Relationships between distance two labellings and circular distance two labellings by group path covering *

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Abstract

For the positive integers j and k with $j \geq k$, $L(j,k)$-labelling is a kind of generalization of the classical graph coloring in which adjacent vertices are assigned integers that are at least j apart, while vertices that are at distance two are assigned integers that are at least k apart. The span of an $L(j,k)$-labelling of a graph G is the difference between the maximum and the minimum integers assigned to its vertices. The *$L(j,k)$-labelling number* of G, denoted by $\lambda_{j,k}(G)$, is the minimum span over all $L(j,k)$-labellings of G. An m-(j,k)-circular labelling of a graph G is a function $f : V(G) \to \{0, 1, \ldots, m-1\}$ such that $|f(u)-f(v)|_m \geq j$ if u and v are adjacent; and $|f(u)-f(v)|_m \geq k$ if u and v are at distance two, where $|x|_m = min\{|x|, m - |x|\}$. The span of an m-(j,k)-circular labelling of a graph G is the difference between the maximum and the minimum integers assigned to its vertices. The *m-(j,k)-circular labelling number* of G, denoted by $\sigma_{j,k}(G)$, is the minimum span over all m-(j,k)-circular labelling of G. The $L'(j,k)$-labelling, is a one-to-one $L(j,k)$-labelling and the m-$(j,k)'$-circular labelling is a one-to-one m-(j,k)-circular labelling. Denoted by $\lambda'_{j,k}(G)$ the $L'(j,k)$-labelling number and $\sigma'_{j,k}(G)$ the m-$(j,k)'$-circular labelling number. When $j = d, k = 1$, $L(j,k)$-labelling becomes $L(d,1)$-labelling. The other labellings are similar. [Discrete Math. 232 (2001) 163-169] determined the relationship between $\lambda_{2,1}(G)$ and $\sigma_{2,1}(G)$ for a graph G. We generalized the concept

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of the path covering to the t-group path covering (Inform Process Lett(2011)) of a graph. In this paper, using the t-group path covering, we establish some relationships between $\lambda'_{d,1}(G)$ and $\sigma'_{d,1}(G)$ and some relationships between $\lambda_{j,k}(G)$ and $\sigma_{j,k}(G)$ of a graph G with diameter 2. Using those results, we can have shorter proofs to obtain the $\sigma_{j,k}$-number of Cartesian products of complete graphs [J Comb Optim (2007) 14: 219-227].

Keywords: distance two labelling; circular distance two labelling; t-group path covering; product graph; assignment problem in MC

1 Introduction

Suppose we have to assign frequencies to a number of transmitters or stations in an area. In order to reduce the interference, "close" transmitters must receive different frequencies and "very close" transmitters must receive frequencies that are at least two apart. Hale firstly formulated this problem into a graph vertex coloring problem which called distance two labelling in [13]. The $L(2,1)$-labelling, introduced by Griggs and Yeh in [7], arouse from a variation of the frequency assignment problem. And then it was been generalized to the $L(j,k)$-labelling. For positive integers j and k with $j \geq k$, an $L(j,k)$*-labelling* of G is a mapping f from $V(G)$ to integers such that:

$$|f(u)-f(v)| \geq \begin{cases} j, & \text{if } d(u,v)=1, \\ k, & \text{if } d(u,v)=2. \end{cases}$$

Without loss of generality, we may assume that the minimum label of an $L(j,k)$-labelling is always 0. Then the span of f is the maximum vertex label. The $L(j,k)$-labelling number of G, denoted by $\lambda_{j,k}(G)$, is the minimum span over all $L(j,k)$-labelling of G. If $span(f) = \lambda_{j,k}(G)$, then we say that f is a $\lambda_{j,k}$-labelling of G. A variation of $L(j,k)$-labelling, namely, $L'(j,k)$-labelling, is a one-to-one $L(j,k)$-labelling. Denote by $\lambda'_{j,k}(G)$ the $L'(j,k)$-labelling number of G.

Circular distance two labelling is another kind lablling which similar to the distance two labelling. Given a real number r, let C^r denote a circular of length r. We fix a point on C^r and label it by 0. We label each point on C^r by a real number $x \in [0,r)$ according to the length of the arc from 0 along reverse clockwise on C^r to this point. The circular difference of two numbers (or two points on C^r) x and y with $0 \leq x, y < r$ on C^r, denoted by $|x-y|_r$, is defined as $\min\{|x-y|, r-|x-y|\}$. For positive integers j

and k with $j \geq k$, an m-(j,k)-circular labelling of a graph G is a function $f : V(G) \to \{0, 1, \ldots, m-1\}$ such that:

$$|f(u) - f(v)|_m \geq \begin{cases} j, & \text{if } d(u,v) = 1, \\ k, & \text{if } d(u,v) = 2. \end{cases}$$

The minimum m such that there exits an m-(j,k)-circular labelling for graph G is called the $\sigma_{j,k}$-number of G and denoted by $\sigma_{j,k}(G)$. A variation of m-(j,k)-circular labelling, namely, m-$(j,k)'$-circular labelling, is a one-to-one m-(j,k)-circular labelling. Denote by $\sigma'_{j,k}(G)$ the m-$(j,k)'$-circular labelling number of G.

When $j = d$ and $k = 1$, $L(j,k)$-labelling is the $L(d,1)$-labelling and m-(j,k)-circular labelling is the m-$(d,1)$-circular labelling. $\lambda_{j,k}(G)$ is $\lambda_{d,1}(G)$ and $\sigma_{j,k}(G)$ is $\sigma_{d,1}(G)$.

The following results are some relationships between $L(j,k)$-labelling and m-(j,k)-circular labelling of graphs.

Theorem 1.1 [2] *If G is a graph , for given $j, k \in Z^+, j \geq k$, then:*

$$\lambda_{j,k}(G) + 1 \leq \sigma_{j,k}(G) \leq \lambda_{j,k}(G) + j.$$

$$\lambda'_{j,k}(G) + 1 \leq \sigma'_{j,k}(G) \leq \lambda'_{j,k}(G) + j,$$

$$\lambda'_{d,1}(G) + 1 \leq \sigma'_{d,1}(G) \leq \lambda'_{d,1}(G) + d.$$

Using the above results, we can see that the numerical difference between $\sigma_{j,k}$-number and $\lambda_{j,k}$-number range from k to j. How to decide the quantitative value about the numerical difference is unsolved problem. Next we will introduce some results under three topics: some relationships between $\lambda_{2,1}(G)$ and $\sigma_{2,1}(G)$; some relationships between $\lambda_{j,k}(G)$ and $\sigma_{j,k}(G)$; some related results.

◇ $\lambda_{j,k}(G)$ and $\sigma_{j,k}(G)$ of G

In 2001, Liu prove the following result:

Theorem 1.2 [2] *Suppose the path covering number of G^c is $p_v(G^c)$,*

$$\sigma_{2,1}(G) \begin{cases} \leq n, & \text{if } G^c \text{ is Hamiltonian}, \\ = n + p_v(G^c), & \text{if } p_v(G^c) \geq 2. \end{cases}$$

Up on the above result, we can get the results as follows:

Theorem 1.3 [2] *If G is a graph on n vertices, the following are equivalent,*
(1) $\sigma_{2,1}(G) = n+1$;
(2) $\sigma_{2,1}(G) = n+1$, and $\lambda_{2,1}(G) = n-1$;
(3) $p_v(G^c) = 1$,and G^c is not Hamiltonian, where $p_v(G)$ is the path covering of G.

Using the above results, we can give some relationships between $\lambda_{2,1}(G)$ and $\sigma_{2,1}(G)$.

Theorem 1.4 *G is a graph on n vertices. If G^c is not Hamiltonian, but it contains a Hamilton path ,then $\sigma_{2,1}(G) = \lambda_{2,1}(G) + 2$.*

Note: If G^c is a Hamiltonian, $\sigma_{2,1}(G) = \lambda_{2,1}(G) + 1$ or $\sigma_{2,1}(G) = \lambda_{2,1}(G) + 2$ cannot be determined.

◇ $\lambda_{j,k}(G)$ and $\sigma_{j,k}(G)$ of G

For a diameter 2 graph G, when $k \le j \le 2k$, Lam, Lin and Wu determined the relationship between $\lambda_{j,k}(G)$ and $\sigma_{j,k}(G)$ by the Hamiltonian of G^c as follows:

Theorem 1.5 [14] *If G is a diameter 2 graph on n vertices, then*

(1) *If G^c contains a Hamilton $r-1$-power path, then for all j, k, when $\frac{j}{k} \le r$, $\lambda_{j,k}(G) = (n-1)k$,*

(2) *If $\lambda_{j,k}(G) = (n-1)k$, then G^c contains a Hamilton r-power path for any $r(2 \le r \le \lceil \frac{j}{k} \rceil)$.*

Theorem 1.6 [14] *If G is a diameter 2 graph on n vertices, then*

(1) *If G^c contains a Hamilton r-power cycle, then for all j, k, when $\frac{j}{k} \le r$, $\sigma_{j,k}(G) = nk$,*

(2)*If $\sigma_{j,k}(G) = nk$, then G^c contains a Hamilton r-power cycle for any $r(2 \le r \le \lceil \frac{j}{k} \rceil)$.*

◇ Some related results

In 2011, We generalized a concept of the path covering to the t-group path covering [3] (Be introduced in section 2). Using this concept, we determined the $\lambda'_{d,1}$-number of a graph and $\lambda_{j,k}$-number of a graph G with diameter 2.

Theorem 1.7 [3] *Let G be a graph of order n, d (≥ 2) is an integer, $GPC_{d-1}(G^c)$ is a $(d-1)$-path covering number of G^c, then $\lambda'_{d,1}(G) = n - d + GPC_{d-1}(G^c)$.*

Theorem 1.8 [4] *Let G be a graph of order n and diameter 2. Suppose j and k are two positive integers with $j \geq k$. Let $j = tk + l$, where $t \geq 1$ and $0 \leq l \leq k-1$. Then $\lambda_{j,k}(G) = nk - j + \min_C\{k \cdot (\sum_{h=1}^{t-1} c_h) + l \cdot c_t\}$, where C takes over all t-group path coverings of G^c.*

For some special graphs, there are some results about the $\lambda_{j,k}$-number and $\sigma_{j,k}$-number. Next, we will give the $\lambda_{j,k}$-number and $\sigma_{j,k}$-number of two kind product graphs.

Given two graphs G and H, the Cartesian product of G and H is the graph $G\Box H$ with vertex set $V(G) \times V(H)$ in which two vertices (x, y) and (x', y') are adjacent if $x = x'$, $yy' \in E(H)$ or $y = y'$, $xx' \in E(G)$. The Direct product of G and H is the graph $G \times H$ with vertex set $V(G) \times V(H)$ in which two vertices (x, y) and (x', y') are adjacent if $xx' \in E(G)$ and $yy' \in E(H)$. For convenience's sake, we call this two kind graphs the product graph of G and H. And denote their vertex (x_i, y_j) be (i, j), where $V(G) = \{x_0, x_1, \ldots, x_{|V(G)|-1}\}$ and $V(H) = \{y_0, y_1, \ldots, y_{|V(H)|-1}\}$, $0 \leq i \leq |V(G)| - 1, 0 \leq j \leq |V(H)| - 1$.

K_n is the Complete Graph with n vertices.

$\lambda_{j,k}$ and $\sigma_{j,k}$ for graphs $K_n \Box K_m$ and $K_n \Box K_n$ have been determined in [6] and [11] as stated in the following:

Theorem 1.9 [6] *Let j, k, m, n be positive integers with $2 \leq n < m$ and $j \geq k$. Then*

$$\lambda_{j,k}(K_n \Box K_m) = \begin{cases} (m-1)j + (n-1)k, & \text{if} j/k \geq n, \\ (mn-1)k, & \text{if} j/k < n. \end{cases}$$

Theorem 1.10 [6] *Let j, k, n be positive integers with $n \geq 2$ and $j \geq k$. Then*

$$\lambda_{j,k}(K_n \Box K_n) = \begin{cases} (n-1)j + (2n-2)k, & \text{if} j/k \geq n-1, \\ (n^2-1)k, & \text{if} j/k < n-1. \end{cases}$$

Theorem 1.11 [11] *Let j, k, m, n are positive integers, and $2 \leq n < m$, $j \geq k$, then*

$$\sigma_{j,k}(K_n \Box K_m) = \begin{cases} mj, & if \frac{j}{k} \geq n \\ mnk, & if \frac{j}{k} < n \end{cases}$$

Theorem 1.12 [11] *Let j, k, n are positive integers, and $2 \le n$, $j \ge k$, then*

$$\sigma_{j,k}(K_n \Box K_n) = \begin{cases} nj + nk, & if \frac{j}{k} \ge n-1 \\ n^2 k, & if \frac{j}{k} < n-1 \end{cases}$$

In this paper, we are going to established some relationships between distance two labelling and circular distance two labelling and give shorter proofs about the $\sigma_{j,k}$-number of the product graphs of the complete graph based on the above results. Firstly, we list some related preliminary definitions, example and lemma about the t-group path covering number and Hamilton t-power cycle of a graph [3] in the section 2 . In the section 3, we establish the relationship between the $L'(d,1)$-labelling number ($d \ge 2$) and the m-$(d,1)'$-circular-labelling number G based on the above definitions. In the section 4, we obtain a more general result about the relationship between $\lambda_{j,k}(G)$ and $\sigma_{j,k}(G)$ for diameter 2 graphs when $j \ge k$. Using this result, we finally give shorter proofs to determine the $\sigma_{j,k}(K_n \Box K_m)$ and $\sigma_{j,k}(K_n \Box K_n)$.

2 Preliminary definitions and lemmas

Throughout this paper, only finite simple graphs are considered.

Definition 2.1 *Let the vertex set of graph P be $V(P) = \{v_1, \dots, v_n\}$, the edge set be $\{v_i v_j, 1 \le |i-j| \le t\}$. We call this kind graph be t-power path with n vertices, where $t(\ge 1)$ is an integer.*

Definition 2.2 *Let G be a graph with vertex set $V(G) = \{v_1, \dots, v_n\}$ and edge set $E(G)$. A collection of vertex disjoint t-power paths $\{P_1, P_2, \dots, P_q\}$ in G is called a t-power path covering of G if each vertex of G belongs to exactly one of P_is.*

Definition 2.3 [3] *Suppose G is a graph with vertex set $\{v_1, v_2, \dots, v_n\}$. Let t be a positive integer not less than 2. A permutation π of $\{1, 2, \dots, n\}$ defines an order of vertices of G as $v_{\pi(1)}, v_{\pi(2)}, \dots, v_{\pi(n)}$.*
Then $C = \{C_1(G), \dots, C_t(G)\}$ is called a t-group path covering of G if the following two rules are satisfied.

(R1) *For $i = 1, 2, \dots, t$, each $C_i(G) = \{P_1^i, P_2^i, \dots, P_{c_i}^i\}$ is an i-power path covering of G, where each P_j^i ($1 \le j \le c_i$) is an i-power path of G spanned by a consecutive subsequence of $(v_{\pi(1)}, v_{\pi(2)}, \dots, v_{\pi(n)})$.*

(R2) *Let* $C_i(G) = \{P_1^i, P_2^i, \ldots, P_{c_i}^i\}$, $C_{i+1}(G) = \{P_1^{i+1}, P_2^{i+1}, \ldots, P_{c_{i+1}}^{i+1}\}$ $(1 \le i \le t-1)$*, then for each* $j \in \{1, 2, \ldots, c_{i+1}\}$, $V(P_j^{i+1}) \subseteq V(P_k^i)$ *for some* k *with* $1 \le k \le c_i$.

Definition 2.4 [3] *Given a graph* G *and a positive integer* t *with* $t \ge 2$. *The* t*-group path covering number of* G*, denoted by* $GPC_t(G)$*, is defined as the minimum number of* $\sum_{i=1}^{t} c_i$ *for all* t*-group path coverings* $C = \{C_1(G), \ldots, C_t(G)\}$ *of* G*, where* $c_i = |C_i(G)|$ *for* $i = 1, 2, \ldots, t$. *The order of the vertex sequence along the 1-power path covering of* C *is called the optimal vertex order. Suppose* $C' = \{C_1'(G), \ldots, C_t'(G)\}$ *of* G *is a* t*-group path covering of* G*, where* $c_i' = |C_i'(G)|$ *for* $i = 1, 2, \ldots, t$. *If* $\sum_{i=1}^{t} c_i' = GPC_t(G)$*, then we call* C' *the* optimal t-group path covering *of* G.

From the above definition, it is obvious to see $c_1 \le c_2 \le \cdots \le c_t$. A graph G has a Hamilton t-power path if and only if G has a t-group path covering $C = \{C_1(G), \ldots, C_t(G)\}$ with $c_1 = c_2 = \cdots = c_t = 1$.

We use the same example in [3] to illustrates the above definitions.

Example: Let $\{v_1v_2v_3, v_4v_5v_6v_7, v_8v_9v_{10}v_{11}v_{12}\}$ is a 2-power path covering of H in Figure 1.

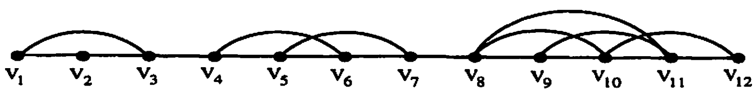

Figure 1: The graph H in the example[3].

Consider the 3-group path covering of H. Let $C_1(H) = \{v_1v_2 \ldots v_{12}\}$, $C_2(H) = \{v_1v_2v_3, v_4v_5v_6v_7, v_8v_9v_{10}v_{11}v_{12}\}$, $C_3(H) = \{v_1v_2v_3, v_4v_5, v_6v_7, v_8v_9v_{10}v_{11}, v_{12}\}$ and $C_3'(H) = \{v_1v_2v_3, v_4v_5, v_6v_7, v_8v_9v_{10}, v_{11}v_{12}\}$. $\{C_1(H), C_2(H), C_3(H)\}$ and $\{C_1(H), C_2(H), C_3'(H)\}$ are two 3-group path coverings of H with $c_1 = 1$, $c_2 = 3$ and $c_3 = 5$. It is not hard to see that they are both optimal 3-group path coverings of H. And $GPC_3(H) = 9$.

$\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}\}$ is the optimal t-group path covering of H. ■

Next, we introduce another related definition.

Definition 2.5 *If* F *is a circular distance two labelling of* G*, define the following for* $0 \le s \le r$:

$$F_s := \{v : F(v) = s\},$$

$$H(F) := \{s : F_s = \varnothing\}.$$

All the indices above are taken mod r. If $\{s+1, s+2, \ldots, s+l\} \subseteq H(F)$ *is a consecutive subsequence, and* s, $s+l+1 \notin H(F)$, *then* $\{s+1, s+2, ..., s+l\}$ *is called an l-hole of F.*

3 $\lambda'_{d,1}(G)$ and $\sigma'_{d,1}(G)$

From the above definition, we can obtain the following lemma:

Lemma 3.1 *If G is the graph on n vertices, then* $\sigma'_{d,1}(G) = n$ *if and only if* G^c *contains a Hamilton* $(d-1)$*-power cycle.*

Proof. Suppose G^c contains a Hamilton $(d-1)$-power cycle, i.e. $v_0, v_1, \ldots, v_{n-1}, v_0$, then the function $L(v_x) = x$ is an n-$(d,1)'$-circular labelling of G. So $\sigma'_{d,1}(G) \leq n$. The $(d,1)'$-circular labelling is a one-to-one function, and $V(G) = n$, so $\sigma'_{d,1}(G) \geq n$. Hence $\sigma'_{d,1}(G) = n$.

Suppose $\sigma'_{d,1}(G) = n$, L is an n-$(d,1)'$-circular labelling of G. Let $v_x = L^{-1}(x)$, where $0 \leq x \leq n-1$. Then we can obtain a Hamilton $(d-1)$-power cycle of G^c along the vertex order $v_0, v_1, \ldots, v_{n-1}, v_0$. ∎

We give the main result of this paper as stated in the following Theorem:

Theorem 3.2 *If G is a graph on n vertices, then*

(1) *If* G^c *is not a Hamiltonian, then* $\sigma'_{d,1}(G) = \lambda'_{d,1}(G) + d$;

(2) *If* G^c *contains a Hamilton* $(d-1)$*-power cycle, then* $\sigma'_{d,1}(G) = \lambda'_{d,1}(G) + 1$;

(3) *If* G^c *hasn't a Hamilton* $(i+1)$*-power cycle, but it contains a Hamilton i-power cycle along the optimal vertex order of G, then* $\sigma'_{d,1}(G) = \lambda'_{d,1}(G) + d - i$, *where* $1 \leq i \leq d-2$.

Proof. (1) By theorem 1.1, it is obvious that $\sigma'_{d,1}(G) \leq \lambda'_{d,1}(G) + d$.

Suppose G^c is not Hamiltonian. Let $\sigma'_{d,1}(G) = p$ and f be a p-$(d,1)'$-circular labelling of G. Let A be the labels used by f, then $|A| = n$ and $A \subset \{0, 1, \ldots, p-1\}$. Because of $p \geq n$, there must be some holes of f. There must be a $(d-1)$-hole of $\{0, 1, \ldots, p-1\}$. Otherwise we can obtain a Hamilton cycle of G^c as follows: for any label i, $i \in \{0, 1, \ldots, p-1\}$, if i is used by f, then $f^{-1}(i)$ is a vertex of G; if i is not used by f, let $f^{-1}(i) = \Phi$. We can see that the sequence $S = (f^{-1}(0), f^{-1}(1), \ldots, f^{-1}(p-1), f^{-1}(0))$

contains all the vertices of G and several Φs. Obviously, any maximal consecutive Φ subsequence is the hole of f. Delete all Φs of S, then S is divided into several parts, i.e. $S_1, S_2, \ldots, S_c$. The sub-graph of G with the vertices of S_i must be contain a path, where $1 \leq i \leq c$. We prove this result as follows. Along the vertices of $P_1, P_2, \ldots, P_c$, we can establish a vertex order of G. In this vertex order, each vertex order of P_i is a sequences of $\{0, 1, \ldots, p-1\}$. Because there is not a $(d-1)$-hole of $\{0, 1, \ldots, p-1\}$, so there must not be a $(d-1)$-hole of any vertex order of P_i. It is easy to see that: the difference between any two vertex labels on P_i is at most $d-1$, so the two vertex is not adjacent in G. They must be adjacent in G^c. Hence, each P_i contains a path of G^c. For any adjacent paths, i.e. P_i, $P_{i+1} (1 \leq i \leq c-1)$, the label of the last vertex on P_i is at most differ by $d-1$ from the label of the first vertex on P_{i+1}, then the two vertices are not adjacent in G but adjacent in G^c. Along the vertex order of $P_1, P_2, \ldots, P_c$, G^c contains a Hamilton cycle which is contradict the assumption. Hence , in $\{0, 1, \ldots, p-1\}$, there must be a $(d-1)$-hole, i.e. $\{h+1, \ldots, h+d-1\}, 0 \leq h \leq p-2$, where all the indices are taken *mod p*. Let:

$$f'(v) = f(v) + p - h - d \mod p$$

It is easy to see that f' is also a p-$(d,1)'$-circular labelling of G with $f'^{-1}(p-d+1) = f'^{-1}(p-d+2) = \cdots = f'^{-1}(p-1) = \Phi$ is an $L'(d,1)$-lablling of G. Then $\lambda'_{d,1}(G) \leq p - d = \sigma'_{d,1}(G) - d$.

Hence $\sigma'_{d,1}(G) = \lambda'_{d,1}(G) + d$.

(2) If G^c contains a Hamilton $(d-1)$-power cycle , then $\sigma'_{d,1}(G) = n$ based on lemma 3.1. G^c contains a Hamilton $(d-1)$-power path, then $\lambda'_{d,1}(G) = n-1$ based on theorem 1.7.

The result is proved .

(3)Let $\lambda'_{d,1}(G) = p$, f be a p-$L'(d,1)$-lablling of G, then the labels of f must be defined along the optimal vertex sequence of G, $v_0, v_1, \ldots, v_{n-1}, v_0$. We will prove f is a $(p+d-i)$-$(d,1)'$-circular labelling of G. G^c does not contain a Hamilton $(i+1)$-power cycle, but it contains a Hamilton i-power cycle along the optimal vertex order of G, there must exist that $1 \leq |f(x) - f(y)| \leq p-1$ for any two vertices x, y when $d(xy) = 2$ and $|f(x) - f(y)|_{p+d-i} \geq 1$; $d \leq |f(u) - f(v)| \leq p - i$ for any adjacent vertices u, v and $|f(u) - f(v)|_{p+d-i} \geq d$. Hence $\sigma'_{d,1}(G) \leq \lambda'_{d,1}(G) + (d-i)$.

Let's prove $\sigma'_{d,1}(G) \geq \lambda'_{d,1}(G) + (d-i)$. Let $\sigma'_{d,1}(G) = p$, f be a p-$(d,1)'$-circular labelling of G and A be the labels used by f, then $|A| = n$ and

$A \subset \{0, 1, \ldots, p-1\}$. Because of $p \geq n$, there must be a $(d-i-1)$-hole of $\{0, 1, \ldots, p-1\}$. Otherwise, we can construct a Hamilton $(i+1)$-power cycle of G^c as follows: for any label $i (i \in \{0, 1, \ldots, p-1\})$, if i is used by f, then $f^{-1}(i)$ is a vertex of G; if i is not used by f, then let $f^{-1}(i) = \Phi$. The sequence $S = (f^{-1}(0), f^{-1}(1), \ldots, f^{-1}(p-1), f^{-1}(0))$ contains all vertices of G and several Φs. It is obvious that any maximal consecutive Φ subsequence is a hole of f. Delete all Φs of S, then S is divided into c parts ,$S_1, S_2, \ldots, S_c$, we can obtain a vertex order $P_1, P_2, \ldots, P_c$ of G from these parts. Each path, P_i, contains an $(i+1)$-power path of G^c. And for any $j, 1 \leq j \leq c-1$, if $x \in P_j, y \in P_{j+1}$ and the distance between x, y be $\leq (i+1)$, then there must be $|f(x) - f(y)|_p \geq d-i-1$. So x, y are not adjacent in G and adjacent in G^c. and for each j, $1 \leq j \leq c-1$, the vertices whose distance differs at most $i+1$ on P_j and P_{j+1}, their labels is at most $d-i-1$ apart. So they are not adjacent in G, but adjacent in G^c. G^c contains a Hamilton $(i+1)$-power cycle along the vertex order of $P_1, P_2, \ldots, P_c$, contradicting the assumption. Hence there must be a $(d-i-1)$-hole of $\{0, 1, \ldots, p-1\}$, i.e. $\{h+1, \ldots, h+d-i-1\}$, $0 \leq h \leq p-2$, where all indices are taken *mod p*. Let:

$$f'(v) = f(v) + p - h - d + i \mod p$$

We can see that f' is also a p-$(d,1)'$-circular labelling of G with $f'^{-1}(p-d+i+1) = f'^{-1}(p-d+i+2) = \cdots = f'^{-1}(p-1) = \Phi$. Since f' is an $L'(d,1)$-labelling of G, $\lambda'_{d,1}(G) \leq p - d + i = \sigma'_{d,1}(G) - d + i$.

The theorem is proved. ∎

4 $\lambda_{j,k}(G)$ and $\sigma_{j,k}(G)$ for diameter 2 graphs

For a diameter 2 graph G, when $k \leq j \leq 2k$, Lam, Lin and Wu determined the relationship between $\lambda_{j,k}(G)$ and $\sigma_{j,k}(G)$ by the Hamiltonian of G^c. In this section, we establish an extension of the above results when $j \geq k$.

Theorem 4.1 [11] *Let G be a diameter 2 graph on n vertices, $p_v(G^c)$ be the path covering number of G^c. When $k \leq j \leq 2k$, there must be:*

(1) *If G^c is not Hamiltonian, then $\lambda_{j,k}(G) = (n - p_v(G^c))k + (p_v(G^c) - 1)j$ and $\sigma_{j,k}(G) = (n - p_v(G^c))k + p_v(G^c)j$;*

(2) *If G^c is Hamiltonian, then $\sigma_{j,k}(G) = nk$.*

Proof. (1) Let's firstly prove $\lambda_{j,k}(G)=(n-p_v(G^c))k+(p_v(G^c)-1)j$. Let $p_v(G^c)=t(\geq 2)$, $\{P_1,P_2,\cdots,P_t\}$ be the path covering of G^c. Denote p_i be the vertex number of P_i, where $i=1,2,\ldots,t$. Suppose v_l^i be the l-th vertex of path P_i, then we give a labelling L as following:

$$L(v_l^i)=[\sum_{s=1}^{i-1}(p_s-1)]k+(l-1)k+(i-1)j$$

It is not hard to see that L is an $L(j,k)$-labelling of G, then there must be $\lambda_{j,k}(G)\leq(n-p_v(G^c))k+(p_v(G^c)-1)j$. When $p_v(G^c)=t=1$, G^c must contain a path covering which contains only one path, namely P. We can give the above labelling L for P. Because of $k\leq j\leq 2k$, L is also an $L(j,k)$-labelling of G. So the result is right when $p_v(G^c)=1$.

Next we prove the opposite inequality. Let L be an $L(j,k)$-labelling of G. Without loss of generality, suppose $L(v_i)=l_i(i=1,2,\ldots,n)$ and $l_1<l_2<\cdots<l_n$. Because G is a diameter 2 graph, so the labels of any two vertices of G differ from at least k. If $|l_s-l_t|<j$, then $v_sv_t\in E(G^c)$. Let p_1 is the minimum number for $l_{p_1}\leq l_{p_1+1}-j$, then we can see that $v_1v_2\ldots v_{p_1}$ is a path of G^c. Let p_2 is the minimum number for $p_2>p_1$ and $l_{p_2}\leq l_{p_2+1}-j$, then $v_{p_1+1}v_{p_1+2}\cdots v_{p_1+p_2}$ is the second path of G^c. Carrying on the above way, we can get a path covering of G^c. Suppose the path covering contains t paths ,then we can see that the span of the labelling L is at least $(n-t)k+(t-1)j\geq(n-p_v(G^c))k+(p_v(G^c)-1)j$. When $t=1$, the result is also right. Hence, $\lambda_{j,k}(G)=(n-p_v(G^c))k+(p_v(G^c)-1)j$ is proved.

To prove $\sigma_{j,k}(G)=(n-p_v(G^c))k+p_v(G^c)j$ is to prove $\sigma_{j,k}(G)=\lambda_{j,k}(G)+j$ when $k\leq j\leq 2k$. By theorem 1.1, $\sigma_{j,k}(G)\leq\lambda_{j,k}(G)+j$. We will prove the opposite. Suppose G^c is not Hamiltonian. Let $\sigma_{j,k}(G)=l$, f be an l-(j,k)-circular labelling of G. There must be an $(j-1)$-hole, $\{h+1,\ldots,h+j-1\}(0\leq h\leq l-2)$ of $\{0,1,\ldots,l-1\}$, where all indices are taken *mod* l. The proof is similar to the same part of the theorem 3.2. Let:

$$f'(v)=f(v)+l-h-j \mod l$$

We can see that f' is also an l-(j,k)-circular labelling of G with $f'^{-1}(l-j+1)=f'^{-1}(l-j+2)=\cdots=f'^{-1}(l-1)=\Phi$. Since f' is an $L(j,k)$-labelling of G, $\lambda_{j,k}(G)\leq l-j=\sigma_{j,k}(G)-j$. G^c is not Hamiltonian, so the first vertex is not adjacent with the last vertex of P. The result is also right for $p_v(G^c)=1$ when $k\leq j\leq 2k$.

Hence $\sigma_{j,k}(G) = \lambda_{j,k}(G) + j$.

(2) Using the similar way and the $p_v(G^c)$ of G^c, we can also prove $\sigma_{j,k}(G) = nk$. ∎

Based on the above theorem, we can get the relationship between $\lambda_{j,k}(G)$ and $\sigma_{j,k}(G)$ for diameter 2 graph G when $k \leq j \leq 2k$.

Theorem 4.2 *Let G be a diameter 2 graph on n vertices. When $k \leq j \leq 2k$,*

(1) If G^c is not Hamiltonian, then $\sigma_{j,k}(G) = \lambda_{j,k}(G) + j$;

(2) If G^c is Hamiltonian, then $\sigma_{j,k}(G) = \lambda_{j,k}(G) + k$.

Next we establish an extension of the above results when $j \geq k$.

Theorem 4.3 *If G is a diameter 2 graph on n vertices, $j \geq k$,*

(1) If G^c is not Hamiltonian, then $\sigma_{j,k}(G) = \lambda_{j,k}(G) + j$,

(2) If G^c contains a Hamilton $\lceil \frac{j}{k} \rceil$-power cycle, then $\sigma_{j,k}(G) = \lambda_{j,k}(G) + k$,

(3) If G^c doesn't contain a Hamilton $(i+1)$-power cycle, but G^c contains a Hamilton i-power cycle along the optimal $L(j,k)$-labelling vertex order of G, then $\sigma_{j,k}(G) = \lambda_{j,k}(G) + j - ik$, where $1 \leq i \leq \lceil \frac{j}{k} \rceil - 1$.

Proof. (1) Suppose G is a diameter 2 graph on n vertices,

By theorem 1.1, $\sigma_{j,k}(G) \leq \lambda_{j,k}(G) + j$. We will prove the opposite.

Suppose G^c is not Hamiltonian. Let $\sigma_{j,k}(G) = l$, f be an l-(j,k)-circular labelling of G. There must be an $(j-1)$-hole, $\{h+1, \ldots, h+j-1\}(0 \leq h \leq l-2)$ of $\{0, 1, \ldots, l-1\}$, where all indices are taken *mod l*. The proof is similar to the same part of the theorem 3.2. Let:

$$f'(v) = f(v) + l - h - j \mod l$$

We can see that f' is also an l-(j,k)-circular labelling of G with $f'^{-1}(l-j+1) = f'^{-1}(l-j+2) = \cdots = f'^{-1}(l-1) = \Phi$. Since f' is an $L(j,k)$-labelling of G, $\lambda_{j,k}(G) \leq l - j = \sigma_{j,k}(G) - j$.

Hence $\sigma_{j,k}(G) = \lambda_{j,k}(G) + j$.

(2) If G^c contains an $\lceil \frac{j}{k} \rceil$-power cycle, then G^c contains an $\lceil \frac{j}{k} \rceil$-power path. By theorem 1.6, we can obtain $\sigma_{j,k}(G) = nk$. Hence $\lambda_{j,k}(G) = (n-1)k$.

The result is proved.

(3) The proof of $\sigma_{j,k}(G) \leq \lambda_{j,k}(G) + (j - ik)$ is similar to the proof of theorem 3.2. Let's prove $\sigma_{j,k}(G) \geq \lambda_{j,k}(G) + (j - ik)$. Let $\sigma_{j,k}(G) = p$, f be a p-(j,k)-circular labelling of G, there must be an $(j - ik - 1)$-hole, $\{h+1, \ldots, h+j-ik-1\}$, $0 \leq h \leq p-1$, of $\{0, 1, \ldots, p-1\}$, where all indices are taken *mod* p. Let:

$$f'(v) = f(v) + p - h - j + ik \mod p$$

It is easy to see that f' is also a p-(j,k)-circular labelling of G with $f'^{-1}(p-j+ik+1) = f'^{-1}(p-j+ik+2) = \cdots = f'^{-1}(p-1) = \Phi$. So f' is an $L(j,k)$-labelling of G, $\lambda_{j,k}(G) \leq p - j + ik = \sigma_{j,k}(G) - j + ik$. ∎

5 The application of the theorem

Lam, Lin and Wu [11] have gotten the $\sigma_{j,k}$-number of the complete products(theorem 1.11 and 1.12). Using the results of this paper (theorem 4.3), we can give a shorter proof of $\sigma_{j,k}$-number of the complete products with the $\lambda_{j,k}$-number (theorem 1.9, theorem 1.10).

Let $c_r(G)$ be the number of r-power path covering of a graph G.

Lemma 5.1 [6] *Let* $2 \leq n < m$.
(1) If $r > n-1$, *then* $c_r((K_n \Box K_m)^c) = m$, *and*
(2) If $1 \leq r \leq n-1$, *then* $c_r((K_n \Box K_m)^c) = 1$.

Lemma 5.2 [6] *Let* $2 \leq n$.
(1) If $r > n-2$, *then* $c_r((K_n \Box K_n)^c) = n$, *and*
(2) If $1 \leq r \leq n-2$, $c_r((K_n \Box K_n)^c) = 1$.

Lemma 5.3 *Let* $2 \leq n < m$, $(K_n \Box K_m)^c$ *hasn't Hamilton* l*-power cycle, where* $l > n-1$, *but* $(K_n \Box K_m)^c$ *contains a Hamilton* r*-power cycle along the optimal* $L(j,k)$*-labelling vertex order of* $K_n \Box K_m$, *where* $r \leq n-1$.

Proof. Since $(K_n \Box K_m)^c = K_n \times K_m$. Put the vertices of $K_n \times K_m$ into $n \times m$ matrix, the vertices which is not in the same line and the same row must be adjacent(in $K_n \times K_m$). By lemma 5.1 and theorem 1.9, when $r \leq n-1$, $(K_n \Box K_m)^c$ contains a Hamilton r-power path, $v_0, v_1, \ldots, v_{mn-1}$ along the optimal $L(j,k)$-labelling vertex order of $K_n \Box K_m$, where $v_t = (a,b)$, $t = ((n+1)a - nb) \bmod mn$. We can find that: along the vertex order of $v_0, v_1, \ldots, v_{mn-1}$, v_i and $v_{i \pm j \ mod \ mn}$ are not in the same row and same line of $K_n \times K_m$, so these vertices are adjacent in $K_n \times K_m$, where $1 \le j \le r$. $(K_n \Box K_m)^c$ contains a r-power cycle, i.e. $v_0, v_1, \ldots, v_{mn-1}, v_0$, along the optimal $L(j,k)$-labelling vertex order of $K_n \Box K_m$. By Lemma 5.1, $(K_n \Box K_m)^c$ contains no Hamilton l-power path, so it doesn't contain a Hamilton l-power cycle, where $l > n-1$. ∎

Lemma 5.4 *Let $2 \le n$, $(K_n \Box K_n)^c$ hasn't Hamilton l-power cycle, where $l > n-2$, but $(K_n \Box K_n)^c$ contains a Hamilton r-power cycle along the optimal $L(j,k)$-labelling vertex order of $K_n \Box K_n$, where $r \le n-2$.*

Proof. By lemma 5.2 and theorem 1.10, the proof is similar to lemma 5.3. ∎

By theorem 1.9, 1.10 and theorem 4.3, the theorems 1.11, 1.12 can easily be proved.

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