SOME ALGEBRAIC RELATIONS ON INTEGER SEQUENCES INVOLVING OBLONG AND BALANCING NUMBERS

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ABSTRACT. Let $k \geq 0$ be an integer. Oblong (pronic) numbers are numbers of the form $O_k = k(k+1)$. In this work, we set a new integer sequence $B = B_n(k)$ defined as $B_0 = 0$, $B_1 = 1$ and $B_n = O_k \ B_{n-1} - B_{n-2}$ for $n \geq 2$ and then derived some algebraic relations on it. Later, we give some new results on balancing numbers via oblong numbers.

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1. Introduction

Fibonacci, Lucas, Pell and the other special numbers and their generalizations arise in the examination of various areas of science and art. In fact, these numbers are special cases of a sequence which is defined as a linear combination as follows:

$$a_{n+k} = c_1 a_{n+k-1} + c_2 a_{n+k-2} + \cdots + c_k a_n, \tag{1.1}$$

where $c_1, c_2, \cdots, c_k$ are real constants. The applications and identities related with these numbers can be seen in [3, 5, 6, 7, 9, 13].

Fibonacci numbers form a sequence defined by the following recurrence relation: $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. The characteristic equation of F_n is $x^2 - x - 1 = 0$ and hence the roots of it are $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Johannes Kepler pointed out that the ratio of consecutive Fibonacci numbers converges to the golden ratio as the limit, that is,

$$\lim_{n \to \infty} \frac{F_n}{F_{n-1}} = \alpha.$$

Like every sequence defined by a second order linear recurrence, the Fibonacci numbers F_n have a closed–form solution. It is known as Binet formula

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

for $n \geq 0$. Lucas numbers L_n are defined by $L_0 = 2, L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. Its Binet formula is $L_n = \alpha^n + \beta^n$. There are a lot of algebraic identities between Fibonacci and Lucas numbers [7, 9]. The Pell numbers are defined by the recurrence relation $P_0 = 0$, $P_1 = 1$ and $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$. Some identities for Pell numbers can be found in [11, 20]. Also the Pell–Lucas, Jacobsthal and Pell–Jacobsthal integer sequences are also famous integer sequences.

In fact all of them are the special cases of the following two integer sequences: Let P and Q be non–zero integers and let $D = P^2 - 4Q$ be called the discriminant and assume that $D \neq 0$ (to exclude a degenerate case). For each $n \geq 0$, define $U_n = U_n(P, Q)$ and $V_n = V_n(P, Q)$ to be

(1.2)
$$\begin{aligned} U_n &= U_n(P, Q) = PU_{n-1} - QU_{n-2} \\ V_n &= V_n(P, Q) = PV_{n-1} - QV_{n-2} \end{aligned}$$

for $n \geq 2$ with initial values $U_0 = 0, U_1 = 1$ and $V_0 = 2, V_1 = P$. The characteristic equation of them is $x^2 - Px + Q = 0$ and hence the roots of it are

$$\alpha = \frac{P + \sqrt{P^2 - 4Q}}{2} \quad \text{and} \quad \beta = \frac{P - \sqrt{P^2 - 4Q}}{2}.$$

So Binet formulas are hence

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n.$$

Note that in (1.2), one has the following table:

P	Q	U_n	V_n
1	−1	Fibonacci sequence	Lucas sequence
2	−1	Pell sequence	Pell–Lucas sequence
1	−2	Jacobsthal sequence	Pell–Jacobsthal sequence

There are a lot of algebraic identities on U_n and V_n (see [19]). Also for the companion matrix $M = \begin{bmatrix} P & -Q \\ 1 & 0 \end{bmatrix}$, one has

$$\begin{bmatrix} U_n \\ U_{n-1} \end{bmatrix} = M^{n-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} V_n \\ V_{n-1} \end{bmatrix} = M^{n-1} \begin{bmatrix} P \\ 2 \end{bmatrix}.$$

Further the generating function for U_n and V_n is

$$\frac{x}{1-Px+Qx^2}=\sum_{n=0}^{\infty}U_nx^n \quad\text{and}\quad \frac{2-Px}{1-Px+Qx^2}=\sum_{n=0}^{\infty}V_nx^n,$$

respectively (for further details see also [3, 7, 9, 12, 13, 19]).

2. Integer Sequence via Oblong Numbers.

Let $k\geq 0$ be an integer. Then it is known that oblong numbers are numbers of the form

$$O_k=k(k+1). \tag{2.1}$$

The first few oblong numbers are $0,2,6,12,20,30,42,56,72,90,110,\cdots$. So the $k-$th oblong number represents the number of points in a rectangular array having k columns and $k+1$ rows. Further the product of two oblong numbers O_{k-1} and O_k is another oblong number O_{k^2-1} (see [2]), that is,

$$O_{k-1}O_k[(k-1)k][k(k+1)]=(k^2-1)k^2=O_{k^2-1}.$$

Also the half of O_k is a triangular number, that is, $T_k=\frac{O_k}{2}$. The connections between oblong and triangular numbers were studied in [8].

In this section, we try to define a new integer sequence associated with O_k and then derive some algebraic identities on it. For this reason, we set $P=O_k$ and $Q=1$. Then we define the sequence $B=B_n(k)$ as $B_0=0$, $B_1=1$ and

$$B_n=O_kB_{n-1}-B_{n-2} \tag{2.2}$$

for $n\geq 2$. The characteristic equation of (2.2) is $x^2-O_kx+1=0$ and hence the roots of it are

$$\alpha=\frac{O_k+\sqrt{O_k^2-4}}{2}\quad\text{and}\quad\beta=\frac{O_k-\sqrt{O_k^2-4}}{2}. \tag{2.3}$$

So for $n\geq 0$, Binet formula is

$$B_n=\frac{\alpha^n-\beta^n}{\alpha-\beta} \tag{2.4}$$

for $k\neq 1$. Now we can give the following results.

Theorem 2.1. *Let B_n denote the n^{th} number.*

(1) *If $k=1$, then*

$$\sum_{i=1}^{n}B_i=T_n.$$

(2) *If $k > 1$, then*

$$\sum_{i=1}^{n} B_i = \frac{(O_k - 1)B_n - B_{n-1} - 1}{O_k - 2}.$$

Proof. (1) Let $k = 1$. Then $O_1 = 2$ and $B_n = 2B_{n-1} - B_{n-2}$. So $B_n = n$ and hence the sum of first n terms of B_n is $\frac{n(n+1)}{2} = T_n$.

(2) Let $k > 1$. Since $B_n = O_k B_{n-1} - B_{n-2}$, we get $B_n + B_{n-2} = O_k B_{n-1}$ and so

$$\begin{aligned} B_2 + B_0 &= O_k B_1 \\ B_3 + B_1 &= O_k B_2 \\ B_4 + B_2 &= O_k B_3 \\ &\cdots \\ B_{n-1} + B_{n-3} &= O_k B_{n-2} \\ B_n + B_{n-2} &= O_k B_{n-1}. \end{aligned} \tag{2.5}$$

If we sum of both sides of (2.5), then we obtain

$$B_0 + B_1 + \cdots + B_{n-2} + B_2 + B_3 + \cdots + B_n = O_k(B_1 + B_2 + \cdots + B_{n-1}).$$

Adding $B_{n-1} + B_n + B_0 + B_1 + O_k(B_0 + B_n)$ to both sides of above equation, we get the desired result. □

Now we can give the following theorem concerning the recurrence relation on B_n numbers.

Theorem 2.2. *Let B_n denote the n^{th} number. Then for $n \geq 2$, we have*

(1) $B_{2n} = O_1 B_{2n-2} - B_{2n-4}$ *and* $B_{2n+1} = O_1 B_{2n-1} - B_{2n-3}$ *for* $k = 1$;
(2) $B_{2n} = (O_k^2 - 2)B_{2n-2} - B_{2n-4}$ *and* $B_{2n+1} = (O_k^2 - 2)\ B_{2n-1} - B_{2n-3}$ *for* $k > 1$.

Proof. (1) Let $k = 1$. Then $B_n = 2B_{n-1} - B_{n-2}$. So $B_{2n} = O_1 B_{2n-2} - B_{2n-4}$ and $B_{2n+1} = O_1 B_{2n-1} - B_{2n-3}$.

(2) Let $k > 1$. Since $B_{2n} = O_k B_{2n-1} - B_{2n-2}$, we get

$$\begin{aligned} B_{2n} &= O_k(O_k B_{2n-2} - B_{2n-3}) - B_{2n-2} \\ &= (O_k^2 - 1)B_{2n-2} - O_k B_{2n-3} \\ &= (O_k^2 - 1)B_{2n-2} - O_k(O_k B_{2n-4} - B_{2n-5}) \end{aligned}$$

$$
\begin{aligned}
&= (O_k^2 - 1)B_{2n-2} - O_k^2 B_{2n-4} + O_k B_{2n-5} \\
&= (O_k^2 - 2)B_{2n-2} + B_{2n-2} - O_k^2 B_{2n-4} + O_k B_{2n-5} \\
&= (O_k^2 - 2)B_{2n-2} + O_k B_{2n-3} - B_{2n-4} - O_k^2 B_{2n-4} + O_k B_{2n-5} \\
&= (O_k^2 - 2)B_{2n-2} + O_k^2 B_{2n-4} - O_k B_{2n-5} - (1 + O_k^2)B_{2n-4} + O_k B_{2n-5} \\
&= (O_k^2 - 2)B_{2n-2} - B_{2n-4}.
\end{aligned}
$$

The other assertion can be proved similarly. □

Now we can give the following theorem related to powers of α and β.

Theorem 2.3. *Let B_n denote the $n-th$ number.*

(1) *If $k = 1$, then $\alpha^n + \beta^n = O_1$.*
(2) *If $k > 1$, then*

$$
\alpha^n + \beta^n = \begin{cases} B_{n+1} - B_{n-1} & \text{for } n \geq 1 \\ O_1 B_{n+1} - O_k B_n & \text{for } n \geq 0 \\ O_k B_n - O_1 B_{n-1} & \text{for } n \geq 0. \end{cases}
$$

Proof. (1) Let $k = 1$. Then $\alpha = \beta = \frac{O_1}{2} = 1$. So $\alpha^n + \beta^n = 2 = O_1$.

(2) Note that $B_{n+1} + B_{n-1} = O_k B_n$. So $B_{n+1} - B_{n-1} = O_k B_n - O_1 B_{n-1}$ and hence

$$
\begin{aligned}
B_{n+1} - B_{n-1} &= O_k \left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right) - O_1 \left(\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}\right) \\
&= \alpha^n \left[\frac{O_k}{\alpha - \beta} - \frac{O_1}{\alpha(\alpha - \beta)}\right] + \beta^n \left[\frac{-O_k}{\alpha - \beta} - \frac{O_1}{\beta(\alpha - \beta)}\right] \\
&= \alpha^n + \beta^n.
\end{aligned}
$$

We see as above that $B_{n+1} - B_{n-1} = \alpha^n + \beta^n$. So

$$
\begin{aligned}
\alpha^n + \beta^n &= B_{n+1} - B_{n-1} \\
&= 2B_{n+1} - (O_k B_n - B_{n-1}) - B_{n-1} \\
&= 2B_{n+1} - O_k B_n.
\end{aligned}
$$

For the last assertion, we easily get

$$
\begin{aligned}
O_k B_n - O_1 B_{n-1} &= O_k \left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right) - O_1 \left(\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}\right) \\
&= \alpha^n \left[\frac{O_k}{\alpha - \beta} - \frac{O_1}{\alpha(\alpha - \beta)}\right] + \beta^n \left[\frac{-O_k}{\alpha - \beta} + \frac{O_1}{\beta(\alpha - \beta)}\right] \\
&= \alpha^n + \beta^n.
\end{aligned}
$$

□

Now we consider the circulant matrix for B_n numbers. Recall that a circulant matrix (see [4]) is a matrix A defined as

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \\ a_n & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_n & a_1 & \cdots & a_{n-3} & a_{n-2} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ a_3 & a_4 & a_5 & \cdots & a_1 & a_2 \\ a_2 & a_3 & a_4 & \cdots & a_n & a_1 \end{bmatrix},$$

where a_i are constant. The eigenvalues of A are

$$\lambda_j(A) = \sum_{k=0}^{n-1} a_k w^{-jk}, \tag{2.6}$$

where $w = e^{\frac{2\pi i}{n}}$, $i = \sqrt{-1}$ and $j = 0, 1, \cdots, n-1$. Therefore the circulant matrix for B_n numbers is hence

$$B = B(B_n) = \begin{bmatrix} B_0 & B_1 & B_2 & \cdots & B_{n-1} \\ B_{n-1} & B_0 & B_1 & \cdots & B_{n-2} \\ B_{n-2} & B_{n-1} & B_0 & \cdots & B_{n-3} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ B_1 & B_2 & B_3 & \cdots & B_0 \end{bmatrix}. \tag{2.7}$$

Then we can give the following theorems.

Theorem 2.4. *Let B_n denote the $n-$th number.*

(1) *If $k = 1$, then B has no eigenvalues.*
(2) *If $k > 1$, then the eigenvalues of B are*

$$\lambda_j(B) = \frac{(B_{n-1}+1)w^{-j} - B_n}{w^{-2j} - O_k w^{-j} + 1}$$

for $j = 0, 1, 2, \cdots, n-1$.

Proof. (1) Let $k = 1$. Then $\alpha = \beta = 1$. So Binet's formula is invalid. Therefore B has no eigenvalues.

(2) If $k \neq 1$, then applying (2.6), we get

$$\lambda_j(B) = \sum_{k=0}^{n-1} B_k w^{-jk}$$

$$
\begin{aligned}
&= \sum_{k=0}^{n-1} \left(\frac{\alpha^k - \beta^k}{\alpha - \beta} \right) w^{-jk} \\
&= \frac{1}{\alpha - \beta} \left[\sum_{k=0}^{n-1} (\alpha w^{-j})^k - \sum_{k=0}^{n-1} (\beta w^{-j})^k \right] \\
&= \frac{1}{\alpha - \beta} \left[\frac{\alpha^n - 1}{\alpha w^{-j} - 1} - \frac{\beta^n - 1}{\beta w^{-j} - 1} \right] \\
&= \frac{1}{\alpha - \beta} \left[\frac{(\alpha^n - 1)(\beta w^{-j} - 1) - (\beta^n - 1)(\alpha w^{-j} - 1)}{(\alpha w^{-j} - 1)(\beta w^{-j} - 1)} \right] \\
&= \frac{1}{\alpha - \beta} \left[\frac{w^{-j}(\alpha^n \beta - \alpha \beta^n - \beta + \alpha) - \alpha^n + \beta^n}{\alpha \beta w^{-2j} - w^{-j}(\alpha + \beta) + 1} \right] \\
&= \frac{1}{w^{-2j} - O_k w^{-j} + 1} w^{-j} \left(\frac{\alpha^n \beta - \alpha \beta^n - \beta + \alpha}{\alpha - \beta} \right) \\
&- \frac{1}{w^{-2j} - O_k w^{-j} + 1} \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right) \\
&= \frac{(B_{n-1} + 1) w^{-j} - B_n}{w^{-2j} - O_k w^{-j} + 1}.
\end{aligned}
$$

□

The spectral norm for a matrix $A = [a_{ij}]_{n \times m}$ is defined to be

$$||A||_{spec} = \max\{\sqrt{\lambda_i}\}, \tag{2.8}$$

where λ_i are the eigenvalues of $A^H A$ for $0 \leq j \leq n-1$ and A^H denotes the conjugate transpose of A. Hence we can give the following theorem which can be proved by induction on n.

Theorem 2.5. *Let B_n denote the n−th number.*

(1) *If $k = 1$, then the spectral norm of B is*

$$||B||_{spec} = \frac{n(n-1)}{2}.$$

(2) *If $k > 1$, then the spectral norm of B is*

$$||B||_{spec} = \frac{(O_k - 1)B_{n-1} - B_{n-2} - 1}{O_k - 2}.$$

Example 2.1. 1) Let $k = 1$ and $n = 5$. Then the eigenvalues of $B_5^H B_5$ are

$$\lambda_0 = 100,\ \lambda_1 = \frac{25 + 5\sqrt{5}}{2},\ \lambda_2 = \frac{25 - 5\sqrt{5}}{2},\ \lambda_3 = \lambda_1,\ \lambda_4 = \lambda_2.$$

The spectral norm is $||B||_{spec} = \sqrt{\lambda_0} = 10$. Also $\frac{5 \cdot 4}{2} = 10$. So $||B||_{spec} = 10$.

2) Let $k = 7$ and $n = 6$. Then $O_7 = 56$, $B_4 = 175504$ and $B_5 = 9825089$. The eigenvalues of $B_6^H B_6$ are

$$\lambda_0 = 100075714326225,\ \lambda_1 = 93173941602225,$$
$$\lambda_2 = \lambda_4 = 98256156072372,\ \lambda_3 = \lambda_5 = 94808571993300.$$

The spectral norm is $||B||_{spec} = 10003785$. Also $\frac{(O_7-1)B_5-B_4-1}{O_7-2} = 10003785$. So

$$||B||_{spec} = \frac{(O_7-1)B_5-B_4-1}{O_7-2} = 10003785$$

as we claimed.

Now we set the following matrices:

$$M = M(B_n) = \begin{bmatrix} B_2 & -B_1 \\ B_1 & B_0 \end{bmatrix} = \begin{bmatrix} O_k & -1 \\ 1 & 0 \end{bmatrix} \quad \text{(companion matrix)}$$
$$W = W(B_n) = \begin{bmatrix} B_2 & B_1 \\ B_1 & B_0 \end{bmatrix} = \begin{bmatrix} O_k & 1 \\ 1 & 0 \end{bmatrix}$$
$$A = A(B_n) = \begin{bmatrix} B_1 & B_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Then we can give the following theorem.

Theorem 2.6. *Let O_k denote the $k-$th oblong number and B_n denote the $n-$th B_n number. Then*

(1)
$$M^n = \begin{bmatrix} B_{n+1} & -B_n \\ B_n & -B_{n-1} \end{bmatrix}$$
for $n \geq 1$.

(2) $B_{n+1} = AM^nA^t$ *for $n \geq 1$.*

(3) $B_n = AM^{n-2}WA^t$ *for $n \geq 2$.*

(4)
$$M^{n-1}W = \begin{bmatrix} B_{n+1} & B_n \\ B_n & B_{n-1} \end{bmatrix}$$
for $n \geq 1$.

(5) *If $n \geq 3$ is odd, then*

$$W^n = \begin{bmatrix} \sum_{i=0}^{\frac{n-1}{2}} C(n-i,i)O_k^{n-2i} & \sum_{i=0}^{\frac{n-1}{2}} C(n-1-i,i)O_k^{n-1-2i} \\ \sum_{i=0}^{\frac{n-1}{2}} C(n-1-i,i)O_k^{n-1-2i} & \sum_{i=0}^{\frac{n-3}{2}} C(n-2-i,i)O_k^{n-2-2i} \end{bmatrix}$$

and if $n \geq 2$ is even, then

$$W^n = \begin{bmatrix} \sum_{i=0}^{\frac{n}{2}} C(n-i,i)O_k^{n-2i} & \sum_{i=0}^{\frac{n-2}{2}} C(n-1-i,i)O_k^{n-1-2i} \\ \sum_{i=0}^{\frac{n-2}{2}} C(n-1-i,i)O_k^{n-1-2i} & \sum_{i=0}^{\frac{n-2}{2}} C(n-2-i,i)O_k^{n-2-2i} \end{bmatrix},$$

here $C(n,i)$ denotes the binomial coefficient.

Proof. (1) We prove it by induction on n. Let $n = 1$. Then

$$M = \begin{bmatrix} O_k & -1 \\ 1 & 0 \end{bmatrix}.$$

So it is true for $n = 1$. Let us assume that this relation is satisfied for $n - 1$, that is,

$$M^{n-1} = \begin{bmatrix} B_n & -B_{n-1} \\ B_{n-1} & -B_{n-2} \end{bmatrix}.$$

We will show that this relation is satisfied for n. Since $M^n = M^{n-1} \cdot M$, we get

$$M^n = \begin{bmatrix} O_kB_n - B_{n-1} & -B_n \\ O_kB_{n-1} - B_{n-2} & -B_{n-1} \end{bmatrix} = \begin{bmatrix} B_{n+1} & -B_n \\ B_n & -B_{n-1} \end{bmatrix}.$$

(2)

$$\begin{aligned} AM^nA^t &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} B_{n+1} & -B_{n+1} \\ B_n & -B_{n-1} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} B_{n+1} \\ B_n \end{bmatrix} \\ &= B_{n+1}. \end{aligned}$$

(3)

$$\begin{aligned} AM^{n-2}WA^t &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} B_{n-1} & -B_{n-2} \\ B_{n-2} & -B_{n-3} \end{bmatrix} \begin{bmatrix} O_k & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} B_{n-1} & -B_{n-2} \\ B_{n-2} & -B_{n-3} \end{bmatrix} \begin{bmatrix} O_k \\ 1 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} O_kB_{n-1} - B_{n-2} \\ O_kB_{n-2} - B_{n-3} \end{bmatrix}$$
$$= O_kB_{n-1} - B_{n-2}$$
$$= B_n.$$

(4)

$$M^{n-1}W = \begin{bmatrix} B_n & -B_{n-1} \\ B_{n-1} & -B_{n-2} \end{bmatrix} \begin{bmatrix} O_k & 1 \\ 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} O_kB_n - B_{n-1} & B_n \\ O_kB_{n-1} - B_{n-2} & B_{n-1} \end{bmatrix}$$
$$= \begin{bmatrix} B_{n+1} & B_n \\ B_n & B_{n-1} \end{bmatrix}.$$

(5) Let n be odd and let $n = 3$. Then

$$\sum_{i=0}^{1} C(3-i,i)O_k^{3-2i} = O_k^3 + 2O_k$$
$$\sum_{i=0}^{1} C(2-i,i)O_k^{2-2i} = O_k^2 + 1$$
$$\sum_{i=0}^{0} C(1-i,i)O_k^{1-2i} = O_k.$$

Hence

$$W^3 = \begin{bmatrix} O_k^3 + 2O_k & O_k^2 + 1 \\ O_k^2 + 1 & O_k \end{bmatrix}.$$

Also by simple calculation, we see that

$$W^3 = \begin{bmatrix} O_k^3 + 2O_k & O_k^2 + 1 \\ O_k^2 + 1 & O_k \end{bmatrix}.$$

So it is true for $n = 3$. Let us assume that this relation is satisfied for $n - 2$, that is,

$$W^{n-2} = \begin{bmatrix} \sum_{i=0}^{\frac{n-3}{2}} C(n-2-i,i)O_k^{n-2-2i} & \sum_{i=0}^{\frac{n-3}{2}} C(n-3-i,i)O_k^{n-3-2i} \\ \sum_{i=0}^{\frac{n-3}{2}} C(n-3-i,i)O_k^{n-3-2i} & \sum_{i=0}^{\frac{n-5}{2}} C(n-4-i,i)O_k^{n-4-2i} \end{bmatrix}.$$

Since $W^n = W^{n-2}W^2$, we set

$$W^n = \begin{bmatrix} W_{11}^n & W_{12}^n \\ W_{21}^n & W_{22}^n \end{bmatrix}.$$

Hence we easily deduce that

$$\begin{aligned}
W_{11}^n &= (O_k^2+1)\left(\sum_{i=0}^{\frac{n-3}{2}} C(n-2-i,i)O_k^{n-2-2i}\right) + O_k\left(\sum_{i=0}^{\frac{n-3}{2}} C(n-3-i,i)O_k^{n-3-2i}\right)\\
&= (O_k^2+1)\left(\begin{array}{c} O_k^{n-2} + C(n-3,1)O_k^{n-4} + C(n-4,2)O_k^{n-6} + \cdots \\ +C(\frac{n-1}{2},\frac{n-5}{2})O_k^3 + C(\frac{n-1}{2},\frac{n-3}{2})O_k \end{array}\right)\\
&+ O_k\left(\begin{array}{c} O_k^{n-3} + C(n-4,1)O_k^{n-5} + C(n-5,2)O_k^{n-7} + \cdots \\ +C(\frac{n-1}{2},\frac{n-5}{2})O_k^2 + C(\frac{n-3}{2},\frac{n-3}{2})O_k^0 \end{array}\right)\\
&= O_k^n + [C(n-3,1)+2]\,O_k^{n-2} + [C(n-4,2)+C(n-3,1)+C(n-4,1)]\,O_k^{n-4}\\
&+ \cdots + \left[C(\frac{n-1}{2},\frac{n-3}{2}) + C(\frac{n+1}{2},\frac{n-5}{2}) + C(\frac{n-1}{2},\frac{n-5}{2})\right] O_k^3\\
&+ \left[C(\frac{n-1}{2},\frac{n-3}{2}) + C(\frac{n-3}{2},\frac{n-3}{2})\right] O_k\\
&= O_k^n + C(n-1,1)O_k^{n-2} + C(n-2,2)O_k^{n-4} + \cdots + C(\frac{n+1}{2},\frac{n-1}{2})O_k\\
&= \sum_{i=0}^{\frac{n-1}{2}} C(n-i,i)O_k^{n-2i}.
\end{aligned}$$

Similarly it can be shown that

$$\begin{aligned}
W_{12}^n &= O_k\left(\sum_{i=0}^{\frac{n-3}{2}} C(n-2-i,i)O_k^{n-2-2i}\right) + \left(\sum_{i=0}^{\frac{n-3}{2}} C(n-3-i,i)O_k^{n-3-2i}\right)\\
&= \sum_{i=0}^{\frac{n-1}{2}} C(n-1-i,i)O_k^{n-1-2i}\\
W_{21}^n &= (O_k^2+1)\left(\sum_{i=0}^{\frac{n-3}{2}} C(n-3-i,i)O_k^{n-3-2i}\right) + O_k\left(\sum_{i=0}^{\frac{n-5}{2}} C(n-4-i,i)O_k^{n-4-2i}\right)\\
&= \sum_{i=0}^{\frac{n-1}{2}} C(n-1-i,i)O_k^{n-1-2i}\\
W_{22}^n &= O_k\left(\sum_{i=0}^{\frac{n-3}{2}} C(n-3-i,i)O_k^{n-3-2i}\right) + \left(\sum_{i=0}^{\frac{n-5}{2}} C(n-4-i,i)O_k^{n-4-2i}\right)\\
&= \sum_{i=0}^{\frac{n-3}{2}} C(n-2-i,i)O_k^{n-2-2i}.
\end{aligned}$$

So it is true for n. The other case can be proved similarly. □

2.1. Simple Continued Fraction Expansion of B_n Numbers. In this section, we want to consider the continued fraction expansion of the ratio of two B_n numbers.

Theorem 2.7. *Let B_n denote the n^{th} number.*

(1) *If $k = 1$, then*

$$\frac{B_{n+1}}{B_n} = \begin{cases} [2] & \text{for } n = 1 \\ [1; n] & \text{for } n > 1 \end{cases}$$

$$\frac{B_{2n+1}}{B_{2n-1}} = \begin{cases} [3] & \text{for } n = 1 \\ [1; n-1, 2] & \text{for } n > 1 \end{cases}$$

$$\frac{B_{2n}}{B_{2n-2}} = \begin{cases} [2] & \text{for } n = 2 \\ [1; n-1] & \text{for } n > 2. \end{cases}$$

(2) *If $k > 1$, then*

$$\frac{B_{n+1}}{B_n} = \begin{cases} [O_k] & \text{for } n = 1 \\ [O_k - 1; \underbrace{1, O_k - 2}_{n-2 \text{ times}}, 1, O_k - 1] & \text{for } n > 1 \end{cases}$$

$$\frac{B_{2n+1}}{B_{2n-1}} = \begin{cases} [O_k^2 - 1] & \text{for } n = 1 \\ [O_k^2 - 3; \underbrace{1, O_k^2 - 4}_{n-2 \text{ times}}, 1, O_k^2 - 2] & \text{for } n > 1 \end{cases}$$

$$\frac{B_{2n}}{B_{2n-2}} = \begin{cases} [O_k^2 - 2] & \text{for } n = 2 \\ [O_k^2 - 3; \underbrace{1, O_k^2 - 4}_{n-3 \text{ times}}, 1, O_k^2 - 3] & \text{for } n > 2. \end{cases}$$

Proof. (1) Let $k = 1$. Then $B_n = 2B_{n-1} - B_{n-2}$ and hence $B_n = n$. So

$$[1; n] = 1 + \frac{1}{n} = \frac{n+1}{n} = \frac{B_{n+1}}{B_n}$$

$$[1; n-1, 2] = 1 + \frac{1}{n - 1 + \frac{1}{2}} = \frac{2n+1}{2n-1} = \frac{B_{2n+1}}{B_{2n-1}}$$

and

$$[1; n-1] = 1 + \frac{1}{n-1} = \frac{n}{n-1} = \frac{2n}{2n-2} = \frac{B_{2n}}{B_{2n-2}}.$$

(2) Let $k > 1$. If $n = 1$. Then $B_2 = O_k$ and $B_1 = 1$. So $\frac{B_2}{B_1} = [O_k]$. Let us assume that is satisfied for $n - 1$, that is,

$$\frac{B_n}{B_{n-1}} = [O_k - 1; \underbrace{1, O_k - 2}_{n-3 \text{ times}}, 1, O_k - 1].$$

Hence

$$[O_k-1;\underbrace{1,O_k-2,1,O_k-1}_{n-2 \text{ times}}]=O_k-1+\cfrac{1}{1+\cfrac{1}{O_k-2+\cfrac{1}{\cfrac{1+\cdots}{O_k-2+\cfrac{1}{1+\frac{1}{O_k-1}}}}}}$$

$$=O_k-1+\cfrac{1}{1+\cfrac{1}{-1+O_k-1+\cfrac{1}{\cfrac{1+}{\cfrac{\cdots}{O_k-2+\cfrac{1}{1+\frac{1}{O_k-1}}}}}}}$$

$$=O_k-1+\cfrac{1}{1+\cfrac{1}{-1+\frac{B_n}{B_{n-1}}}}$$

$$=\frac{B_{n+1}}{B_n}.$$

The other cases can be proved similarly. □

3. Balancing Numbers.

Recently, Behera and Panda [1] introduced balancing numbers $n \in \mathbb{Z}^+$ as solutions of the Diophantine equation

$$1+2+\cdots+(n-1)=(n+1)+(n+2)+\cdots+(n+r) \tag{3.1}$$

for some positive integer r which is called balancer or cobalancing number. If n is a balancing number with balancer r, then from (3.1) one has

$$r=\frac{-(2n+1)+\sqrt{8n^2+1}}{2} \quad \text{and} \quad n=\frac{2r+1+\sqrt{8r^2+8r+1}}{2}. \tag{3.2}$$

Let B_n denote the n^{th} balancing number and let b_n denote the n^{th} cobalancing number. Then they satisfy the recurrence relation $B_{n+1}=6B_n-B_{n-1}$ and $b_{n+1}=6b_n-b_{n-1}+2$ for $n \geq 2$, with initial values $B_1=1, B_2=6$ and $b_1=0, b_2=2$. From (3.2), we see that B_n is a balancing number iff $8B_n^2+1$ is a perfect square and b_n is a cobalancing number iff $8b_n^2+8b_n+1$ is a perfect square. So we set $C_n=\sqrt{8B_n^2+1}$ and $c_n=\sqrt{8b_n^2+8b_n+1}$ which are called the n^{th} Lucas–balancing number and n^{th} Lucas–cobalancing number, respectively (for further details see [14, 15, 16, 17, 18]).

Using the recurrence relation, Panda and Behera derived the following results.

Theorem 3.1. [1, Theorem 2.1] *For any balancing number x, the functions*

$$F(x)=2x\sqrt{8x^2+1},\quad G(x)=3x+\sqrt{8x^2+1} \text{ and } H(x)=17x+\sqrt{8x^2+1}$$

are also balancing numbers.

Applying the above theorem, one can say that $F(x)$ is always even, whereas $G(x)$ is even when x is odd and $G(x)$ is odd when x is even. Further

Theorem 3.2. [1, Theorem 4.1] *If x and y are balancing numbers, then*

$$f(x,y) = x\sqrt{8y^2+1} + y\sqrt{8x^2+1}$$

is also balancing number.

The recurrence relation on balancing number is given below.

Theorem 3.3. [1, Theorem 5.1]

$$\begin{aligned} B_{n+1}B_{n-1} &= (B_n+1)(B_n-1) \\ B_n &= B_kB_{n-k} - B_{k-1}B_{n-k-1} \text{ for any positive integer } k < n \\ B_{2n} &= B_n^2 - B_{n-1}^2 \\ B_{2n+1} &= B_n(B_{n+1} - B_{n-1}). \end{aligned}$$

Like in Fibonacci numbers, $F_{m+k}F_{m-k} = F_m^2 - (-1)^{m+k}F_k^2$, the balancing numbers satisfy the following property.

Theorem 3.4. [15, Theorem 2.1] *If m and k are natural numbers and $k < m$, then $(B_m + B_k)(B_m - B_k) = B_{m+k}B_{m-k}$.*

Also,

Theorem 3.5. [15, Theorem 2.2] *If B_m is the $m-th$ balancing number, then*

$$\begin{aligned} B_1 + B_3 + \cdots + B_{2m-1} &= B_m^2 \\ B_2 + B_4 + \cdots + B_{2m} &= B_mB_{m+1} \\ B_1 + B_2 + \cdots + B_{2m} &= B_m(B_m + B_{m+1}). \end{aligned}$$

Theorem 3.6. [1, Theorem 6.1] *The generating function of the sequence B_n of balancing numbers is*

$$g(s) = \frac{s}{1-6s+s^2}$$

and consequently

$$B_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k C(n-k,k) 6^{n-2k}.$$

Theorem 3.7. [1, Theorem 8.1] *If B_n is the $n-th$ balancing number, then*

$$\lim_{n\to\infty} \frac{B_{n+1}}{B_n} = 3 + \sqrt{8}.$$

There is a connection between the sequence $\{R_n\}_{n=0}^{\infty}$ (which is called a second order linear recurrence if the recurrence relation

$$R_n = AR_{n-1} + BR_{n-2}$$

for $n \geq 1$ holds for its terms, where $A, B \neq 0$, R_0 and R_1 are fixed rational integers and $|R_0| + |R_1| > 0$) and the Pell equation $x^2 - dy^2 = 1$ which is given below.

Theorem 3.8. [10, Theorem 1] *The terms of the second order linear recurrence* $R(6, -1, 1, 6)$ *are the solutions of the equation* $x^2 - 8y^2 = 1$ *for some integer* y.

Recall that in the previous section, we derived some algebraic identities for integer sequences involving the oblong numbers O_k. If we take $k = 2$ in that sequence, then we get the sequence of balancing numbers $\{B_n\}_{n=0}^{\infty}$, that is, $B_0 = 0$, $B_1 = 1$ and

$$B_n = 6B_{n-1} - B_{n-2} \tag{3.3}$$

for $n \geq 2$ since $O_2 = 6$. Consequently, all results obtained in the previous section are valid for $k = 2$. Thus we can give the following result without giving its proof.

Theorem 3.9. *Let* B_n *denote the* $n-th$ *balancing number. Then*

(1) *The sum of balancing numbers from* 1 *to* n *is*

$$\sum_{i=1}^{n} B_i = \frac{5B_n - B_{n-1} - 1}{4}.$$

(2) $B_{2n} = 34B_{2n-2} - B_{2n-4}$ *and* $B_{2n+1} = 34B_{2n-1} - B_{2n-3}$ *for* $n \geq 2$.

(3)

$$\alpha^n + \beta^n = \begin{cases} B_{n+1} - B_{n-1} & \text{for } n \geq 1 \\ 2B_{n+1} - 6B_n & \text{for } n \geq 0 \\ 6B_n - 2B_{n-1} & \text{for } n \geq 0, \end{cases}$$

where $\alpha = 3 + \sqrt{8}$ *and* $\beta = 3 - \sqrt{8}$.

(4) *The eigenvalues of* B *are*

$$\lambda_j(B) = \frac{(B_{n-1} + 1)w^{-j} - B_n}{w^{-2j} - 6w^{-j} + 1}$$

for $j = 0, 1, 2, \cdots, n-1$.

(5) *The spectral norm of* B *is*

$$||B||_{spec} = \frac{5B_{n-1} - B_{n-2} - 1}{4}.$$

(6) *For the matrices,*

$$M = \begin{bmatrix} 6 & -1 \\ 1 & 0 \end{bmatrix}, W = \begin{bmatrix} 6 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } A = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

we have

i)

$$M^n = \begin{bmatrix} B_{n+1} & -B_n \\ B_n & -B_{n-1} \end{bmatrix}$$

for $n \geq 1$.

ii) $B_{n+1} = AM^nA^t$ *for* $n \geq 1$.

iii) $B_n = AM^{n-2}WA^t$ *for* $n \geq 2$.

iv)

$$M^{n-1}W = \begin{bmatrix} B_{n+1} & B_n \\ B_n & B_{n-1} \end{bmatrix}$$

for $n \geq 1$.

v) *If* $n \geq 3$ *is odd, then*

$$W^n = \begin{bmatrix} \sum_{i=0}^{\frac{n-1}{2}} C(n-i,i)6^{n-2i} & \sum_{i=0}^{\frac{n-1}{2}} C(n-1-i,i)6^{n-1-2i} \\ \sum_{i=0}^{\frac{n-1}{2}} C(n-1-i,i)6^{n-1-2i} & \sum_{i=0}^{\frac{n-3}{2}} C(n-2-i,i)6^{n-2-2i} \end{bmatrix}$$

and if $n \geq 2$ *is even, then*

$$W^n = \begin{bmatrix} \sum_{i=0}^{\frac{n}{2}} C(n-i,i)6^{n-2i} & \sum_{i=0}^{\frac{n-2}{2}} C(n-1-i,i)6^{n-1-2i} \\ \sum_{i=0}^{\frac{n-2}{2}} C(n-1-i,i)6^{n-1-2i} & \sum_{i=0}^{\frac{n-2}{2}} C(n-2-i,i)6^{n-2-2i} \end{bmatrix}.$$

(7)

$$\frac{B_{n+1}}{B_n} = \begin{cases} [6] & \text{for } n = 1 \\ [5; \underbrace{1,4,}_{n-2 \text{ times}} 1,5] & \text{for } n > 1 \end{cases}$$

$$\frac{B_{2n+1}}{B_{2n-1}} = \begin{cases} [35] & \text{for } n = 1 \\ [33; \underbrace{1,32,}_{n-2 \text{ times}} 1,34] & \text{for } n > 1 \end{cases}$$

$$\frac{B_{2n}}{B_{2n-2}} = \begin{cases} [34] & \text{for } n = 2 \\ [33; \underbrace{1,32,}_{n-3 \text{ times}} 1,33] & \text{for } n > 2. \end{cases}$$

The rank of an integer N is defined to be

$$\rho(N) = \begin{cases} p & \text{if } p \text{ is the smallest prime with } p|N \\ \infty & \text{if } N \text{ is prime.} \end{cases}$$

Thus we can give the following theorem.

Theorem 3.10. *The rank of B_n is*

$$\rho(B_n) = \begin{cases} 2 & \text{if } n = 2t+2 \\ 5 & \text{if } n = 6t+3 \\ 13 & \text{if } n = 42t+7 \text{ or } n = 42t+35 \end{cases}$$

for every $t \geq 0$.

Proof. Let $n = 2t+2$. We prove it by induction. Let $t = 0$ then $B_2 = 6$, so $\rho(B_2) = 2$. Let us assume that B_n is 2 for $n = t-1$, that is, $\rho(B_{2t}) = 2$. So

$$B_{2(t-1)+2} = B_{2t} = 2^a \cdot u_1$$

for some integers $a \geq 1$ and $u_1 > 0$. For $n = t$, we get

$$B_{2t+2} = 6B_{2t+1} - B_{2t} = 6B_{2t+1} - 2^a \cdot u_1 = 2(3B_{2t+1} - 2^{a-1} \cdot u_1).$$

Consequently, we get $\rho(B_{2t+2}) = 2$.

Let $n = 6t+3$ and $t = 0$ then $B_3 = 35$, so $\rho(B_3) = 5$. Let us assume that B_n is 5 for $n = t-1$, that is,

$$B_{3+6(t-1)} = B_{3+6t-6} = B_{6t-3} = 5^b \cdot u_2$$

for some integers $b \geq 1$ and $u_2 > 0$. For $n = t$, we get

$$\begin{aligned} B_{6t+3} &= 6B_{6t+2} - B_{6t+1} \\ &= 6(6B_{6t+1} - B_{6t}) - B_{6t+1} \\ &= 36B_{6t+1} - 6B_{6t} - B_{6t+1} \\ &= 35B_{6t+1} - 5B_{6t} - B_{6t} \\ &= 35B_{6t+1} - 5B_{6t} - 6B_{6t-1} + B_{6t-2} \\ &= 35B_{6t+1} - 5B_{6t} - 6(6B_{6t-2} + B_{6t-3}) + B_{6t-2} \\ &= 35B_{6t+1} - 5B_{6t} - 35B_{6t-2} + 6B_{6t-3} \\ &= 5(7B_{6t+1} - B_{6t} - 7B_{6t-2} + 65^{b-1} \cdot u_2). \end{aligned}$$

So $\rho(B_{6t+3}) = 5$. The other assertion can be proved similarly. □

Finally we can give the following theorem related to the limit of cross–ratio of four consecutive balancing numbers B_n, B_{n+1}, B_{n+2} and B_{n+3}.

Theorem 3.11. *Let B_n, B_{n+1}, B_{n+2} and B_{n+3} be four consecutive balancing numbers. Then*

$$\lim_{n\to\infty}[B_n, B_{n+1}; B_{n+2}, B_{n+3}] = \frac{8}{7}.$$

Proof. Recall that the cross–ratio of four numbers a, b, c and d is defined to be

$$[a, b; c, d] = \frac{(a-c)(b-d)}{(b-c)(a-d)}.$$

So the cross–ratio of four consecutive B_n, B_{n+1}, B_{n+2} and B_{n+3} numbers is hence

$$[B_n, B_{n+1}; B_{n+2}, B_{n+3}] = \frac{(B_n - B_{n+2})(B_{n+1} - B_{n+3})}{(B_{n+1} - B_{n+2})(B_n - B_{n+3})}. \tag{3.4}$$

Since $B_n = 6B_{n-1} - B_{n-2}$, we get $B_{n+2} = 6B_{n+1} - B_n$ and $B_{n+3} = 35B_{n+1} - 6B_n$. So

$$\begin{aligned} B_n - B_{n+2} &= -6B_{n+1} + 2B_n \\ B_{n+1} - B_{n+3} &= -34B_{n+1} + 6B_n \\ B_{n+1} - B_{n+2} &= -5B_{n+1} + B_n \\ B_n - B_{n+3} &= -35B_{n+1} + 6B_n. \end{aligned}$$

Therefore (3.4) becomes

$$[B_n, B_{n+1}; B_{n+2}, B_{n+3}] = \frac{(-6B_{n+1} + 2B_n)(-34B_{n+1} + 6B_n)}{(-5B_{n+1} + B_n)(-35B_{n+1} + 6B_n)}$$

and clearly we conclude that

$$\lim_{n\to\infty}[B_n, B_{n+1}; B_{n+2}, B_{n+3}] = \frac{8}{7}$$

as we wanted. □

Using these symmetries, we can give the following result.

Corollary 3.12. *Let B_n, B_{n+1}, B_{n+2} and B_{n+3} be four consecutive balancing numbers. Then*

$$\begin{aligned} \lim_{n\to\infty}[B_n, B_{n+1}; B_{n+3}, B_{n+2}] &= \frac{7}{8} \\ \lim_{n\to\infty}[B_n, B_{n+2}; B_{n+3}, B_{n+1}] &= -7 \\ \lim_{n\to\infty}[B_n, B_{n+2}; B_{n+1}, B_{n+3}] &= \frac{-1}{7} \\ \lim_{n\to\infty}[B_n, B_{n+3}; B_{n+2}, B_{n+1}] &= 8 \\ \lim_{n\to\infty}[B_n, B_{n+3}; B_{n+1}, B_{n+2}] &= \frac{1}{8}. \end{aligned}$$

The following two theorems are from [21] related to the integer solutions of the Pell equation

$$x^2 - 6y^2 = 1$$

depending on the balancing numbers B_1 and B_2.

Theorem 3.13. [21, Theorem 2.1] *For the Pell equation $x^2 - 6y^2 = 1$;*

(1) *The continued fraction expansion of $\sqrt{6}$ is*

$$\sqrt{6} = \left[\frac{\sqrt{4B_2+1}-1}{2}; \overline{2B_1, \sqrt{4B_2+1}-1}\right].$$

(2) *The fundamental solution is*

$$(x_1, y_1) = (\sqrt{4B_2+1}, 2B_1)$$

and the other solutions are (x_n, y_n), where

$$\frac{x_n}{y_n} = \left[\frac{\sqrt{4B_2+1}-1}{2}; \underbrace{2B_1, \sqrt{4B_2+1}-1}_{n-1 \text{ times}}, 2B_1\right]$$

for $n \geq 2$.

(3) *The solutions satisfy the recurrence relations*

$$x_n = \sqrt{4B_2+1}x_{n-1} + 2B_2y_{n-1}$$
$$y_n = 2B_1x_{n-1} + \sqrt{4B_2+1}y_{n-1}$$

for $n \geq 2$ and

$$x_n = (2\sqrt{4B_2+1} - 1)(x_{n-1} + x_{n-2}) - x_{n-3}$$
$$y_n = (2\sqrt{4B_2+1} - 1)(y_{n-1} + y_{n-2}) - y_{n-3}$$

for $n \geq 4$.

Theorem 3.14. [21, Theorem 2.3] *The integer solutions of $x^2 - 6y^2 = 1$ are (x_n, y_n), where*

$$x_n = \begin{cases} \sum_{i=0}^{\frac{n}{2}} C(n, 2i)(\sqrt{4B_2+1})^{n-2i}2^{i+1}(B_1B_2)^i & \text{if } n \text{ is even} \\ \sum_{i=0}^{\frac{n-1}{2}} C(n, 2i)(\sqrt{4B_2+1})^{n-2i}2^{i+1}(B_1B_2)^i & \text{if } n \text{ is odd} \end{cases}$$

and

$$y_n = \begin{cases} \sum_{i=0}^{\frac{n-2}{2}} C(n,2i+1)(\sqrt{4B_2+1})^{n-1-2i}2^{i+1}B_1^{i+1}B_2^{i} & \textit{if } n \textit{ is even} \\ \sum_{i=0}^{\frac{n-1}{2}} C(n,2i+1)(\sqrt{4B_2+1})^{n-1-2i}2^{i+1}B_1^{i+1}B_2^{i} & \textit{if } n \textit{ is odd} \end{cases}$$

for $n \geq 2$.

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