\rho + 1 - \rho(\beta+\beta') \\ &= 2\rho^2 - 2\rho + 1 - \rho(1-2\rho) = 4\rho^2 - 3\rho + 1. \end{aligned}$$

Therefore, we obtain

$$\alpha \le -4\rho^2 + 3\rho = -4\left(\rho - \frac{3}{8}\right)^2 + \frac{9}{16}$$

and the desired estimate $\alpha \le \frac{9}{16}$. □

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