A Note On n-Connected Splitting-Off Matroids

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Abstract. The splitting-off operation has important applications for graph connectivity problems. Shikare, Dalvi, and Dhotre [*splitting-off operation for binary matroids and its applications, Graphs and Combinatorics, 27(6) (2011), 871-882*] extended this operation to binary matroids. In this paper, we provide a sufficient condition for preserving n-connectedness of a binary matroid under splitting-off operation.

Keywords: binary matroid, connected matroid, splitting-off, cocircuit

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1. Introduction

For notations and undefined concepts, we refer to Oxley [10]. Throughout this paper, we consider only loopless graphs and loopless matroids with at least three elements. Let G be a graph and let $x = vv_1$ and $y = vv_2$ be two edges of G. We denote by G_{xy} the graph obtained from G by deleting x, y and adding the new edge $a = v_1v_2$. The transition from G to G_{xy} is called a splitting-off operation.

The splitting-off operation has important applications to connectivity problems. Lovász [6] proved that if a graph $G = (V \cup s, E)$ is k-edge connected in V ($k \geq 2$) and $d(s)$ is even, then given an edge su there exists an edge sv such that splitting-off the pair su, sv maintains the k-edge connectedness. For applications of splitting-off operation for graphs see Lovász [6], Mader [7], Frank [4] and [5], Nagamochi, Nishimura, and Ibaraki [8]. This operation is extended to binary matroids by Shikare, Dhotre, and Dalvi [13] as follows.

Definition 1.1. Let M be a binary matroid represented by a matrix A over $GF(2)$. Let $x, y \in E(M)$. We denote by A_{xy} the matrix obtained from A as follows: if $\{x, y\}$ is a 2-circuit in M, then delete columns labeled by x, y; otherwise add a new column labeled by a which is the sum of columns of x, y over $GF(2)$ and then delete the columns of x and y. Denote by M_{xy} the matroid represented by the matrix A_{xy}. The transition from M to M_{xy} is called a splitting-off operation. The ground set of the matroid M_{xy} is $E(M) - \{x, y\}$ if $\{x, y\}$ is a 2-circuit of M, otherwise it is $(E(M) - \{x, y\}) \cup \{a\}$.

The circuits of the splitting-off matroid are characterized in [13] as below.

Lemma 1.2 [13]. *Let M be a binary matroid with the ground set $E(M)$ and the collection of circuits $\mathcal{C}$. Suppose $\{x,y\} \subset E(M)$ is not a 2-circuit of M and $a \in E(M_{xy}) - E(M)$. Let*
$\mathcal{C}_0 = \{C \in \mathcal{C} : C$ *contains neither of x and y* $\}$,
$\mathcal{C}_1 = \{(C - \{x,y\}) \cup \{a\} : C \in \mathcal{C}$ *and* $x, y \in C\}$, *and*
$\mathcal{C}_2 = \{(C_1 \cup C_2 - \{x,y\}) \cup \{a\} : C_1,\ C_2 \in \mathcal{C}, C_1 \cap C_2 = \phi,\ x \in C_1,\ y \in C_2$ *and $C_1 \cup C_2$ contains no circuit of M containing both x and y, or neither of them* $\}$.
Then the circuit collection of the matroid M_{xy} is $\mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2$.

The splitting-off operation is closely related to the *splitting operation* which is defined by Raghunathan et al. [11] for binary matroids as a natural extension of the corresponding operation for graphs introduced by Fleischner [3].

Definition 1.3. Let M be a binary matroid represented by a matrix A over $GF(2)$ and let $x, y \in E(M)$. Denote by $A_{x,y}$ the matrix obtained by adjoining an extra row to A with this row being zero everywhere except in the columns corresponding to x and y, where it takes the value 1. Let $M_{x,y}$ be the matroid represented by $A_{x,y}$. We say that $M_{x,y}$ is obtained from M by splitting away the elements x and y.

The splitting operation is well studied in [1], [2], [9], [11], [12] and [14]. The two matroids M_{xy} and $M_{x,y}$ are obviously related to each other as follows.

Lemma 1.4. *Let M be a binary matroid and let $x, y \in E(M)$. Then $M_{xy} = M \setminus \{x,y\} = M_{x,y} \setminus \{x,y\}$ if $\{x,y\}$ is a 2-circuit of M, otherwise $M_{xy} \cong M_{x,y}/x \cong M_{x,y}/y$.*

In what follows we assume that n is an integer greater than 1. We obtain a sufficient condition to preserve n-connectedness of a matroid under splitting-off operation. The following result is the main theorem of this paper.

Main Theorem 1.5. *Let M be an n-connected, vertically $(n+1)$-connected binary matroid with $|E(M)| \geq 2n-1$ and $x, y \in E(M)$. Suppose every circuit of M containing x, y has size at least $n+1$, and every cocircuit containing x, y has size at least $n+2$ and further, such a cocircuit does not contain an n-circuit. Then the splitting-off matroid M_{xy} is n-connected.*

Since every k-connected matroid is vertically k-connected, the next result follows immediately.

Corollary 1.6. *Let M be an $(n+1)$-connected binary matroid with $|E(M)| \geq 2n-1$ and let $x, y \in E(M)$. Suppose every cocircuit in M containing x, y has size at least $n+2$. Then the matroid M_{xy} is n-connected.*

The special case for $n = 3$ of the above corollary is proved in [13]. Also, a similar result for $n = 2$ is obtained by Borse and Dhotre [2] for splitting matroid $M_{x,y}$.

We prove the main theorem in the second section, and discuss the sharpness and other consequences of this theorem in the last section.

2. Proof of the Main Theorem

We need the following well-known results.

Lemma 2.1 [10, pp 75]. *Let M be a matroid and let Q be a cocircuit of M. Then Q is a nonempty subset of $E(M)$ such that $|C \cap Q| \neq 1$ for each circuit C of M.*

Lemma 2.2 [10, pp 273]. *If M is an n-connected matroid with $|E(M)| \geq 2(n-1)$, then all circuits and all cocircuits of M have at least n elements.*

Lemma 2.3 [10, pp 275]. *Let (X, Y) be a k-separation of a k-connected matroid and suppose $|X| = k$. Then X is either a coindependent circuit or an independent cocircuit.*

The next two results are related to the rank function of the matroid M_{xy}.

Lemma 2.4 [13]. *Let M be a binary matroid and let $\{x, y\}$ be an independent set in M. Suppose r and r' are the rank functions of the matroids M and M_{xy}, respectively. Then, for $X \subseteq E(M_{xy})$*

$$r'(X) = \begin{cases} r(X) & \text{if } a \text{ does not belong to } X \\ r(X-a) & \text{if } a \text{ is not a coloop of } M_{xy}|X \\ r(X-a)+1 & \text{if } a \text{ is a coloop of } M_{xy}|X \end{cases}$$

Lemma 2.5 [13]. *Let M be a binary matroid and let $x, y \in E(M)$. Then $r(M) = r'(M_{xy})$ if $\{x, y\}$ does not contain a cocircuit of M.*

A *hyperplane* of a matroid M is a flat of rank $r(M) - 1$. By [10, Proposition 2.1.6], a subset Y of $E(M)$ is a hyperplane if and only if $E(M) - Y$ is a cociruit in M. The next lemma follows immediately.

Lemma 2.6. *Let M be a matroid and $X \subset E(M)$ such that $r(M \setminus X) = r(M) - 1$. Then X contains a cocircuit of M.*

We need some properties of cocircuits of M_{xy} which can be obtained from the corresponding properties of cocircuits of the splitting matroid $M_{x,y}$ due to Mills [9]. The following lemma is a consequence of Theorems 2.7 and 2.8 of [9].

Lemma 2.7. *Let M be a binary matroid and let x, y be elements of M such that $\{x, y\}$ does not contain a cocircuit of M. Then*

(i) $\{x, y\}$ *is a cocircuit of* $M_{x,y}$*;*
(ii) if Q *is a cocircuit of* M *with* $\{x, y\} \subset Q$*, then* $Q - \{x, y\}$ *is a cocircuit of* $M_{x,y}$*;*
(iii) if Q' *is a cocircuit of* $M_{x,y}$ *with* $Q' \cap \{x, y\} = \phi$ *such that* Q' *does not contain a cocircuit of* M*, then* $Q' \cup \{x, y\}$ *is a cocircuit of* M*, or* Q' *is union of two disjoint cocircuits of* M *each containing* x *or* y*.*

Lemma 2.8. *Let* M *be a binary matroid and let* x, y *be elements of* M *such that* $\{x, y\}$ *does not contain a cocircuit of* M*. Then*
(i) if Q *is a cocircuit of* M *with* $\{x, y\} \subset Q$*, then* $Q - \{x, y\}$ *is a cocircuit of* M_{xy}*;*
(ii) if Q' *is a cocircuit of* M_{xy} *with* $a \notin Q'$ *such that* Q' *does not contain a cocircuit of* M*, then* $Q' \cup \{x, y\}$ *is a cocircuit of* M*, or* Q' *is union of two disjoint cocircuits of* M *each containing* x *or* y*.*

Proof. (i). By Lemma 2.7(i), $\{x, y\}$ is a 2-cocircuit of $M_{x,y}$. Suppose Q is a cocircuit of M with $\{x, y\} \subset Q$. If $\{x, y\}$ is a 2-circuit of M, then $M_{xy} = M \setminus \{x, y\}$ and hence $Q - \{x, y\}$ is a cocircuit of M_{xy}. Suppose $\{x, y\}$ is not a 2-circuit of M. Then, by Lemma 1.4, $M_{xy} \cong M_{x,y}/x \cong M_{x,y}/y$. Therefore, by Lemma 2.7(ii), $Q - \{x, y\}$ is a cocircuit of M_{xy}.

(ii). Suppose Q' is a cocircuit of M_{xy} with $a \notin Q'$. Then $Q' \subset E(M) - \{x, y\}$. We prove that Q' is a cocircuit of the matroid $M_{x,y}$. Let C be a circuit of $M_{x,y}$ intersecting Q'. If $C \cap \{x, y\} = \phi$, then C is a circuit of M. Hence, by Lemma 1.2, C is a circuit of M_{xy}. Therefore $|C \cap Q'| \neq 1$. Suppose $C \cap \{x, y\} \neq 1$. Then, by Lemma 2.7(i), $\{x, y\} \subset C$. By [Theorem 2.2, 11], C is a circuit of M or it is disjoint union of circuits of M each containing x or y. It follows from Lemma 1.2 that $C' = (C - \{x, y\}) \cup \{a\}$ is a circuit of M_{xy}. Hence $|C \cap Q'| = |C' \cap Q'| \neq 1$. Thus, by Lemma 2.1, Q' contains a cocircuit Q'' of $M_{x,y}$. Obviously, Q'' is disjoint from the 2-cocircuit $\{x, y\}$ of $M_{x,y}$. It follows from Lemma 1.4 that Q'' is a cocircuit of M_{xy}. Hence $Q'' = Q'$. Thus Q' is a cocircuit of $M_{x,y}$. Now, the result follows from Lemma 2.7(iii). □

Lemma 2.9. *Let* M *be a binary matroid,* $\{x, y\}$ *be an independent set in* M *and* (X, Y) *be a partition of* $E(M_{xy})$ *with* $a \in X$*. Let* $X' = (X - a) \cup \{x, y\}$*. Then* $r(X') \leq r'(X) + 1$*, where* r *and* r' *are rank functions of the matroids* M *and* M_{xy}*, respectively. Equality holds if and only if either both* x, y *are coloops or they form a 2-cocircuit in* $M|X'$*.*

Proof. Suppose $\{x, y\}$ is a cocircuit of the matroid $M|X'$. Then x, y belong to a circuit C of $M|X'$. By Lemma 1.2, $(C - \{x, y\}) \cup \{a\}$ is a circuit in M_{xy} and hence in $M_{xy}|X$. Thus a is not a coloop in $M_{xy}|X$. Therefore $r'(X) = r'(X - a)$. By Lemma 2.4, $r(X - a) = r'(X - a)$. Hence $r(X') = r(X' - \{x, y\}) + 1 = r(X - a) + 1 = r'(X - a) + 1 = r'(X) + 1$. Suppose both x and y are coloops in $M|X'$. Then, by Lemma 1.2, there is no circuit in $M_{xy}|X$ containing a. Therefore a is a coloop in $M_{xy}|X$. Hence $r(X') = r(X' - \{x,y\}) + 2 = r(X - a) + 2 = r'(X - a) + 2 = r'(X) + 1$.

Suppose only one of x and y, say x is a coloop in $M|X'$. Then x does not belong to any circuit but y belongs to a circuit in $M|X'$. By Lemma 1.2, a is a coloop in $M_{xy}|X$. Therefore $r(X') = r(X' - \{x,y\}) + 1 = r(X - a) + 1 = r'(X)$. Finally, suppose $\{x,y\}$ does not contain a cocircuit in the matroid $M|X'$. Then there exist circuits C_x and C_y in $M|X'$ containing x and y, respectively such that $y \notin C_x$ and $x \notin C_y$. Hence, by Lemma 2.4, $r(X') = r(X' - \{x,y\}) = r(X - a) = r'(X - a) \leq r'(X)$. □

Proof of Theorem 1.5. Suppose M_{xy} is not n-connected. Then it has an $(n-1)$-separation (X,Y). Therefore $min\{|X|,|Y|\} \geq n-1$ and $r'(X) + r'(Y) - r'(M_{xy}) \leq n-2$, where r' is the rank function of M_{xy}. Since $n \geq 2$ and every circuit containing x, y has at least $n+1$ elements, $\{x,y\}$ is not a 2-circuit of M. Hence the ground set of M_{xy} is $(E(M) - \{x,y\}) \cup \{a\}$. We may assume that $a \in X$. Let $X' = (X - a) \cup \{x,\ y\}$. Then $|X'| \geq n$. Suppose $|Y| = n-1$. By Lemma 2.2, Y is independent in M. Also, by Lemma 1.2, Y does not contain any circuit of M_{xy}. Therefore $r'(Y) = n-1$. Consequently, $r'(X) \leq r'(M_{xy}) - r'(Y) + n - 2 = r'(M_{xy}) - 1$. By Lemma 2.5, Y contains a cocircuit Q of M_{xy}. As $|Q| \leq n-1$, by Lemma 2.2, Q does not contain a cocircuit of M. Further, $Q \cup \{x,y\}$ cannot be cocircuit of M because it has size less than $n+2$. Therefore, by Lemma 2.8, $Q \cup \{x,y\}$ is disjoint union of two cocircuits of M each containing x or y. By Lemma 2.2, $n+1 \geq |Q \cup \{x,y\}| \geq 2n$. Hence $n = 1$, a contradiction. Thus $|Y| \geq n$. This implies that $min\{|X'|,|Y|\} \geq n$.

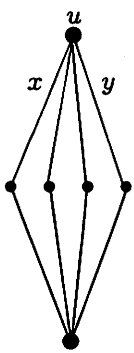
Since M is n-connected with $n \geq 2$, it does not have a coloop. Further, every cocircuit of M containing both x and y has size at least 4. Therefore $\{x,y\}$ does not contain a cocircuit of M. By Lemma 2.5, $r(M) = r'(M_{xy})$. Suppose $r(X') \leq r'(X)$. Then $r(X') + r(Y') - r(M) \leq r'(X) + r'(Y) - r'(M_{xy}) \leq n-2$. Thus (X',Y) is an $(n-1)$-separation of M, which is a contradiction. Therefore $r(X') > r'(X)$. By Lemma 2.9, $r(X') = r'(X)+1$. This implies that (X',Y) is an n-separation of M. If $r(X'), r(Y) \geq n$, then (X',Y) is a vertical n-separation of M, a contradiction. Therefore $r(X') = n-1$ or $r(Y) = n-1$. Suppose $r(X') = n-1$. Let X_1 be a subset of X' with $|X_1| = n$ and $x, y \in X_1$. Then X_1 is dependent in M. By Lemma 2.2, X_1 is a circuit in M containing both x and y, a contradiction to the hypothesis. Thus $r(X') \geq n$ and $r(Y) = n-1$. Therefore, by Lemma 2.2, every subset Y_1 of Y with $|Y_1| = n$ is a circuit in $M|Y$. If $|Y| \geq n+1$, then it follows that $U_{4,2}$ is a minor of $M|Y$, which is a contradiction. Hence $|Y| = n$ and Y is a circuit of M. As (X',Y) is an n-separation of M, by Lemma 2.3, Y does not contain any cocircuit of M.

Since $r'(Y) = n-1$, $r'(X) \leq r'(M_{xy}) - r'(Y) + n - 2 = r'(M_{xy}) - 1$. By Lemma 2.6, Y contains a cocircuit Q of M_{xy}. As Y does not contain any cocircuit of M, Q also does not contain a cocircuit of M. Obviously, $a \notin Q$ and $Q \cap \{x, y\} = \phi$. By Lemma 2.8, $Q \cup \{x, y\}$ is a cocircuit of M or it is union of two disjoint cocircuits of M each containing x or y. Suppose $Q \cup \{x, y\}$ is a cocircuit of M. Since every cocircuit of M containing both x and y has size at least $n + 2$, $n = |Y| \geq |Q| \geq n$. Therefore $Y = Q$. Thus $Y \cup \{x, y\}$ is a cocircuit of M such that Y is an n-circuit of M, a contradiction to the hypothesis. Suppose $Q \cup \{x, y\} = Q_x \cup Q_y$, where Q_x and Q_y are disjoint cocircuits of M containing x and y, respectively. By Lemma 2.2, $|Q_x|, |Q_y| \geq n$. Hence $n + 2 \geq |Q \cup \{x, y\}| = |Q_x| + |Q_y| \geq 2n$. Therefore $2 \geq n$. As M does not have coloops, each cocircuit of M has at least two elements. This implies that $|Q| = |Q_x| = |Q_y| = 2 = n$. Thus Y is a 2-circuit in M such that $|Y \cap Q_x| = |Q \cap Q_x| = 1$, which is a contradiction by Lemma 2.1. □

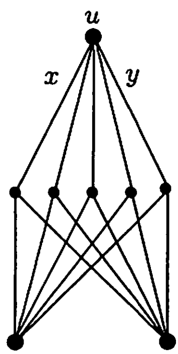
3. Remarks

In this section, we discuss the sharpness of Theorem 1.5.

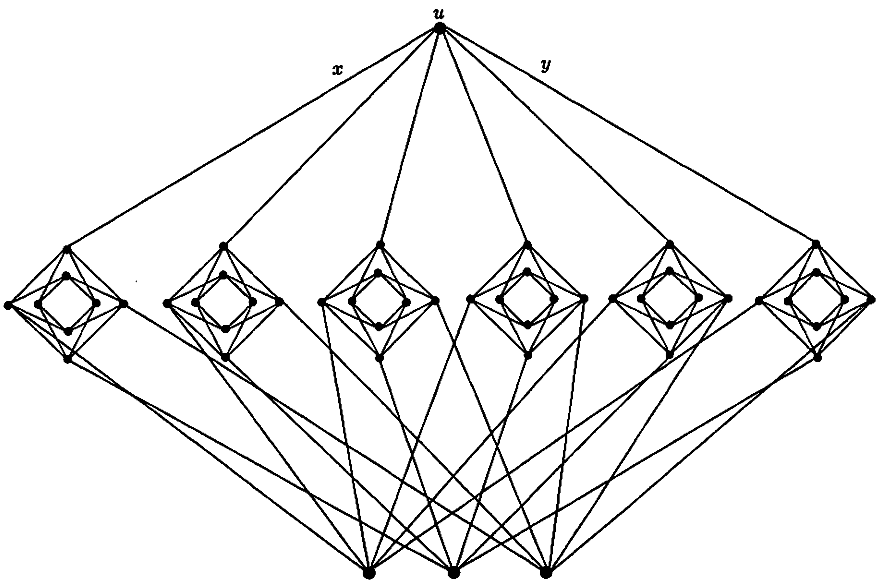
Remark 3.1. Splitting-off operation does not preserve connectedness of a binary matroid in general. We give some examples here. Let G_2, G_3 and G_4 be the graphs of Figures 1, 2 and 3, respectively. Then G_i is i-connected but not $(i+1)$-connected. Therefore the cycle matroid $M(G_i)$ is i-connected but not vertically $(i + 1)$-connected for $i = 2, 3, 4$. Let x and y be any two edges of G_i that are incident to the vertex u as shown in figures. Then the matroid $M(G_i)_{xy}$ is not i-connected for $i = 2, 3, 4$. Hence the condition of vertical connectivity in Theorem 1.5 is necessary.



G_2
Figure-1



G_3
Figure-2



G_4
Figure-3

Remark 3.2. Let M be a binary n-connected matroid with $|E(M)| \geq 2n$ and let $\{x, y\}$ be an independent set in M. Suppose there is an n-circuit C in M containing x, y. By Lemma 1.2, $(C - \{x, y\}) \cup a$ is a circuit of the matroid M_{xy} of size less than n. Hence, by Lemma 2.2, M_{xy} is not n-connected. Similarly, if M has a cocircuit Q containing both x and y with $2 < |Q| < n+2$, then, by Lemma 2.8, $Q - \{x, y\}$ is a cocircuit of M_{xy} of size less than n and therefore, by Lemma 2.2, M_{xy} is not n-connected. Suppose M has a cocircuit Q of size $n+2$ containing both x and y such that $Y = Q - \{x, y\}$ is an n-circuit in M. By Lemma 2.8, Y is a cocircuit of M_{xy}. Let $X = E(M_{xy}) - Y$. Then X is a hyperplane in M_{xy} and therefore $r'(X) = r'(M_{xy}) - 1$. Thus (X, Y) is a partition of M_{xy} with $|X|, |Y| \geq n-1$ and further, $r'(X) + r'(Y) - r'(M_{xy}) = r'(M_{xy}) - 1 + n - 1 - r'(M_{xy}) = n-2$. Hence (X, Y) is an $(n-1)$-separation of M_{xy}. Therefore M_{xy} is not n-connected. This shows that the conditions on circuits and cocircuits of M containing x, y in the hypothesis of Theorem 1.5 are necessary.

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